ON AN INVESTMENT-CONSUMPTION MODEL WITH TRANSACTION COSTS

MARIANNE AKIAN[†§], JOSÉ LUIS MENALDI[‡], AND AGNÈS SULEM[†¶]

Abstract. This paper considers the optimal consumption and investment policy for an investor who has available one bank account paying a fixed interest rate and n risky assets whose prices are log-normal diffusions. We suppose that transactions between the assets incur a cost proportional to the size of the transaction. The problem is to maximize the total utility of consumption. Dynamic Programming leads to a Variational Inequality for the value function. Existence and uniqueness of a viscosity solution is proved. The Variational Inequality is solved by using a numerical algorithm based on policies iterations and multigrid methods. Numerical results are displayed for $n = 1$ and $n = 2$.

Key words. portfolio selection, transaction costs, viscosity solution, variational inequality, multigrid methods.

AMS subject classifications. 90A09, 93E20, 49L20, 49L25, 65N55, 35R45

1. Introduction.

This paper concerns the theoretical and numerical study of a portfolio selection problem. Consider an investor who has available one riskless bank account paying a fixed rate of interest r and n risky assets modeled by log-normal diffusions with expected rates of return $\alpha_i > r$ and rates of return variation σ_i^2. The investor consumes at rate $c(t)$ from the bank account. Any movement of money between the assets incurs a transaction cost proportional to the size of the transaction, paid from the bank account. The investor is allowed to have a short position in one of the holdings, but his position vector must remain in the closed solvency region $\mathcal{S}$ defined as the set of positions for which the net wealth is nonnegative. The investor's objective is to maximize over an infinite horizon the expected discounted utility of consumption with a HARA (Hyperbolic Absolute Risk Aversion) type utility function.

This problem was formulated by Magill and Constantinides [21] for $n = 1$ who conjectured that the no-transaction region is a cone in the two-dimensional space of position vectors. This fact was proved in a discrete-time setting by Constantinides [8] who proposed an approximate solution based on making some assumptions on the consumption process. Davis and Norman proved, in continuous time, without this restriction, that the optimal strategy confines indeed the investor's portfolio to a wedge-shaped region in the portfolio plane [10]. An analysis of the optimal strategy together with regularity results for the value function can be found in Fleming and Soner [13, chapter 8.7] and Shreve and Soner [28]. Taksar, Klass and Assaf [31] consider a model without consumption and study the problem of maximizing the long-run average growth of wealth. A deterministic model is solved by Shreve, Soner and Xu [29] with a general utility function which is not necessarily a HARA type function. A stochastic model driven by a finite state Markov chain rather than a Brownian motion and with a general but bounded utility function has been investigated in Zariphopoulou [32]. She supposes that the amount of money allocated in the assets must remain non-negative and shows that the value function is the unique

† INRIA, Domaine de Voluceau Rocquencourt, BP105, 78153 Le Chesnay Cedex, France
‡ Wayne State University, Dept. of Mathematics, Detroit, Michigan, 48202, USA (jlm@math.wayne.edu). Supported in part under NSF grant DMS-9101360.
§ marianne.akian@inria.fr
¶ agnes.sulem@inria.fr

constrained viscosity solution of a system of variational inequalities with gradient constraints. Fitzpatrick and Fleming [12] study numerical methods for the optimal investment-consumption model with possible borrowing. They examine a Markov chain discretization of the original continuous problem similar to Kushner's numerical schemes [18]. The convergence arguments rely on viscosity solution techniques.

We consider here Davis and Norman's model [10] in the case where more than one risky asset is allowed. We restrict to power utility functions of the form $\frac{c^\gamma}{\gamma}$ with $0 < \gamma < 1$.

The purpose of the paper is to prove an existence and uniqueness result for the dynamic programming equation associated to this problem and then solve this equation by using an efficient numerical method, the convergence of which is ensured by the uniqueness result.

The mathematical formulation of the problem is given in §2. In §3, we prove that the value function is the unique viscosity solution of a variational inequality. Since the utility and the drift functions are not bounded, uniqueness does not derive from classical results. For the numerical study, an adequate change of variables is performed in §4 which reduces the dimension of the problem. Then, in §5, the variational inequality is discretized by finite difference schemes and solved by using an algorithm based on the "Howard algorithm" (policy iteration) and the multigrid method. Numerical results are presented in §6 in the case of one bank account and one or two risky asset(s). They provide the optimal strategy and indicate the shape of the transaction and no-transaction regions. Finally, in §7, a theoretical study of the optimal strategy is done by using properties of the variational inequality; this analysis corroborates the numerical results.

2. Formulation of the problem. Let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space and $(\mathcal{F}_t)_{t\geq 0}$ a given filtration. We denote by $s_0(t)$ (resp. $s_i(t)$ for $i = 1 \dots n$) the amount of money in the bank account (resp. in the i^{th} risky asset) at time t and refer as $s(t) = (s_i(t))_{i=0,\dots,n}$ the investor position at time t. We suppose that the evolution equations of the investor holdings are

$$(1) \qquad \begin{cases} ds_0(t) = (rs_0(t) - c(t))dt + \sum_{i=1}^{n}(-(1+\lambda_i)d\mathcal{L}_i(t) + (1-\mu_i)d\mathcal{M}_i(t)), \\ ds_i(t) = \alpha_i s_i(t)dt + \sigma_i s_i(t)dW_i(t) + d\mathcal{L}_i(t) - d\mathcal{M}_i(t), \quad i = 1, \dots, n, \end{cases}$$

with initial values

$$(2) \qquad s_i(0^-) = x_i, \quad i = 0, \dots, n$$

where $W_i(t)$, $i = 1, \dots, n$, are independent Wiener processes, $\mathcal{L}_i(t)$ and $\mathcal{M}_i(t)$ represent cumulative purchase and sale of stock i on $[0, t]$ respectively and $s(t^-)$ denotes the left hand limit of the process s at time t. The coefficients λ_i and μ_i represent the proportional transaction costs.

A policy for investment and consumption is a set $(c(t), (\mathcal{L}_i(t), \mathcal{M}_i(t))_{i=1,\dots,n})$ of adapted processes such that

1. $c(t, \omega) \geq 0$, $\displaystyle\int_0^t c(s, \omega)ds < \infty$, for a.e. (t, ω),
2. $\mathcal{L}_i(t)$, $\mathcal{M}_i(t)$ are right-continuous, non-decreasing and $\mathcal{L}_i(0^-) = \mathcal{M}_i(0^-) = 0$.

The process $s(t)$ is thus right continuous with left-hand limit and equations (1) and

(2) are equivalent to :

$$\begin{cases} s_0(t) &= x_0 + \int_0^t (rs_0(\theta) - c(\theta))d\theta + \sum_{i=1}^n \left(-(1+\lambda_i)\mathcal{L}_i(t) + (1-\mu_i)\mathcal{M}_i(t)\right), \\ s_i(t) &= x_i + \int_0^t \alpha_i s_i(\theta)d\theta + \int_0^t \sigma_i s_i(\theta)dW_i(\theta) + \mathcal{L}_i(t) - \mathcal{M}_i(t), \quad i = 1, \ldots, n \end{cases}$$

for $t \geq 0$.

We define the solvency region as :

$$\mathcal{S} = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, \quad \mathcal{W}(x) \geq 0\}$$

where

$$\mathcal{W}(x) = x_0 + \sum_{i=1}^n \min((1-\mu_i)x_i, (1+\lambda_i)x_i) \tag{3}$$

represents the net wealth, that is the amount of money obtained in the bank account after performing the transactions which bring the holdings in the risky assets to zero.

Suppose that the investor is given an initial endowment x in $\mathcal{S}$. A policy is admissible if the bankruptcy time $\overline{\tau}$ defined as

$$\overline{\tau} = \inf\{t \geq 0, s(t) \notin \mathcal{S}\} \tag{4}$$

is infinite. We denote by $\mathcal{U}(x)$ the set of admissible policies. The investor's objective is to maximize over all policies $\mathcal{P}$ in $\mathcal{U}(x)$ the discounted utility of consumption

$$J_x(\mathcal{P}) = E_x \int_0^\infty e^{-\delta t} u(c(t))dt \tag{5}$$

where E_x denotes expectation given the initial endowment x, δ is a positive discount factor and $u(c)$ is a utility function defined by

$$u(c) = \frac{c^\gamma}{\gamma}, \quad 0 < \gamma < 1. \tag{6}$$

We define the value function V as

$$V(x) = \sup_{\mathcal{P} \in \mathcal{U}(x)} J_x(\mathcal{P}). \tag{7}$$

We are facing here a singular control problem. We refer to Menaldi and Robin [23], Chow, Menaldi and Robin [7] for various treatments of singular stochastic control problems.

Remark 2.1. When the process $s(t)$ reaches the boundary $\partial\mathcal{S}$ at time t, i.e. $s(t^-) \in \partial\mathcal{S}$, the only admissible policy is to jump immediately to the origin and remain there with a null consumption (see Shreve, Soner and Xu [29]). Consequently, if the initial endowment x is on the boundary, then $V(x) = 0$.

Remark 2.2. Let τ denote the exit time of the interior of $\mathcal{S}$, defined as

$$\tau = \inf\left\{t \geq 0, s(t) \notin \overset{\circ}{\mathcal{S}}\right\}. \tag{8}$$

For all admissible policies $\mathcal{P}$, we have

$$J_x(\mathcal{P}) = E_x \int_0^\tau e^{-\delta t} u(c(t))dt. \tag{9}$$

On the other hand, for any policy $\mathcal{P}$, we can construct an admissible policy which coincides with $\mathcal{P}$ until time τ (it is such that the process $s(t)$ jumps to the origin at time τ). The value function can then be rewritten as

$$V(x) = \sup_{\mathcal{P}\in\mathcal{U}} E_x \int_0^\tau e^{-\delta t} u(c(t))dt \tag{10}$$

where $\mathcal{U}$ is the set of all policies.

We make the assumptions

$$\delta > \gamma \left(r + \frac{1}{2(1-\gamma)} \sum_{i=1}^n (\frac{\alpha_i - r}{\sigma_i})^2 \right), \tag{$A.1$}$$

$$0 \leq \mu_i < 1, \quad \lambda_i \geq 0, \quad \lambda_i + \mu_i > 0, \quad \forall i = 1, \ldots, n. \tag{$A.2$}$$

Remark 2.3. When the transaction costs are equal to zero (Merton's problem), the value function V is finite iff assumption (A.1) is satisfied (see Davis and Norman [10] for $n = 1$, Karatzas, Lehoczky, Sethi and Shreve [17], and §7 below).

3. The Variational Inequality. We state the main theorem.

THEOREM 3.1. *Under assumptions* (A.1) *and* (A.2),

(i) *the value function V defined in* (7) *or* (10) *is γ-Hölder continuous and concave in $\mathcal{S}$ and non decreasing with respect to x_i for $i = 0, \ldots n$.*

(ii) *V is the unique viscosity solution of the variational inequality (VI) :*

$$\max \left\{ AV + u^*(\frac{\partial V}{\partial x_0}), \max_{1\leq i\leq n} L_i V, \max_{1\leq i\leq n} M_i V \right\} = 0 \qquad \text{in } \mathring{\mathcal{S}} \tag{11}$$

$$V = 0 \qquad \text{on } \partial\mathcal{S} \tag{12}$$

where

$$AV = \frac{1}{2} \sum_{i=1}^n \sigma_i^2 x_i^2 \frac{\partial^2 V}{\partial {x_i}^2} + \sum_{i=1}^n \alpha_i x_i \frac{\partial V}{\partial x_i} + r x_0 \frac{\partial V}{\partial x_0} - \delta V, \tag{13}$$

$$L_i V = -(1+\lambda_i) \frac{\partial V}{\partial x_0} + \frac{\partial V}{\partial x_i}, \tag{14}$$

$$M_i V = (1-\mu_i) \frac{\partial V}{\partial x_0} - \frac{\partial V}{\partial x_i}, \tag{15}$$

and u^* is the convex Legendre transform of u defined by

$$\begin{aligned} u^*(\nu) &= \max_{c\geq 0}(-c\nu + u(c)) \\ &= (\frac{1}{\gamma} - 1)\nu^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{16}$$

The solvency region $\mathcal{S}$ is divided as follows :

$$B_i = \{x \in \mathcal{S},\ L_i V(x) = 0\}, \tag{17}$$

$$S_i = \{x \in \mathcal{S},\ M_i V(x) = 0\}, \tag{18}$$

$$NT_i = \mathcal{S} \setminus (B_i \cup S_i), \tag{19}$$

$$NT = \bigcap_{i=1}^{n} NT_i. \tag{20}$$

NT is the no transaction region. Outside NT, an instantaneous transaction brings the position to the boundary of NT : buy stock i in B_i, sell stock i in S_i. After the initial transaction, the agent position remains in

$$\overline{NT} = \{x \in \mathcal{S}, AV + u^* \left(\frac{\partial V}{\partial x_0}\right) = 0\},$$

and further transactions occur only at the boundary (see [10]).

We shall first recall the definition of viscosity solutions, then prove points (i) and (ii) of Theorem 3.1 in §3.2 and §3.3 respectively.

3.1. Viscosity solutions of non linear elliptic equations. For simplicity, we restrict ourselves to equations with Dirichlet boundary conditions. Consider fully nonlinear elliptic equations of the form

$$F(D^2 v, Dv, v, x) = 0 \quad \text{in } \mathcal{O} \tag{21}$$

$$v = 0 \quad \text{on } \partial\mathcal{O} \tag{22}$$

where F is a given continuous function in $S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathcal{O}$, S^N is the space of symmetric $N \times N$ matrices, $\mathcal{O}$ is an open domain of $\mathbb{R}^N$, and the ellipticity of (21) is expressed by

$$F(A, p, v, x) \geq F(B, p, v, x) \quad \text{if } A \geq B,\ A, B \in S^N,\ p \in \mathbb{R}^N,\ v \in \mathbb{R},\ x \in \mathcal{O}. \tag{23}$$

A special case of (21) is given by

$$F(X, p, v, x) = \max_{\nu \in U}\{\sum_{i,j=1}^{N} a_{ij}(x, \nu) X_{ij} + \sum_{i=1}^{N} b_i(x, \nu) p_i - \beta(x, \nu) v + u(x, \nu)\} \tag{24}$$

where (23) is satisfied when the matrix $(a_{ij}(x, \nu))_{i,j}$ is symmetric non negative in $\mathcal{O} \times U$.

Bellman equations are clearly equations of this type, whereas variational inequalities like (11-12) can also be formulated in this form by using an additive discrete control which selects the equation which satisfies the maximum.

Definition 3.2. *Let $v \in \mathcal{C}(\overline{\mathcal{O}})$. Then v is a viscosity solution of* (21-22) *if the following relations hold, together with* (22) :

$$F(X, p, v(x), x) \geq 0 \qquad \forall (p, X) \in J^{2,+} v(x),\ \ \forall x \in \mathcal{O} \tag{25}$$

$$F(X, p, v(x), x) \leq 0 \qquad \forall (p, X) \in J^{2,-} v(x),\ \ \forall x \in \mathcal{O} \tag{26}$$

where $J^{2,+}$ and $J^{2,-}$ are the second order "superjets" defined by :

$$J^{2,+} v(x) = \{(p, X) \in \mathbb{R}^N \times S^N,$$
$$\limsup_{\substack{y \to x \\ y \in \mathcal{O}}} [v(y) - v(x) - (p, y - x) - \frac{1}{2}(X(y - x), y - x)] |y - x|^{-2} \leq 0\}$$

and

$$J^{2,-}v(x) = \{(p,X) \in \mathbb{R}^N \times S^N, \\ \liminf_{\substack{y\to x\\ y\in\mathcal{O}}} [v(y) - v(x) - (p, y-x) - \frac{1}{2}(X(y-x), y-x)]|y-x|^{-2} \geq 0\}.$$

A viscosity subsolution (resp. supersolution) of (21) is similarly defined as an upper semicontinuous function satisfying (25) (resp. a lower semicontinuous function satisfying (26)) (see Crandall, Ishii and Lions [9]).

3.2. Properties of the value function.

PROPOSITION 3.3. *The value function V is concave in $\mathcal{S}$.*

Proof. The dynamic (1) is linear and the solvency region $\mathcal{S}$ is convex. Hence, for any θ in $[0,1]$, x and x' in $\mathcal{S}$, $\mathcal{P}$ in $\mathcal{U}(x)$ and $\mathcal{P}'$ in $\mathcal{U}(x')$, we have $\theta\mathcal{P} + (1-\theta)\mathcal{P}' \in \mathcal{U}(y)$ for $y = \theta x + (1-\theta)x'$ and

$$V(y) \geq J_y(\theta\mathcal{P} + (1-\theta)\mathcal{P}') = E\int_0^{+\infty} e^{-\delta t} u(\theta c(t) + (1-\theta)c'(t))dt.$$

Since u is concave we infer

$$V(y) \geq \theta J_x(\mathcal{P}) + (1-\theta)J_{x'}(\mathcal{P}').$$

Taking now the supremum over all $\mathcal{P}$ and $\mathcal{P}'$, we obtain that V is concave. □

As a consequence, V is locally Lipschitz continuous in $\mathring{\mathcal{S}}$. The continuity of V at the boundary is a consequence of the Proposition 3.5 below. First, let us state the following lemma.

LEMMA 3.4. *Suppose* (A.1) *holds. Then, there exists a positive constant, a, such that the functions*

$$\varphi_\nu(x) = a\left(x_0 + \sum_{i=1}^n (1-\nu_i)x_i\right)^\gamma \quad \textit{with } \nu = (\nu_1, \ldots, \nu_n), \quad \nu_i = -\lambda_i \textit{ or } \mu_i \tag{27}$$

are classical supersolutions of equation (11)*. Consequently, the function*

$$\varphi(x) = a\left(x_0 + \sum_{i=1}^n \min((1-\mu_i)x_i, (1+\lambda_i)x_i)\right)^\gamma \tag{28}$$

is a viscosity supersolution of equation (11) *such that $\varphi = 0$ on $\partial\mathcal{S}$.*

Proof. Denote

$$\mathcal{W}_\nu(x) = x_0 + \sum_{i=1}^n (1-\nu_i)x_i. \tag{29}$$

Then

$$\varphi_\nu(x) = a\mathcal{W}_\nu(x)^\gamma \tag{30}$$

and we have

$$L_i\varphi_\nu(x) = -(\lambda_i + \nu_i)a\gamma\mathcal{W}_\nu(x)^{\gamma-1} \leq 0, \tag{31}$$

$$M_i\varphi_\nu(x) = (\mu_i - \nu_i)a\gamma\mathcal{W}_\nu(x)^{\gamma-1} \le 0, \tag{32}$$

$$A\varphi_\nu(x) = a\gamma\mathcal{W}_\nu(x)^{\gamma-2}G(x)$$

with

$$\begin{aligned} G(x) &= \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 x_i^2(1-\nu_i)^2(\gamma-1) \\ &+ \mathcal{W}_\nu(x)\left(\sum_{i=1}^{n}(\alpha_i - r)x_i(1-\nu_i)\right) \\ &+ \mathcal{W}_\nu(x)^2\left(r - \frac{\delta}{\gamma}\right). \end{aligned}$$

From Assumption (A.1), there exists $\eta > 0$ such that

$$\frac{\delta}{\gamma} - r - \eta \ge \frac{1}{2(1-\gamma)}\sum_{i=1}^{n}\left(\frac{\alpha_i - r}{\sigma_i}\right)^2.$$

This implies

$$G(x) \le -\eta\mathcal{W}_\nu(x)^2$$

and

$$A\varphi_\nu(x) \le -a\gamma\eta\mathcal{W}_\nu(x)^\gamma = -\gamma\eta\varphi_\nu(x). \tag{33}$$

Moreover

$$u^*(\frac{\partial\varphi_\nu}{\partial x_0}(x)) = u^*(a\gamma\mathcal{W}_\nu(x)^{\gamma-1}) = (\frac{1}{\gamma} - 1)(a\gamma)^{\frac{\gamma}{\gamma-1}}\mathcal{W}_\nu(x)^\gamma = (1-\gamma)(a\gamma)^{\frac{1}{\gamma-1}}\varphi_\nu(x).$$

The constant a can then be chosen such that

$$A\varphi_\nu + u^*(\frac{\partial\varphi_\nu}{\partial x_0}) \le 0 \quad \text{in } \overset{\circ}{\mathcal{S}}. \tag{34}$$

Since (31), (32), (34) hold and $\varphi_\nu \ge 0$ on $\partial\mathcal{S}$, φ_ν is a classical supersolution of (11) (continuous in $\mathcal{S}$ and twice continuously differentiable in $\overset{\circ}{\mathcal{S}}$).

Now, φ can be rewritten as

$$\varphi = \min_\nu \varphi_\nu.$$

Consequently, φ is a viscosity supersolution of (11) as the minimum of continuous supersolutions and clearly vanishes on $\partial\mathcal{S}$. □

Proposition 3.5. *Suppose* (A.1) *holds. The value function V satisfies*

$$0 \le V(x) \le \varphi(x) \qquad \forall x \in \mathcal{S}, \tag{35}$$

where φ is the supersolution defined by (28). *Consequently, V is continuous in $\mathcal{S}$.*

Proof. Consider $x \in \mathcal{S}$ and $\mathcal{P} \in \mathcal{U}$ and denote by τ the first exit time of $\overset{\circ}{\mathcal{S}}$ of the process $s(t)$ defined by (1) with $s(0^-) = x$. The function φ_ν defined in (27) has $\mathcal{C}^2$-regularity and is a classical supersolution of (11). Denote by A^c the operator $A - c\frac{\partial}{\partial x_0}$ and by $\mathcal{L}_i^c(t)$ and $\mathcal{M}_i^c(t)$ the continuous parts of $\mathcal{L}_i(t)$ and $\mathcal{M}_i(t)$. We apply Ito's formula for càdlàg processes (see Meyer [25]) to $e^{-\delta t}\varphi_\nu(s(t))$. For any stopping time θ, the process

$$\begin{aligned} & e^{-\delta(\theta\wedge\tau)}\varphi_\nu(s(\theta\wedge\tau)) \\ & \quad - \int_0^{\theta\wedge\tau} e^{-\delta t}\left\{A^{c(t)}\varphi_\nu(s(t))dt + \sum_{i=1}^n [L_i\varphi_\nu(s(t))d\mathcal{L}_i^c(t) + M_i\varphi_\nu(s(t))d\mathcal{M}_i^c(t)]\right\} \\ & \quad - \sum_{0\le t\le\theta\wedge\tau} e^{-\delta t}[\varphi_\nu(s(t)) - \varphi_\nu(s(t^-))] \end{aligned}$$

is a martingale.

Since $s(t)$ has a jump only when $\mathcal{L}_i(t)$ or $\mathcal{M}_i(t)$ is discontinuous, we have

$$\begin{aligned} \mathcal{W}_\nu(s(t)) &= \mathcal{W}_\nu(s(t^-)) \\ & \quad - \sum_{i=1}^n [(\lambda_i + \nu_i)(\mathcal{L}_i(t) - \mathcal{L}_i(t^-)) + (\mu_i - \nu_i)(\mathcal{M}_i(t) - \mathcal{M}_i(t^-))] \\ &\le \mathcal{W}_\nu(s(t^-)). \end{aligned}$$

Hence,

$$\varphi_\nu(s(t)) \le \varphi_\nu(s(t^-)).$$

In addition, $\mathcal{L}_i^c$ and $\mathcal{M}_i^c$ are non decreasing. Consequently

$$M_t^\nu = e^{-\delta(t\wedge\tau)}\varphi_\nu(s(t\wedge\tau)) + \int_0^{t\wedge\tau} e^{-\delta\theta}u(c(\theta))d\theta \tag{36}$$

is a supermartingale, as well as the process $\min_\nu M_t^\nu$. Therefore

$$E\int_0^\tau e^{-\delta t}u(c(t))dt \le \varphi(x).$$

Taking the supremum over all policies $\mathcal{P}\in\mathcal{U}$, we get $V(x)\le\varphi(x)$. As $V(x)\ge 0$, we conclude that $V(x) = 0$ on ∂S and that V is continuous on ∂S. Since V is locally Lipschitz continuous in $\mathring{\mathcal{S}}$, V is continuous in $\mathcal{S}$. □

The regularity of V can be precised as follows:

PROPOSITION 3.6. *Suppose* (A.1) *holds. Then, V is uniformly γ-Hölder continuous in $\mathcal{S}$, that is*

$$\exists C > 0, \quad |V(x) - V(x')| \le C\|x - x'\|^\gamma \qquad \forall x, x' \in \mathcal{S}. \tag{37}$$

Proof. Consider two initial positions x and x', and denote by τ (resp. τ') the first exit time of $\mathring{\mathcal{S}}$ of the process $s(t)$ (resp. $s'(t)$) defined by (1) and $s(0^-) = x$ (resp. $s'(0^-) = x'$). We have

$$V(x) - V(x') = \sup_{\mathcal{P}\in\mathcal{U}} E\int_0^\tau e^{-\delta t}u(c(t))dt - \sup_{\mathcal{P}\in\mathcal{U}} E\int_0^{\tau'} e^{-\delta t}u(c(t))dt$$

$$\leq \sup_{\mathcal{P}\in\mathcal{U}} E\left(\int_0^{\tau} e^{-\delta t}u(c(t))dt - \int_0^{\tau'} e^{-\delta t}u(c(t))dt\right)$$
$$\leq \sup_{\mathcal{P}\in\mathcal{U}} E\int_{\tau\wedge\tau'}^{\tau} e^{-\delta t}u(c(t))dt.$$

Using the supermartingale property of $\min_{\nu} M_t^{\nu}$ defined in (36), we get

$$E\int_{\tau\wedge\tau'}^{\tau} e^{-\delta t}u(c(t))dt \leq E(e^{-\delta(\tau\wedge\tau')}\varphi(s(\tau\wedge\tau')) - e^{-\delta\tau}\varphi(s(\tau)))$$

and since φ vanishes on $\partial\mathcal{S}$, we have

$$V(x) - V(x') \leq \sup_{\mathcal{P}\in\mathcal{U}} E(e^{-\delta\tau'}(\varphi(s(\tau')) - \varphi(s'(\tau')))1_{\tau'<\tau})$$

where 1_A denotes the characteristic function of the set A.

Let us fix for instance $\|x\| = \sup_{i=0,\ldots,n}|x_i|$. The function φ is γ-Hölder continuous, that is

$$|\varphi(x) - \varphi(x')| \leq C\|x-x'\|^{\gamma}$$

for some positive constant C. We thus get

$$V(x) - V(x') \leq C\sup_{\mathcal{P}\in\mathcal{U}} E(e^{-\delta\tau'}\|(s(\tau') - s'(\tau')\|^{\gamma}1_{\tau'<\tau}). \tag{38}$$

The process $\Sigma(t) = s(t) - s'(t)$ is a diffusion process with generator $A + \delta I$ and initial value $\Sigma(0) = x - x'$.

If the function $\psi(x) = \|x\|^{\gamma}$ would satisfy $A\psi \leq 0$, then $\psi(\Sigma(\tau\wedge t))e^{-\delta(\tau\wedge t)}$ would be a supermartingale which would readily lead to (37). As ψ is not smooth, we consider the function $\psi_\beta(x) = \sum_{i=0}^n (x_i^2+\beta)^{\gamma/2}$ with $\beta > 0$. We have :

$$A\psi_\beta = \gamma(x_0^2+\beta)^{(\frac{\gamma}{2}-1)}((r-\frac{\delta}{\gamma})x_0^2 - \frac{\delta}{\gamma}\beta) + \sum_{i=1}^n \gamma(x_i^2+\beta)^{(\frac{\gamma}{2}-2)}f_i$$

with

$$f_i = x_i^4(\frac{1}{2}\sigma_i^2(\gamma-1) + \alpha_i - \frac{\delta}{\gamma}) + x_i^2\beta(\frac{1}{2}\sigma_i^2 + \alpha_i - \frac{2\delta}{\gamma}) - \frac{\delta}{\gamma}\beta^2.$$

Assumption (A.1) implies

$$r - \frac{\delta}{\gamma} < 0 \quad \text{and} \quad \frac{1}{2}\sigma_i^2(\gamma-1) + \alpha_i - \frac{\delta}{\gamma} < 0.$$

Consequently, there exists a positive constant C such that

$$A\psi_\beta \leq C\beta^{\gamma/2}.$$

Applying now Ito's formula to ψ_β, we obtain

$$E(e^{-\delta(\tau'\wedge\tau)}\psi_\beta(\Sigma(\tau'\wedge\tau))) \leq \psi_\beta(x-x') + \frac{C}{\delta}\beta^{\gamma/2}. \tag{39}$$

Taking the limit of (39) when β goes to zero and using

$$\psi(x) \leq \psi_0(x) \leq (n+1)\psi(x)$$

we get

$$E(e^{-\delta(\tau' \wedge \tau)}\psi(\Sigma(\tau' \wedge \tau))) \leq (n+1)\psi(x - x')$$

which leads together with (38) to the desired estimate (37). □

PROPOSITION 3.7. *V is non decreasing with respect to x_i, for $i = 0, \ldots, n$.*

Proof. Let us denote explicitly by $s(t,x)$ the process $s(t)$ defined in (1) with initial value x and by τ_x the exit time of $\mathring{\mathcal{S}}$ of $s(t,x)$. As

$$V(x) = \sup_{\mathcal{P} \in \mathcal{U}} E \int_0^{\tau_x} e^{-\delta t} u(c(t))dt$$

and u is positive, it is enough to prove the non decreasing property of the stopping time τ_x for any control process $\mathcal{P}$.

Define $y(t,x)$ by

$$\begin{cases} y_0(t,x) &= e^{-rt} s_0(t,x) \\ \\ y_i(t,x) &= e^{-(\alpha_i - \frac{1}{2}\sigma_i^2)t - \sigma_i W_i(t)} s_i(t,x) \quad i = 1, \ldots, n. \end{cases}$$

The process $y(t,x)$ evolves according to

$$(40) \begin{cases} dy_0(t,x) &= e^{-rt}\left(-c(t)dt + \sum_{i=1}^n (-(1+\lambda_i)d\mathcal{L}_i(t) + (1-\mu_i)d\mathcal{M}_i(t))\right) \\ dy_i(t,x) &= e^{-(\alpha_i - \frac{1}{2}\sigma_i^2)t - \sigma_i W_i(t)}(d\mathcal{L}_i(t) - d\mathcal{M}_i(t)) \end{cases}$$

and satisfies $y(0,x) = x$. Hence, we can write

$$y(t,x) = x + Y(t,\mathcal{P})$$

where $Y(t,\mathcal{P})$ depends only on $\mathcal{P}$. Consequently,

$$(41) \qquad s(t,x) = (e^{rt}x_0, (e^{(\alpha_i - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)} x_i)_{i=1,\ldots,n}) + S(t,\mathcal{P})$$

where $S(t,\mathcal{P})$ is a process which is independent of x.

Consider now $\tilde{x} \geq x$ (i.e. $\tilde{x}_i \geq x_i$, $\forall i = 0, \ldots, n$) and fix $\mathcal{P}$ in $\mathcal{U}$. We have from (41),

$$s(t,x) \leq s(t,\tilde{x})$$

and

$$\mathcal{W}(s(t,x)) \leq \mathcal{W}(s(t,\tilde{x}))$$

where $\mathcal{W}$ is defined in (3).

Since

$$\tau_{\tilde{x}} = \inf\{t \geq 0,\ \mathcal{W}(s(t,\tilde{x})) \leq 0\},$$

for any $t > \tau_{\tilde{x}}$, there exists t' such that $\tau_{\tilde{x}} < t' < t$ and $\mathcal{W}(s(t',\tilde{x})) \leq 0$. This implies $\mathcal{W}(s(t',x)) \leq 0$ and $t > t' \geq \tau_x$. Consequently, $\tau_{\tilde{x}} \geq \tau_x$ and $V(\tilde{x}) \geq V(x)$. □

3.3. Existence and Uniqueness results. First, we show that the value function V is a viscosity solution of the variational inequality (11). The problem is reduced to prove a weak dynamic programming principle (see Fleming and Soner [13]).

LEMMA 3.8. *There exists $C > 0$ such that*

$$|J_x(\mathcal{P}) - J_{x'}(\mathcal{P})| \le C\|x - x'\|^\gamma \quad \forall x, x' \in \mathcal{S}, \quad \forall \mathcal{P} \in \mathcal{U}, \tag{42}$$

where $J_x(\mathcal{P})$ is given in (9).

Proof. Estimate (42) is readily obtained from the proof of Proposition 3.6. □

PROPOSITION 3.9. *The weak dynamic programming principle is satisfied for the value function V, that is*

$$V(x) = \sup_{\mathcal{P}\in\mathcal{U}} E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}V(s((\theta\wedge\tau)^-))\right) \quad \forall x \in \mathcal{S}, \tag{43}$$

for any stopping time θ.

Proof. By means of the Markov property, we have for all $\mathcal{P}$ in $\mathcal{U}$

$$E^{\mathcal{F}^{\theta\wedge\tau}}\int_0^{\tau} e^{-\delta t}u(c(t))dt = \int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}J_{s((\theta\wedge\tau)-)}(\mathcal{P}'),$$

with $\mathcal{P}'$ equal to $\mathcal{P}$ "shifted" by $\theta\wedge\tau$. Note that $\mathcal{P}'$ may not be admissible. The correct method would be to proceed with admissible systems composed with a filtration $(\Omega, \mathcal{F}_t, P)$, a Wiener process $W = (W_i)_{i=1,\dots,n}$ in $\mathbb{R}^n$, and an admissible control process $\mathcal{P}$ and consider V as the supremum of $J_x(\mathcal{P})$ over all admissible systems instead of the supremum over all admissible policies. We give here a formal proof. Rigorous proofs are given in Fleming and Soner [13], Nisio [26], El Karoui [11], Lions [19]. Thus,

$$\begin{aligned} J_x(\mathcal{P}) &= E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}J_{s((\theta\wedge\tau)-)}(\mathcal{P}')\right) \\ &\le E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}V(s((\theta\wedge\tau)^-))\right). \end{aligned}$$

By taking the supremum over all policies $\mathcal{P}$, we deduce one inequality side of (43). For the reverse inequality, we need to construct nearly optimal controls for each initial state x in a measurable way. To that purpose, consider $\varepsilon > 0$ and $\{\mathcal{S}^k\}_{k=1}^\infty$ a sequence of disjoint subsets of $\mathcal{S}$ such that

$$\bigcup_{k=1}^{\infty} \mathcal{S}^k = \mathcal{S}, \qquad \text{diameter}(\mathcal{S}^k) < \varepsilon.$$

For any k, take x^k in $\mathcal{S}^k$ and $\mathcal{P}^k = (c^k, (\mathcal{L}_i^k, \mathcal{M}_i^k)_{i=1,\dots,n})$ in $\mathcal{U}$ such that

$$V(x^k) - \varepsilon \le J_{x^k}(\mathcal{P}^k). \tag{44}$$

Now, for a given stopping time θ and an arbitrary policy $\mathcal{P}$ in $\mathcal{U}$, we define $\mathcal{P}^\theta = (c^\theta, (\mathcal{L}_i^\theta, \mathcal{M}_i^\theta)_{i=1,\dots,n})$ with

$$\begin{aligned} c^\theta(t) &= c(t)1_{t<\theta} + c^k(t-\theta)1_{t\ge\theta}, \\ \mathcal{L}_i^\theta(t) &= \mathcal{L}_i(t)1_{t<\theta} + (\mathcal{L}_i(\theta^-) + \mathcal{L}_i^k(t-\theta))1_{t\ge\theta}, \\ \mathcal{M}_i^\theta(t) &= \mathcal{M}_i(t)1_{t<\theta} + (\mathcal{M}_i(\theta^-) + \mathcal{M}_i^k(t-\theta))1_{t\ge\theta}, \end{aligned}$$

for $s(\theta^-) \in \mathcal{S}^k$. Using (42) and (44) we have

$$\begin{aligned} J_{s(\theta-)}(\mathcal{P}^k) &= (J_{s(\theta-)}(\mathcal{P}^k) - J_{x^k}(\mathcal{P}^k)) + J_{x^k}(\mathcal{P}^k) \\ &\geq -C\varepsilon^\gamma - \varepsilon + V(x^k) \\ &\geq -2C\varepsilon^\gamma - \varepsilon + V(s(\theta^-)). \end{aligned}$$

Denote by I the right-hand side of (43). There exists a policy $\mathcal{P}$ such that

$$I - \varepsilon \leq E\left(\int_0^{\theta\wedge\tau} e^{-\delta t} u(c(t))dt + e^{-\delta(\theta\wedge\tau)} V(s((\theta\wedge\tau)^-))\right)$$

and using the Markov property, we get

$$I - \varepsilon \leq J_x(\mathcal{P}^{\theta\wedge\tau}) + (2C\varepsilon^\gamma + \varepsilon)$$

and

$$I - 2C\varepsilon^\gamma - 2\varepsilon \leq J_x(\mathcal{P}^{\theta\wedge\tau}) \leq V(x)$$

which leads to (43). $\square$

COROLLARY 3.10. *The value function* $V(x)$ *defined by* (10) *is a viscosity solution of the variational inequality* (11-12).

In the case of pure diffusion processes, this is a standard result of the theory of viscosity solutions (see Lions [20]). For singular stochastic control problems, we refer to Fleming and Soner [13, chapter 8, Theorem 5.1].

PROPOSITION 3.11. *Under assumptions* (A.1) *and* (A.2)*, the value function* V *is the unique viscosity solution of the variational inequality* (11-12) *in the class of continuous functions in* $\mathcal{S}$ *which satisfy*

$$|V(x)| \leq C(1 + \|x\|^\gamma) \qquad \forall x \in \mathcal{S}. \tag{45}$$

Proof. By Corollary 3.10 and equation (35), the value function V is a viscosity solution of (11-12) and satisfies (45). Uniqueness is a consequence of the following maximum principle :

LEMMA 3.12. *if* v *is a viscosity subsolution and* v' *is a viscosity supersolution of* (11) *which satisfy* (45) *and* $v \leq v'$ *on* $\partial\mathcal{S}$*, then* $v \leq v'$ *in* $\mathcal{S}$.

Indeed, a viscosity solution of (11-12) is both a subsolution and a supersolution with the boundary condition $v = 0$ on $\partial\mathcal{S}$. We prove Lemma 3.12 by using the Ishii technique, in particular we adapt the proofs of Theorems 3.3 and 5.1 of Crandall, Ishii and Lions [9]. They are themselves based on the following corollary of Theorem 3.2 of [9] :

LEMMA 3.13. *Let* V *be an upper semicontinuous function and* V' *a lower semicontinuous function in an open domain* $\mathcal{O}$ *of* $\mathbb{R}^N$. *Consider* $W(x,y) = V(x) - V'(y) - \frac{k}{2}|x-y|^2$ *with* $k > 0$ *and suppose that* $(\hat{x}, \hat{y})$ *is a local maximum of* W. *Then, there exist two matrices* X *and* Y *in* S^N *such that*

$$(k(\hat{x} - \hat{y}), X) \in \overline{J}^{2,+} V(\hat{x}), \quad (k(\hat{x} - \hat{y}), Y) \in \overline{J}^{2,-} V'(\hat{y})$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3k \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{46}$$

In this statement, $|.|$ denotes the euclidian norm, I the identity $N \times N$ matrix and $\overline{J}^{2,+}$ is defined as follows :

$$\overline{J}^{2,+}v(x) = \{(p, X) \in \mathbb{R}^N \times S^N,\ \exists (x_n, p_n, X_n) \in \mathcal{O} \times \mathbb{R}^N \times S^N,$$
$$(p_n, X_n) \in J^{2,+}v(x_n) \text{ and } (x_n, v(x_n), p_n, X_n) \underset{n\to\infty}{\to} (x, v(x), p, X)\}.$$

$\overline{J}^{2,-}$ is similarly defined. If F is a continuous function in $S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathcal{O}$ satisfying the elliptic condition (23), and v is a viscosity subsolution of (21), we have

$$F(X, p, v(x), x) \geq 0 \ \ \forall (p, X) \in \overline{J}^{2,+}v(x), \ \ \forall x \in \mathcal{O}. \tag{47}$$

Consider now v and v' as in Lemma 3.12 and argue by contradiction in order to prove $v \leq v'$ in $\mathcal{S}$. Suppose that there exists z in $\mathcal{S}$ such that $v(z) - v'(z) > 0$. For $k > 0$, define the function w_k in $\mathcal{S} \times \mathcal{S}$ as :

$$w_k(x, y) = v(x) - v'(y) - \frac{k}{2}|x - y|^2 - \varepsilon(\mathcal{W}_\nu(x)^{\gamma'} + \mathcal{W}_\nu(y)^{\gamma'})$$

where

$$\mathcal{W}_\nu(x) = x_0 + \sum_{i=1}^{n}(1 - \nu_i)x_i,$$

and $\nu, \varepsilon, \gamma'$ are parameters which will be chosen further. In addition, denote

$$m_k = \sup_{(x,y)\in\mathcal{S}\times\mathcal{S}} w_k(x, y).$$

In the following, C, C_1, C_2 denote generic constants.

Lemma 3.14. *For $\nu = (\nu_i)_{i=1,\dots n}$ with $-\lambda_i < \nu_i < \mu_i$, there exist C_1 and $C_2 > 0$ such that*

$$C_1|x| \leq \mathcal{W}_\nu(x) \leq C_2|x| \quad \forall x \in \mathcal{S}. \tag{48}$$

Proof. The second inequality of (48) is straightforward. To obtain the first inequality, we use the non negativity of $\mathcal{W}$ (defined in (3)) in $\mathcal{S}$:

$$\mathcal{W}_\nu(x) = \mathcal{W}(x) - \sum_{i=1}^{n} \min((\nu_i - \mu_i)x_i, (\nu_i + \lambda_i)x_i) \geq C\sum_{i=1}^{n}|x_i| \geq 0.$$

Moreover,

$$|x_0| = |\mathcal{W}_\nu(x) - \sum_{i=1}^{n}(1 - \nu_i)x_i| \leq \mathcal{W}_\nu(x) + C\sum_{i=1}^{n}|x_i| \leq C\mathcal{W}_\nu(x).$$

Consequently,

$$|x| \leq C\mathcal{W}_\nu(x). \qquad \square$$

Fix $\gamma' > \gamma$ such that the assumption (A.1) is still valid with γ' instead of γ, and ν as in Lemma 3.14. This guarantees $m_k < +\infty$ (see Lemma 3.15). On the other hand, we have :

$$m_k \geq \overline{m} = \sup_{x\in\mathcal{S}}\{v(x) - v'(x) - 2\varepsilon\mathcal{W}_\nu(x)^{\gamma'}\} \geq v(z) - v'(z) - 2\varepsilon\mathcal{W}_\nu(z)^{\gamma'}.$$

As $v(z) > v'(z)$, there exists $\varepsilon > 0$ such that $m_k \geq \overline{m} > 0$ for any k; in the following, we consider such ε.

LEMMA 3.15. *Consider $\gamma' > \gamma$ and ν as in Lemma 3.14. There exist x_k, y_k in $\mathcal{S}$ such that*

$$m_k = w_k(x_k, y_k) < +\infty,$$

$$k|x_k - y_k|^2 \underset{k\to\infty}{\to} 0, \tag{49}$$

and

$$m_k \underset{k\to\infty}{\to} \overline{m} \equiv \sup_{x\in\mathcal{S}}\{v(x) - v'(x) - 2\varepsilon\mathcal{W}_\nu(x)^{\gamma'}\}. \tag{50}$$

Proof. Since v and v' satisfy (45), we have

$$m_k \leq C + \sup_{x\in\mathcal{S}}(C_1|x|^\gamma - C_2|x|^{\gamma'}) < +\infty.$$

Let (x^n, y^n) be a maximizing sequence :

$$w_k(x^n, y^n) \geq m_k - \frac{1}{n} \geq \overline{m} - \frac{1}{n}$$

which implies that

$$C_2|x^n|^{\gamma'} - C_1|x^n|^\gamma \leq C.$$

Hence, x^n is bounded and similarly y^n is bounded. Consequently, there exists a converging subsequence of (x^n, y^n) and the limit $(x_k, y_k) \in \mathcal{S}\times\mathcal{S}$ realizes the maximum of w_k. As

$$v(x_k) - v'(y_k) - \varepsilon(\mathcal{W}_\nu(x_k)^{\gamma'} + \mathcal{W}_\nu(y_k)^{\gamma'}) = m_k + \frac{k}{2}|x_k - y_k|^2 \geq 0$$

for any k,we conclude that x_k, y_k and $k|x_k - y_k|^2$ are bounded. Moreover, for any subsequence of (x_k, y_k) converging to $(\hat{x}, \hat{y})$ when k goes to infinity, we have $\hat{x} = \hat{y}$ and using $m_k \geq \overline{m}$, we get

$$\limsup_{k\to\infty} \frac{k}{2}|x_k - y_k|^2 \leq v(\hat{x}) - v'(\hat{x}) - 2\varepsilon\mathcal{W}_\nu(\hat{x})^{\gamma'} - \overline{m} \leq 0.$$

Consequently, (49) and (50) are satisfied. □

Now, since $\overline{m} > 0$ and $v \leq v'$ on $\partial\mathcal{S}$, the limit $\hat{x}$ of x_k and y_k is in $\mathring{\mathcal{S}}$; then for any converging subsequence of (x_k, y_k), we have $(x_k, y_k) \in \mathring{\mathcal{S}} \times \mathring{\mathcal{S}}$ for large k. Applying Lemma 3.13 with $V = v - \varepsilon{\mathcal{W}_\nu}^{\gamma'}$ and $V' = v' + \varepsilon{\mathcal{W}_\nu}^{\gamma'}$ at the point (x_k, y_k) in $\mathring{\mathcal{S}} \times \mathring{\mathcal{S}}$, we obtain that there exist X, Y in S^{n+1} satisfying (46) such that

$$\begin{aligned}(p_k, X_k) &\equiv \Big(k(x_k - y_k) + \varepsilon\gamma'\mathcal{W}_\nu(x_k)^{\gamma'-1}\hat{p}, X + \varepsilon\gamma'(\gamma'-1)\mathcal{W}_\nu(x_k)^{\gamma'-2}A\Big)\\ &\in \overline{J}^{2,+}v(x_k)\end{aligned} \tag{51}$$

$$
\begin{aligned}
(p'_k, Y_k) &\equiv \Big(k(x_k - y_k) - \varepsilon\gamma'\mathcal{W}_\nu(y_k)^{\gamma'-1}\hat{p}, Y - \varepsilon\gamma'(\gamma'-1)\mathcal{W}_\nu(y_k)^{\gamma'-2}A\Big) \\
&\in \overline{J}^{2,-}v'(y_k) \qquad (52)
\end{aligned}
$$

with $\hat{p} = (1, 1-\nu_1, \ldots, 1-\nu_n)$ and $A = \hat{p}^t\hat{p}$.

Denote

$$F(X,p,v,x) = \max\left(F_0(X,p,v,x) + u^*(p_0),\ \max_{1\leq i\leq n} G_i(p),\ \max_{1\leq i\leq n} H_i(p)\right),$$

$$F_0(X,p,v,x) = \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 x_i^2 X_{ii} + \sum_{i=1}^{n}\alpha_i x_i p_i + r x_0 p_0 - \delta v,$$

$$G_i(p) = -(1+\lambda_i)p_0 + p_i,$$

$$H_i(p) = (1-\mu_i)p_0 - p_i,$$

where $X = (X_{ij})_{i,j=0,\ldots,n}$, $p = (p_i)_{i=0,\ldots,n}$.

Note that although F is continuous, F takes its values in $\mathbb{R}\cup\{+\infty\}$, since $F = +\infty$ when $p_0 \leq 0$. This leads to a difficulty to obtain a uniform continuity property similar to [9, eq. (3.14)] and consequently straightforward application of the results of [9] can not be used. Moreover, as the discount factor δ appears only in the F_0 component of F and not in G_i and F_i, property [9, eq. (3.13)], that is

$$F(X,p,v,x) - F(X,p,v',x) \leq -\lambda(v-v') \text{ for } v \geq v', \text{ with } \lambda > 0,$$

is not satisfied.

Using that v is a viscosity subsolution and v' is a viscosity supersolution of (11) (that is of $F(D^2v, Dv, v, x) = 0$ in $\mathring{\mathcal{S}}$), and using (51) and (52), we get :

$$
\begin{aligned}
F(X_k, p_k, v(x_k), x_k) &\geq 0, \\
F(Y_k, p'_k, v'(y_k), y_k) &\leq 0.
\end{aligned}
$$

This last inequality implies $G_i(p'_k) \leq 0$ and $H_i(p'_k) \leq 0$ and by linearity of G_i and H_i, we obtain

$$G_i(p_k) = G_i(p'_k) - \varepsilon\gamma'(\mathcal{W}_\nu(x_k)^{\gamma'-1} + \mathcal{W}_\nu(y_k)^{\gamma'-1})(\lambda_i + \nu_i) < 0$$

and

$$H_i(p_k) = H_i(p'_k) + \varepsilon\gamma'(\mathcal{W}_\nu(x_k)^{\gamma'-1} + \mathcal{W}_\nu(y_k)^{\gamma'-1})(\nu_i - \mu_i) < 0.$$

This leads to

$$F_0(X_k, p_k, v(x_k), x_k) + u^*((p_k)_0) \geq 0 \geq F_0(Y_k, p'_k, v'(y_k), y_k) + u^*((p'_k)_0).$$

Using now that u^* is non increasing and $(p'_k)_0 < (p_k)_0$, we obtain

$$F_0(X_k, p_k, v(x_k), x_k) - F_0(Y_k, p'_k, v'(y_k), y_k) \geq 0.$$

Hence,

$$
\begin{aligned}
(53) \qquad 0 \;\le\; & \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2((x_k)_i^2 X_{ii} - (y_k)_i^2 Y_{ii}) \\
& + k(\sum_{i=1}^{n} \alpha_i((x_k)_i - (y_k)_i)^2 + r((x_k)_0 - (y_k)_0)^2) \\
& - \delta(v(x_k) - v'(y_k) - \varepsilon(\mathcal{W}_\nu(x_k)^{\gamma'} + \mathcal{W}_\nu(y_k)^{\gamma'})) \\
& + \varepsilon(f(x_k) + f(y_k))
\end{aligned}
$$

where

$$
\begin{aligned}
f(x) &= \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 x_i^2 \gamma'(\gamma'-1)\mathcal{W}_\nu(x)^{\gamma'-2} A_{ii} \\
&\quad + \sum_{i=1}^{n} \alpha_i x_i \gamma' \mathcal{W}_\nu(x)^{\gamma'-1}\hat{p}_i + r x_0 \gamma' \mathcal{W}_\nu(x)^{\gamma'-1}\hat{p}_0 \\
&\quad - \delta \mathcal{W}_\nu(x)^{\gamma'} \\
&= \gamma' \mathcal{W}_\nu(x)^{\gamma'}[r - \frac{\delta}{\gamma'} + \sum_{i=1}^{n}(\alpha_i - r) y_i + \frac{(\gamma'-1)}{2}\sum_{i=1}^{n}\sigma_i^2 y_i^2]
\end{aligned}
$$

with

$$
y_i = \frac{(1-\nu_i)x_i}{\mathcal{W}_\nu(x)}.
$$

We have

$$
f(x) \le \gamma' \mathcal{W}_\nu(x)^{\gamma'}[r - \frac{\delta}{\gamma'} + \frac{1}{2(1-\gamma')}\sum_{i=1}^{n}(\frac{\alpha_i - r}{\sigma_i})^2]
$$

and since γ' is such that (A.1) is satisfied, $f(x) \le 0 \;\; \forall x \in \mathbb{R}^{n+1}$. Using (46), we see that the first term of the r.h.s. of (53) is bounded by $Ck|x_k - y_k|^2$. Hence,

$$
0 \le Ck|x_k - y_k|^2 - \delta m_k \underset{k\to\infty}{\longrightarrow} -\delta \overline{m} < 0.
$$

We thus get a contradiction and Lemma 3.12 and Proposition 3.11 are proven. □

4. Change of variables.

4.1. Reduction of the state dimension. The value function V defined by (7) has the homothetic property (see [10])

$$
\forall \rho > 0, \; V(\rho x) = \rho^\gamma V(x). \tag{54}
$$

Consequently, the $(n+1)$-dimensional VI (11-12) satisfied by V can be reduced to a n-dimensional VI by using the following homogeneous model, that is by considering the new state variables :

$$
(55)\begin{cases}
\rho = x_0 + \displaystyle\sum_{i=1}^{n}(1-\mu_i)x_i & \text{(net wealth)} \\
y_i = \dfrac{(1-\mu_i)x_i}{\rho}, \; i = 1, \ldots, n & \text{(fraction of net wealth invested in stock } i\text{)}
\end{cases}
$$

and the new control variable

$$C = \frac{c}{\rho} \quad \text{(fraction of net wealth dedicated to consumption).} \tag{56}$$

The function $V(x)$ can be written as

$$\begin{aligned} V(x) &= V(\rho(1-\sum_{i=1}^n y_i), \frac{\rho y_1}{(1-\mu_1)}, \dots, \frac{\rho y_n}{(1-\mu_n)}) \\ &= \rho^\gamma W(y) \end{aligned} \tag{57}$$

where the function

$$W(y) = V(1-\sum_{i=1}^n y_i, \frac{y_1}{(1-\mu_1)}, \dots, \frac{y_n}{(1-\mu_n)}) \tag{58}$$

is defined in

$$\tilde{\mathcal{S}} = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n,\ 1 - \sum_{i=1}^n \frac{\lambda_i + \mu_i}{1-\mu_i}\{y_i\}^- \geq 0\}$$

with $\{y\}^- = \max(0, -y)$.

Using inequality (35) we deduce that the function W is bounded in $\tilde{\mathcal{S}}$:

$$0 \leq W(y) \leq \varphi(1-\sum_{i=1}^n y_i, \frac{y_1}{(1-\mu_1)}, \dots, \frac{y_n}{(1-\mu_n)}) \leq a. \tag{59}$$

The function W is the unique viscosity solution of

$$\max(\tilde{A}W + u^*(BW), \max_{1\leq i\leq n} \tilde{L}_i W, \max_{1\leq i\leq n} \tilde{M}_i W) = 0 \qquad \text{in } \mathring{\tilde{\mathcal{S}}}, \tag{60}$$

$$W = 0 \qquad \text{on } \partial\tilde{\mathcal{S}}, \tag{61}$$

where

$$\tilde{A}W = \sum_{j,k=1}^n a_{jk} \frac{\partial^2 W}{\partial y_j \partial y_k} + \sum_{j=1}^n b_j \frac{\partial W}{\partial y_j} - \beta W, \tag{62}$$

$$BW = \gamma W - \sum_{j=1}^n y_j \frac{\partial W}{\partial y_j}, \tag{63}$$

$$\tilde{L}_i W = \frac{\partial W}{\partial y_i} - \left(\frac{\lambda_i + \mu_i}{1-\mu_i}\right) BW, \tag{64}$$

$$\tilde{M}_i W = -\frac{\partial W}{\partial y_i} \tag{65}$$

and

$$a_{jk} = \frac{y_j y_k}{2} \sum_{i=1}^n \sigma_i^2 (\delta_{ki} - y_i)(\delta_{ji} - y_i), \tag{66}$$

$$b_j = y_j \sum_{i=1}^n [(\gamma - 1)\sigma_i^2 y_i + \alpha_i - r](\delta_{ij} - y_i), \tag{67}$$

$$\beta = \delta - \gamma(r + \sum_{i=1}^n [(\alpha_i - r) y_i + \frac{\gamma - 1}{2}\sigma_i^2 y_i^2]). \tag{68}$$

The symbol δ_{ij} denotes the Kronecker index which is equal to 0 when $i \neq j$ and equal to 1 when $i = j$.

Using the properties of V and (60), we deduce that W is concave, non negative and non decreasing with respect to each coordinate y_i.

Remark 4.1. Equation (60-61) only depends on $\nu = (\nu_i)_{i=1\ldots n}$ with $\nu_i = (\lambda_i + \mu_i)/(1-\mu_i)$, and so does the function W. Denote by $V_{\lambda,\mu}$ the value function (7) in order to express explicitly the dependency of V on the transaction costs and by $W_{\lambda,\mu}$ the solution of (60-61). We have :

$$\begin{aligned} W_{\lambda,\mu}(y) &= W_{\nu,0}(y) \\ &= V_{\nu,0}(1-\sum_{i=1}^{n} y_i, y_1, \ldots, y_n). \end{aligned} \tag{69}$$

Using (54), we get

$$V_{\lambda,\mu}(x) = V_{\nu,0}(x_0, (1-\mu_1)x_1, \ldots, (1-\mu_n)x_n). \tag{70}$$

Consequently, it is sufficient to compute the value function V when the transaction costs on sale are equal to zero.

This remark could have been observed directly from the model. Indeed, the quantity $s_i(t)$ represents the amount of money in the i^{th} risky asset at time t, that is the quantity of the i^{th} asset multiplied by the reference price $P_i(t)$. This reference price is useless in practice unless the transaction costs are time-dependent. What matters for the investor is the buying price $(1+\lambda_i)P_i$ and the selling price (or net price) $(1-\mu_i)P_i$. The relevant quantity to consider is the net value of the i^{th} asset, that is $(1-\mu_i)s_i$. Purchase of dL_i units of the i^{th} asset increases the net value of this asset by $dL'_i = (1-\mu_i)dL_i$ and requires a payment of $(1+\nu_i)dL'_i$, whereas sale of dM_i units reduces the net value by $dM'_i = (1-\mu_i)dM_i$ and realizes effectively dM'_i in cash. Consequently, by using a formulation of the problem based on the net values $(1-\mu_i)s_i$ of the assets, the value function only depends on the coefficients ν_i where ν_i represents the proportional transaction cost on purchase with respect to the net price of the i^{th} asset.

4.2. Additional treatment for numerical purpose. Our purpose is now to solve equation (60-61).

In order to simplify the numerical computation, we restrict the admissible region $\mathcal{S}$ to

$$\mathcal{S}^+ = \{x \in \mathbb{R}^{n+1},\ x_1, \ldots, x_n \geq 0,\ x_0 + \sum_{i=1}^{n}(1-\mu_i)x_i \geq 0\}$$

that is, we suppose that the amounts of money allocated in the risky assets are non negative, while the amount of money in the bank account can be negative as long as the net wealth remains non negative. This is not restrictive since when $\alpha_i > r$ the no-transaction cone is inside $\mathcal{S}^+$ and a trajectory which starts in $\mathcal{S}^+$ remains in $\mathcal{S}^+$ (see [10] for $n = 1$).

This leads to the study of VI (60) in the domain $(\mathbb{R}^+)^n$:

$$\max(\tilde{A}W + u^*(BW), \max_{1\leq i\leq n} \tilde{L}_i W, \max_{1\leq i\leq n,\, y_i>0} \tilde{M}_i W) = 0 \quad \text{in } (\mathbb{R}^+)^n \tag{71}$$

This VI degenerates at the boundary and is valid up to the boundary, but the controls which make the trajectory go out of the domain are not admissible. Note that the function W has bounded derivatives in $(\mathbb{R}^+)^n$.

We proceed with a technical change of variables which brings $(\mathbb{R}^+)^n$ to $[0,1]^n$, namely :

$$
\left\{
\begin{array}{l}
\psi(z) = \theta(z) W(y), \\
\theta(z) = \prod_{i=1}^{n} (1 - z_i), \\
z_i = \dfrac{y_i}{1 + y_i}, \quad i = 1, \dots, n.
\end{array}
\right.
\tag{72}
$$

The function ψ is bounded and concave with respect to z_i, $i = 1, \dots, n$, has bounded derivatives, and satisfies :

$$
\left\{
\begin{array}{l}
\max(\overline{A}\psi + \sup_{C \geq 0}(-C\overline{B}\psi + \theta(z)\dfrac{C^\gamma}{\gamma}), \max_{1 \leq i \leq n} \overline{L}_i \psi, \max_{i,\, z_i > 0} \overline{M}_i \psi) = 0 \quad \text{in } [0,1)^n \\
\psi = 0 \quad \text{on } [0,1]^n \cap \{z_i = 1\} \quad \forall i = 1, \dots, n
\end{array}
\right.
\tag{73}
$$

where

$$
\begin{aligned}
\overline{A}\psi &= \sum_{j,k=1}^{n} a'_{jk} \frac{\partial^2 \psi}{\partial z_j \partial z_k} + \sum_{j=1}^{n} b'_j \frac{\partial \psi}{\partial z_j} - \beta' \psi, \\
\overline{B}\psi &= (\gamma - \sum_{j=1}^{n} z_j)\psi - \sum_{j=1}^{n} z_j (1 - z_j) \frac{\partial \psi}{\partial z_j}, \\
\overline{L}_i \psi &= (1 - z_i)(\psi + (1 - z_i)\frac{\partial \psi}{\partial z_i}) - \lambda_i \overline{B}\psi, \\
\overline{M}_i \psi &= -(1 - z_i)(\psi + (1 - z_i)\frac{\partial \psi}{\partial z_i}),
\end{aligned}
$$

with

$$
\begin{aligned}
a'_{jk} &= \frac{1}{2} z_j (1 - z_j) z_k (1 - z_k) \overline{a}_{jk}, \\
b'_j &= z_j (1 - z_j)(\overline{b}_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} z_k \overline{a}_{jk}), \\
\beta' &= \beta - \sum_{j=1}^{n} z_j \overline{b}_j - \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \overline{a}_{jk} \frac{z_j z_k}{2}, \\
\overline{a}_{jk} &= \sum_{i=1}^{n} \sigma_i^2 (\delta_{ki} - \frac{z_i}{1 - z_i})(\delta_{ji} - \frac{z_i}{1 - z_i}), \\
\overline{b}_j &= \sum_{i=1}^{n} ((\gamma - 1)\sigma_i^2 \frac{z_i}{1 - z_i} + \alpha_i - r)(\delta_{ij} - \frac{z_i}{1 - z_i}),
\end{aligned}
$$

and β defined in (68).

The numerical study is organized as follows : equation (73) is solved by using the numerical methods explained in §5 below. Then a reverse change of variable is performed in order to display the numerical results for equation (71) (see §6).

5. Numerical methods. We consider equations of the form :

$$
\left\{
\begin{array}{ll}
\max\limits_{P \in \mathcal{P}_{ad}} (A^P W + u(P)) = 0 & \text{in } \Omega = [0,1]^m \setminus \Gamma \\
W = 0 & \text{on } \Gamma
\end{array}
\right.
\tag{74}
$$

where A^P is a second order degenerate elliptic operator

$$
A^P W(x) = \sum_{i,j=1}^{m} a_{ij}(x,P) \frac{\partial^2 W}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{m} b_i(x,P) \frac{\partial W}{\partial x_i}(x) - \beta(x,P) W(x)
$$

with

$$
\sum_{i,j=1}^{m} a_{ij}(x,P) \eta_i \eta_j \geq 0, \qquad \beta(x,P) \geq 0 \qquad \forall x \in \Omega,\ \eta \in \mathbb{R}^m,\ P \in \mathcal{P}_{ad}.
$$

$\mathcal{P}_{ad}$ is a closed subset of $\mathbb{R}^k$ (which may depend on x) and Γ is a part of the boundary $\partial\Omega$, which consists of faces of the m-cube $[0,1]^m$. On $\partial\Omega \setminus \Gamma$, the operator A^P is degenerate.

In §3, we have proven that the value function (7), within a change of variables, is the unique viscosity solution of an equation of type (74). This solution can be approximate by the following numerical method : (i) discretize (74) by using a consistent finite difference approximation which satisfies the Discrete Maximum Principle (DMP) (recalled below), (ii) solve the discrete equation by means of the value iteration (successive approximation) algorithm or the Howard algorithm (policy iteration). This method does not require any stronger regularity condition on the viscosity solution (see Barles and Souganidis [3], Fleming and Soner [13]). The algorithms mentioned in (ii) may be replaced by the (Full) Multigrid-Howard algorithm (FMGH), introduced in Akian [1, 2], based on the "Howard algorithm" and the multigrid method. This algorithm is more efficient but proof of convergence has been obtained only when the DMP is satisfied, the feedbacks are regular and the Bellman equation is strongly elliptic.

For the numerical solution of (74), we use a classical finite-difference discretization in a regular grid and the FMGH algorithm. Convergence arguments used in [1, 2] cannot be applied here since the DMP is not satisfied (presence of mixed derivatives), the equation is degenerate and the control is singular. Nevertheless, numerical experiments show that this numerical method converges.

This procedure and the computer implementation are treated by using the expert system *Pandore* (see Chancelier, Gomez, Quadrat and Sulem [6], Akian [2]) that has been developed to automate studies in stochastic control.

5.1. Discretization. Let $h = 1/N$ ($N \in \mathbb{N}^*$) denote the finite difference step in each coordinate direction, e_i the unit vector in the i^{th} coordinate direction, and $x = (x_1, \ldots, x_m)$ a point of the uniform grid $\Omega_h = \Omega \cap (h\mathbb{Z})^m$. Equation (74) is discretized by replacing the first and second order derivatives of W by the following approximation :

$$
\frac{\partial W}{\partial x_i} \sim \frac{W(x+he_i) - W(x-he_i)}{2h}
\tag{75}
$$

or

$$\frac{\partial W}{\partial x_i} \sim \begin{cases} \dfrac{W(x+he_i)-W(x)}{h} & \text{when} \quad b_i(x,P) \geq 0 \\[2ex] \dfrac{W(x)-W(x-he_i)}{h} & \text{when} \quad b_i(x,P) < 0. \end{cases} \tag{76}$$

$$\frac{\partial^2 W}{\partial x_i^2}(x) \sim \frac{W(x+he_i)-2W(x)+W(x-he_i)}{h^2}, \tag{77}$$

$$\frac{\partial^2 W}{\partial x_i \partial x_j}(x) \sim \frac{W(x+he_i+he_j)-W(x+he_i-he_j)}{4h^2} + \frac{W(x-he_i-he_j)-W(x-he_i+he_j)}{4h^2} \quad \text{for } i \neq j. \tag{78}$$

Approximation (75) may be used when A is uniformly elliptic, whereas (76) has to be used when A is degenerate (see Kushner [18]). These differences are computed in the entire grid Ω_h, by extending W to the "boundary" of Ω_h in $(h\mathbb{Z})^m$:

$$\begin{array}{rcll} W(x) & = & 0 & \forall x \in \Gamma \cap (h\mathbb{Z})^m \\ W(x-he_i) & = & W(x) & \forall x \in \{x_i = 0\} \cap \Omega_h \\ W(x+he_i) & = & W(x) & \forall x \in \{x_i = 1\} \cap \Omega_h. \end{array}$$

We obtain a system of N_h non linear equations of N_h unknowns $\{W_h(x),\ x \in \Omega_h\}$:

$$\max_{P \in \mathcal{P}_{ad}} (A_h^P W_h + u(P))(x) = 0 \quad \forall x \in \Omega_h \tag{79}$$

where $N_h = \sharp\Omega_h \sim 1/h^m$. Let $\mathcal{P}_h$ denote the set of control functions $P : \Omega_h \to \mathcal{P}_{ad}$ and $\mathcal{V}_h$ the set of functions from Ω_h into $\mathbb{R}$. Equation (79) can be rewritten :

$$\max_{P \in \mathcal{P}_h} (A_h^P W_h + u(P)) = 0, \quad W_h \in \mathcal{V}_h.$$

Then, the operator A_h^P, depending on P in $\mathcal{P}_h$, maps $\mathcal{V}_h$ into itself (or is a $N_h \times N_h$ matrix).

Because of the degeneracy of the operator A^P at some points of the closed m-cube $\overline{\Omega}$ and the presence of mixed derivatives, A_h^P does not satisfy the usual Discrete Maximum Principle (i.e. $(A_h^P W_h(x) \leq 0 \quad \forall x \in \Omega_h) \Rightarrow (W_h(x) \geq 0 \quad \forall x \in \Omega_h)$). Consequently, equation (79) may not be stable, even for small step h. However A_h^P can be written as the sum of a symmetric negative definite operator and an operator which satisfies the Discrete Maximum Principle; we thus infer the stability of A_h^P which is confirmed by numerical experiments.

We describe below the available algorithms to solve equation (79).

5.2. The Value Iteration method. Suppose that the $N_h \times N_h$ matrix A_h^P satisfies

$$(A_h^P)_{ij} \geq 0 \;\; \forall i \neq j, \;\; \sum_{j=1}^{N_h} (A_h^P)_{ij} = -\lambda < 0 \;\; \forall i \tag{80}$$

which implies that A_h^P satisfies the Discrete Maximum Principle. Equation (79) can be rewritten as

$$W_h = \frac{1}{1+\lambda k} \max_{P \in \mathcal{P}_h} (M^P W_h + ku(P)), \tag{81}$$

where $k > 0$ and $M^P = I + k(A_h^P + \lambda I)$ is a Markov matrix (I is the $N_h \times N_h$ identity matrix). Equation (79) can then be interpreted as the Dynamic Programming equation of a control problem of Markov chain with discount factor $1/(1+\lambda k)$, instantaneous cost $ku(P)$ and transition matrix M^P

$$\max_{(P_n)} \sum_{n=0}^{\tau} \frac{k}{(1+\lambda k)^{n+1}} u(X_n, P_n).$$

The value iteration method (see Bellman [5]) consists in the contraction iteration

$$W^{n+1} = \frac{1}{1+\lambda k} \max_{P \in \mathcal{P}_h} (M^P W^n + ku(P)). \tag{82}$$

The contracting factor is $1/(1+\lambda k) = 1 - \mathcal{O}(h^2)$ and the complexity[1] of the method is

$$C_h = \mathcal{O}(\frac{-\log h}{h^2} N_h) = \mathcal{O}(-h^{-(2+m)} \log h) = \mathcal{O}(N_h^{1+2/m} \log N_h).$$

When the operator A_h^P does not satisfy the Discrete Maximum Principle, equation (79) cannot be interpreted as a discrete Bellman equation. Nevertheless, the iterative method (82) can still be used if we find λ and k such that the L^2 norm of M^P (which is no more a Markov matrix) is lower than 1 for all P. This condition may be obtained for instance when the discount factor $\beta(x, P)$ is large enough.

An example of the use of the value iteration algorithm is given in Sulem [30] for solving the one dimensional investment-consumption problem.

5.3. The Multigrid-Howard algorithm. Another classical algorithm is the Howard algorithm (see Howard [16], Bellman [4, 5]) also named policy iteration. It consists in an iteration algorithm on the control and value functions (starting from P^0 or W^0) :

$$\text{for } n \geq 1, \qquad P^n \in \operatorname*{Argmax}_{P \in \mathcal{P}_h} (A_h^P W^{n-1} + u(P)) \tag{83}$$

$$\text{for } n \geq 0, \qquad W^n \text{ is solution of } A_h^{P^n} W + u(P^n) = 0. \tag{84}$$

When A_h^P satisfies the Discrete Maximum Principle, the sequence W^n decreases and converges to the solution of (79) and the convergence is in general superlinear [4, 5, 1, 2].

The exact computation of step (84) is expensive in dimension $m \geq 2$ (the complexity of a direct method is $\mathcal{O}(N_h^{3-2/m})$). We thus use the multigrid-Howard algorithm introduced in [1, 2] : in (84), W^n is computed by a multigrid method with initial value W^{n-1}. The advantage is that each multigrid iteration takes a computing time of $\mathcal{O}(N_h)$ and contracts the error by a factor independent of the discretization step

[1] The number of elementary operations for computing an approximation of the solution of (79) with an error in the order of the discretization error.

h. For a detailed description of the multigrid algorithm, see for example McCormick [22], Hackbusch [14], Hackbusch and Trottenberg [15].

Let $\mathcal{M}^P$ denote the operator of an iteration of the multigrid method associated to the equation $A_h^P W + u(P) = 0$. Starting from W^0, we proceed the following iteration :

$$\left\{ \begin{array}{l} (83) \\ \text{for } n \geq 1 \left\{ \begin{array}{l} W^{n,0} = W^{n-1} \\ \text{for } i = 1 \text{ to } m_n, \ W^{n,i} = \mathcal{M}^{P^n}(W^{n,i-1}) \\ W^n = W^{n,m_n} \end{array} \right. \end{array} \right. \tag{85}$$

This algorithm converges to the solution W_h^* of (79) if W^0 is sufficiently close to W_h^* and m_n is large enough (independently of the step h) [1, 2].

We introduce now the FMGH algorithm which solves equation (79) from any initial value W^0.

5.4. The Full-Multigrid-Howard algorithm. This algorithm [1, 2] fully uses the idea of the Full-Multigrid method (FMG) (see for example Hackbusch and Trottenberg [15]).

Consider the sequence of grids $(\Omega_k)_{k\geq 1}$ of steps $h_k = 2^{-k}$ and denote by $\mathcal{I}_k^{k+1}$ the operator of the m-linear interpolation from $\mathcal{V}_{h_k}$ into $\mathcal{V}_{h_{k+1}}$.

If $W_k \in \mathcal{V}_{h_k}$, $W_{k+1} = \mathcal{I}_k^{k+1} W_k$ is defined by :

$$\left\{ \begin{array}{rcll} W_{k+1}(x) & = & W_k(x) & \forall x \in \Omega_k \subset \Omega_{k+1} \\ W_{k+1}(\frac{x+y}{2}) & = & \dfrac{W_{k+1}(x) + W_{k+1}(y)}{2} & \forall x, y \in \Omega_{k+1} \text{ such that } \frac{x+y}{2} \in \Omega_{k+1} \\ & & & \text{and } x, \, y \text{ are in the same cell of } \Omega_k, \end{array} \right.$$

where a cell of Ω_h is a m-cube of width h included in $\overline{\Omega}$ and with vertices in $(h\mathbb{Z})^m$.

The Full-Multigrid-Howard (FMGH) algorithm is defined as :

For $1 \leq k \leq \overline{k}$,
$W_k^{\overline{n}}$ is the $\overline{n}$-th iteration of the sequence defined by (85) in the grid Ω_k of initial value W_k^0.
For $1 \leq k < \overline{k}$,
$W_{k+1}^0 = \mathcal{I}_k^{k+1} W_k^{\overline{n}}$.

Under appropriate assumptions (strong ellipticity, Discrete Maximum Principle, regularity of the feedback, see [1, 2]), the error between $W_k^{\overline{n}}$ and the solution W_k^* of (79) with $h = h_k$ is in the order of the discretization error, for any k. This property is realized for any initial value W_1^0, if the numbers m_n and $\overline{n}$ are large enough (but independent of the level k). Consequently, this algorithm solves equation (79) (with an error in the order of the discretization error) with a computing time of $\mathcal{O}(N_h)$.

6. Numerical results. Equation (71) is solved in $(\mathbb{R}^+)^n$ by using the FMGH algorithm for $n = 1$ and $n = 2$ and various numerical values of the parameters.

Remark 6.1. The regions B_i and S_i defined in (17) and (18) are characterized by

$$\begin{array}{rcl} B_i & = & \{x \in \mathcal{S}, \ \tilde{L}_i W(y) = 0, \ y \text{ given by } (55)\}, \\ S_i & = & \{x \in \mathcal{S}, \ \tilde{M}_i W(y) = 0, \ y \text{ given by } (55)\} \end{array}$$

where the operators $\tilde{L}_i$ and $\tilde{M}_i$ are defined in (64) and (65). By extension we use the notation :

$$
\begin{aligned}
&B_i = \{y \in (\mathbb{R}^+)^n,\ \tilde{L}_i W(y) = 0\},\\
&S_i = \{y \in (\mathbb{R}^+)^n,\ \tilde{M}_i W(y) = 0\},\\
&NT_i = (\mathbb{R}^+)^n \backslash (B_i \cup S_i),\\
&NT = \bigcap_{i=1}^{n} NT_i.
\end{aligned}
\tag{86}
$$

6.1. One risky asset. Numerical tests are performed with $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$, $\nu = \nu_1 = (\lambda_1 + \mu_1)/(1 - \mu_1) = 0.1, 0.3, 0.5, 1, 2, 3$ or 4%. These values of ν are obtained for example when $\lambda_1 = \mu_1 \simeq \nu/2 = 0.05,\ 0.15,\ 0.25,\ 0.5,\ 1,\ 1.5,\ 2\%$.

When $\nu > 0$, the regions B_1 and S_1 are of the form (see §7) : $B_1 = [0, \pi^-]$ and $S_1 = [\pi^+, +\infty)$ with $0 < \pi^- < \pi^+$. When $\nu = 0$ (no transaction costs), the optimal policy is to keep a constant proportion of risky asset equal to π_1^* (given by (90) below), that is $\pi^+ = \pi^- = \pi_1^*$. In our example, $\pi_1^* = 0.635$. The values of π^+ and π^- are given in the table below and displayed in Figure 1 as functions of ν.

ν (%)	0.1	0.3	0.5	1	2	3	4
π^-	0.56	0.54	0.52	0.47	0.42	0.39	0.36
π^+	0.68	0.68	0.68	0.68	0.68	0.68	0.68

The graphs of π^+ and π^- are similar to those obtained by Davis and Norman [10] who already observed that the "sell-barrier" is very insensitive to the transaction cost while the "buy-barrier" decreases rapidly as ν increases. Indeed, even if the selling cost is high, the risky asset must be sold before it can be realized for consumption. On the other hand, it may not be worthwhile to invest in the risky asset if the transaction costs are too high.

The value function W, solution of (71), and the optimal consumption C are displayed in Figures 2 and 3.

From equations (71) and (86), we obtain $W(y) = c(1 + \nu y)^\gamma$ in B_1, where c is a constant depending on ν. In S_1, W is constant and seems insensitive to the transaction costs. This means that when the initial proportion in the risky asset is in S_1, the probability of a future purchase is small. On the other hand, if the initial proportion invested in stock is in B_1, loss of profit (when ν increases) is due mainly to the first transaction.

The values of C are not relevant in B_1 and S_1 since the investor makes transactions and thus does not consume. As expected, C decreases in $[\pi^-, \pi^+]$ as well as the fraction of wealth in cash.

6.2. Two risky assets. We set $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$ and fix the parameters of the first risky asset to $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\nu_1 = (\lambda_1 + \mu_1)/(1 - \mu_1) = 1\%$.

Four tests are performed with

test 1 : $\alpha_2 = 15\%$ $\sigma_2 = 35\%$ $\nu_2 = 2\%$,
test 2 : $\alpha_2 = 15\%$ $\sigma_2 = 35\%$ $\nu_2 = 0.5\%$,
test 3 : $\alpha_2 = 15\%$ $\sigma_2 = 35\%$ $\nu_2 = 1\%$,
test 4 : $\alpha_2 = 20\%$ $\sigma_2 = 50\%$ $\nu_2 = 1\%$.

For test 1, the value function W, the optimal consumption C and their contour lines are displayed in Figures 4, 5, 6 and 7.

The partition of the domain is displayed for each test in Figures 8, 9, 10 and 11. As expected, 9 regions appear : buy (resp. sell) asset i when y_i is below (resp. above) a critical level π_i^- (resp. π_i^+) depending on y_j ($j \neq i$), and no transaction between π_i^- and π_i^+.

After the first transaction, the position of the investor evolves as a diffusion process with reflection on the boundary of NT. The direction of the reflection is given by the equation $L_i W = 0$ on the frontier with B_i and $M_i W = 0$ on the frontier with S_i.

Note that the no transaction interval for the first asset $NT_1 \cap \{y_2 = \text{constant}\} \simeq [0.39, 0.78]$ is much larger than the no transaction interval $[0.47, 0.68]$ obtained in dimension 1, when only one asset (with same parameters) is available. This is not surprising since the second asset has larger expected rate of return; it is thus more interesting to make transactions on the second asset.

We observe that the boundaries of the regions B_i and S_i seem at first sight to be straight lines (y_i = constant). This would mean that the investment policies are decoupled although the dynamics are correlated. In fact, when the cost for purchase ν_2 grows, the region NT_2 grows as expected but the boundaries of S_1 and B_1 are also perturbed. Moreover, a variation of α_2 and σ_2 affect both NT_2 and NT_1. A theoretical study of the boundaries is done below in order to confirm these remarks.

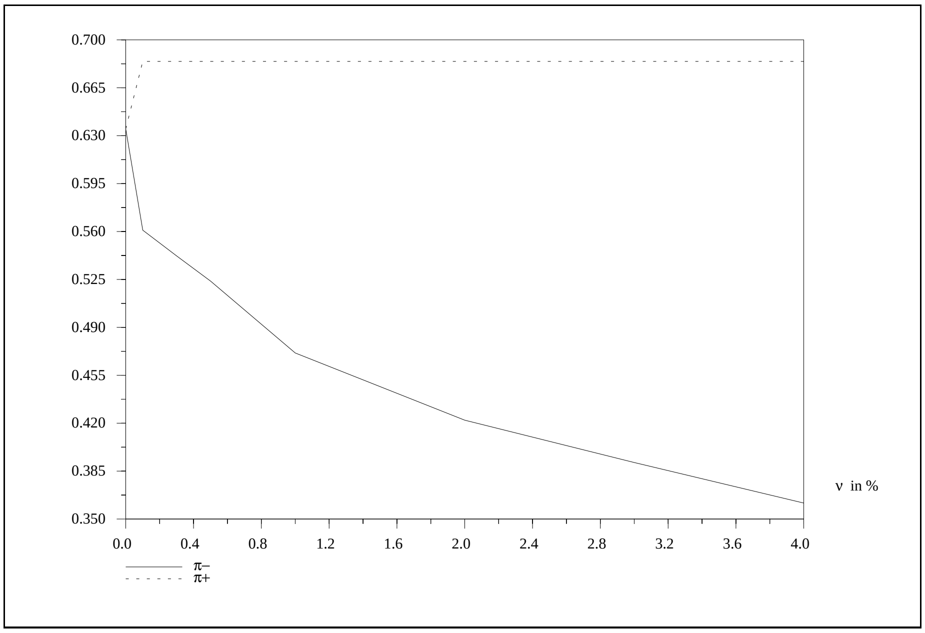

FIG. 1. *Graph of π^+ and π^- for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$.*

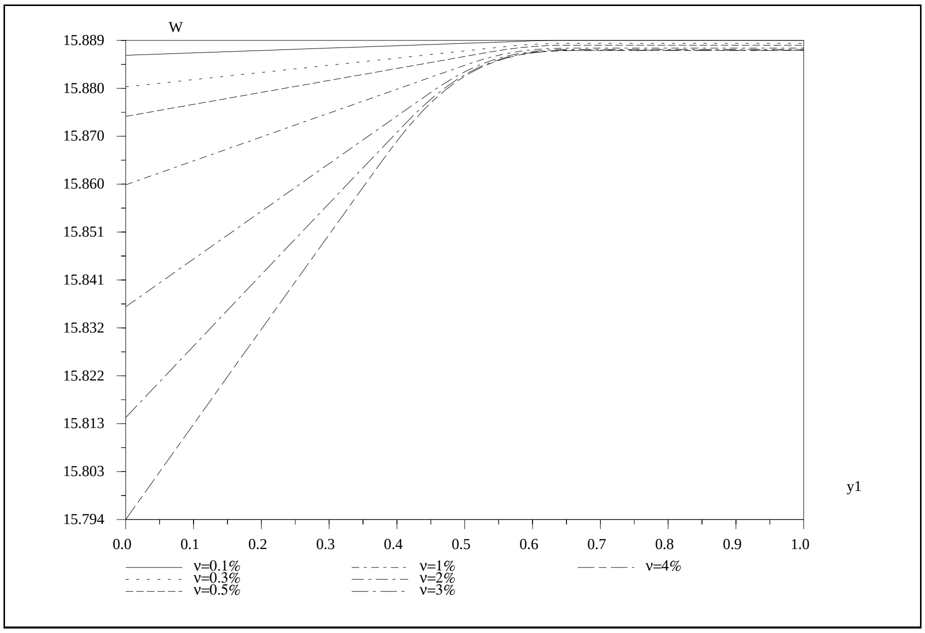

FIG. 2. *Value function W for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$.*

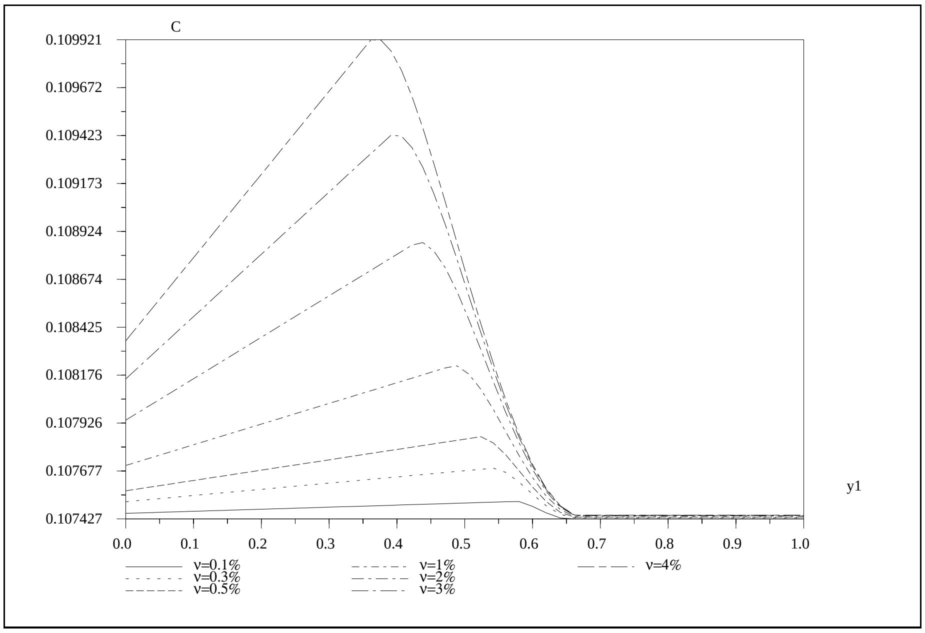

FIG. 3. *Optimal consumption C for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$.*

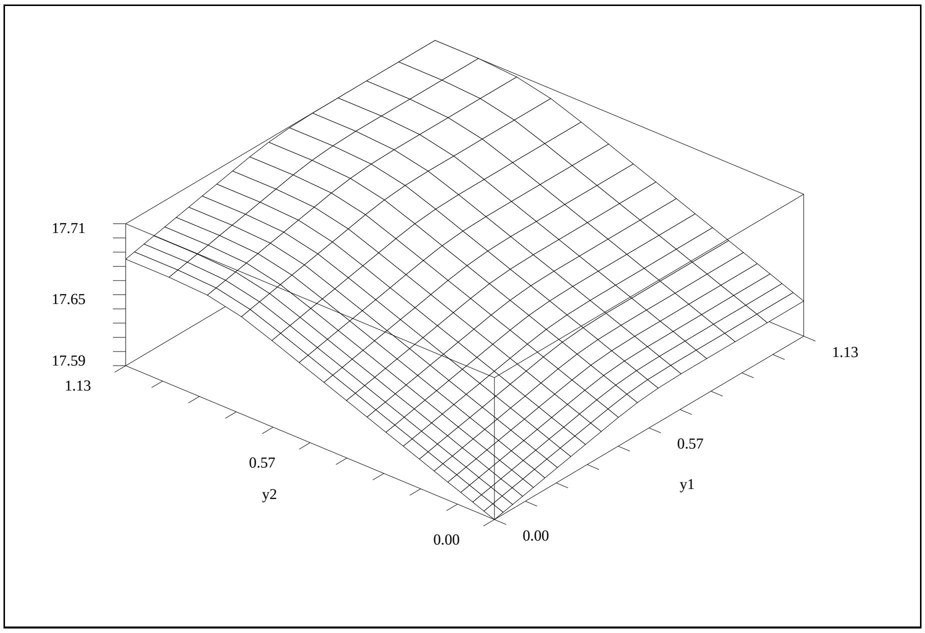

FIG. 4. *Value function W for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 2\%)$.*

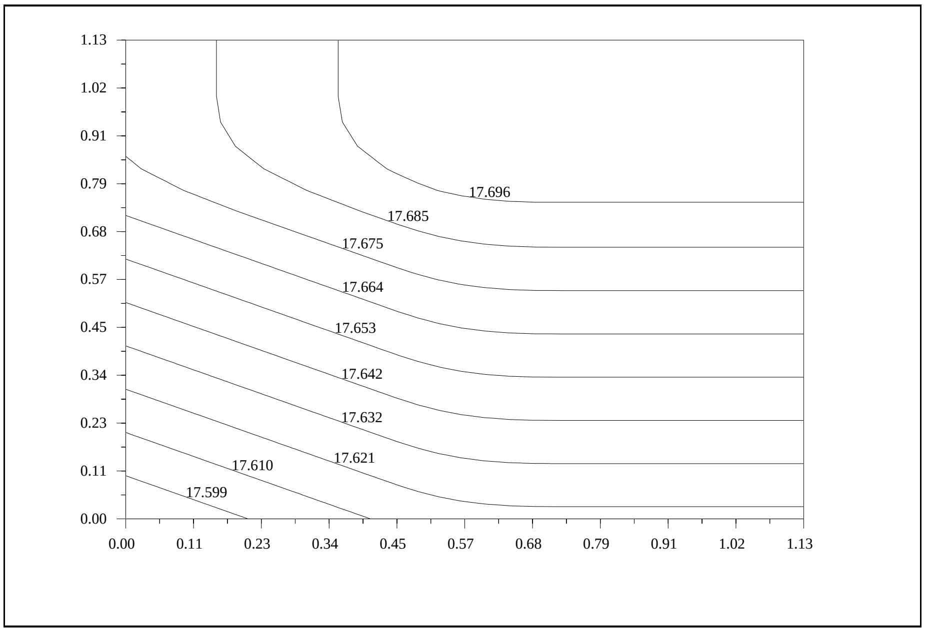

FIG. 5. *Value function W for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 2\%)$.*

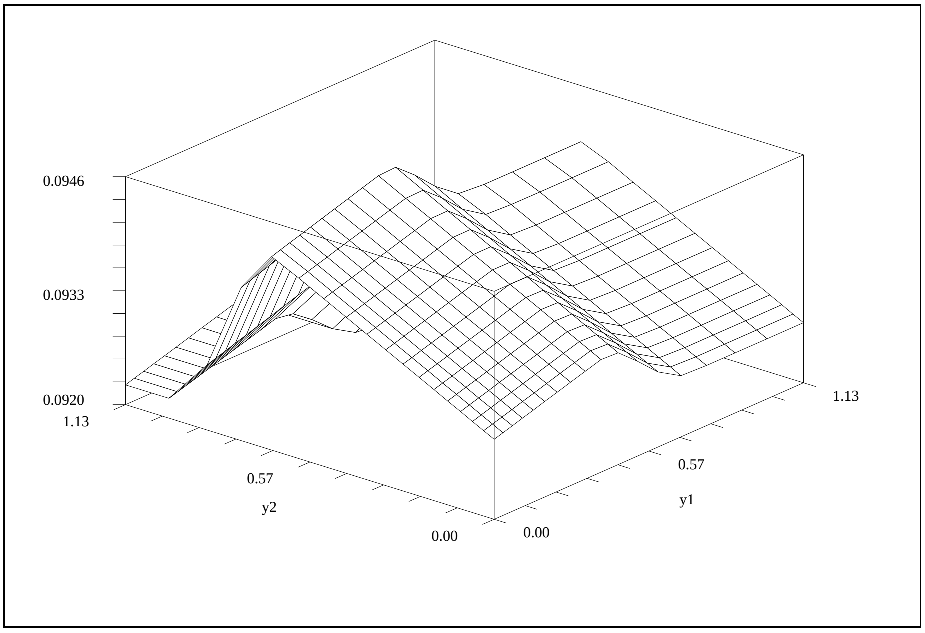

FIG. 6. *Optimal consumption C for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 2\%)$.*

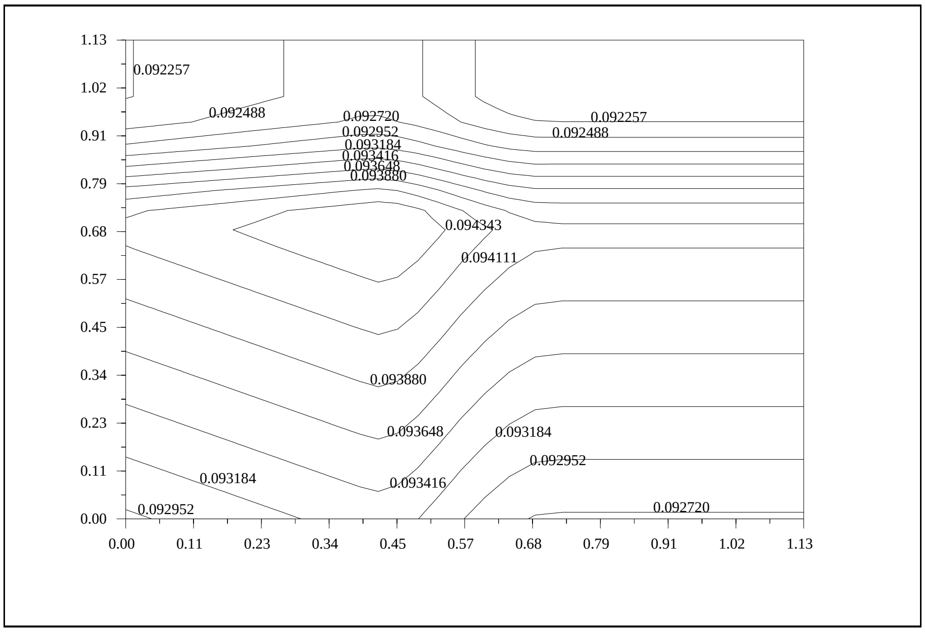

FIG. 7. *Optimal consumption C for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 2\%)$.*

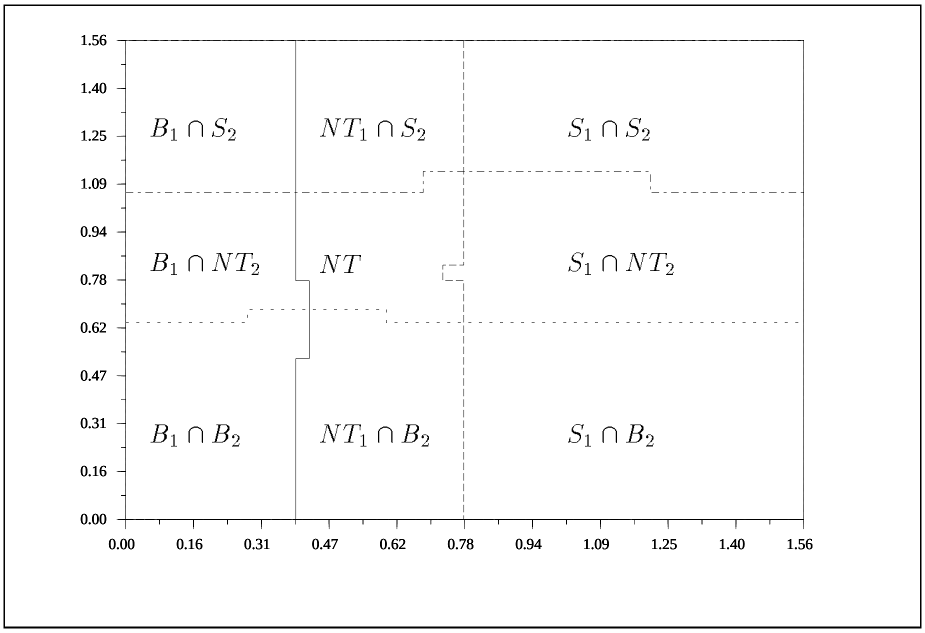

FIG. 8. *Boundaries of the regions B_i, S_i and NT_i for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 2\%)$.*

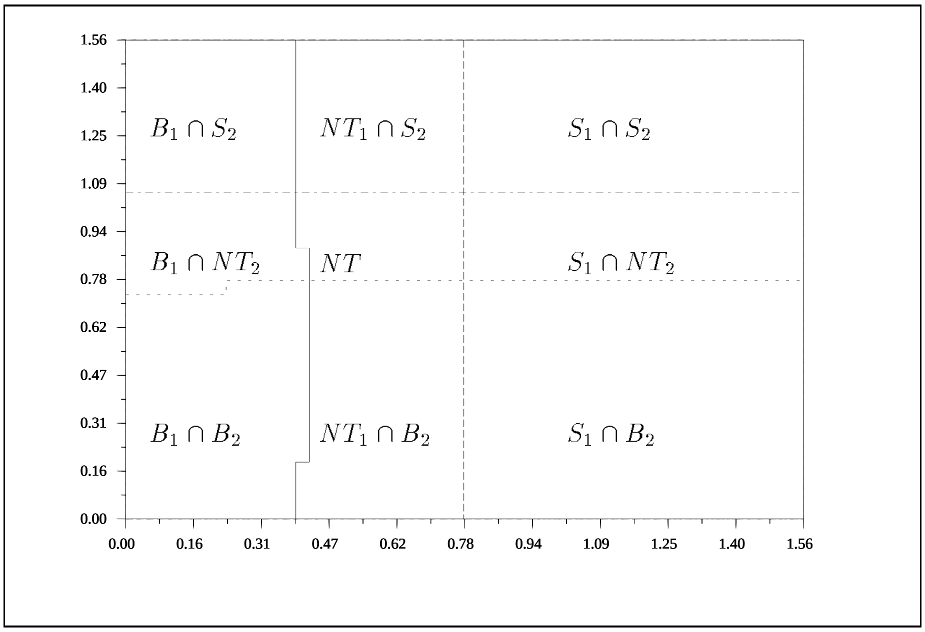

FIG. 9. *Boundaries of the regions B_i, S_i and NT_i for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 0.5\%)$.*

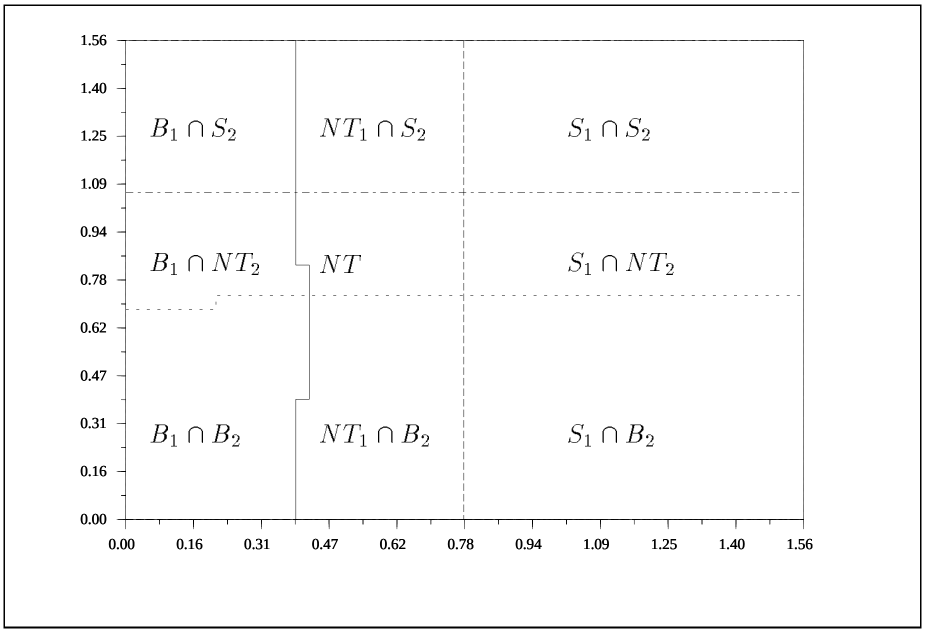

FIG. 10. *Boundaries of the regions B_i, S_i and NT_i for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\nu = (1\%, 1\%)$.*

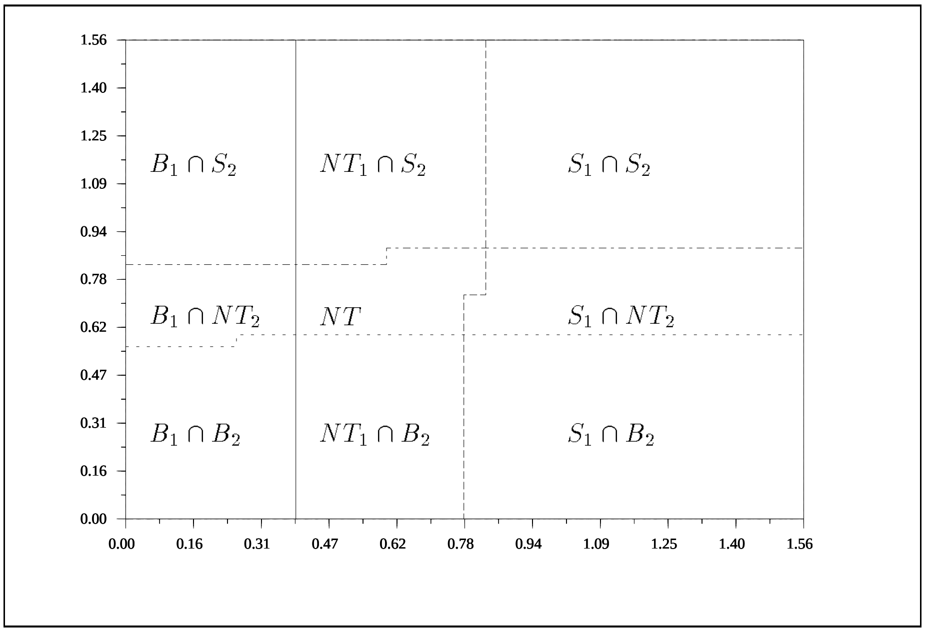

FIG. 11. *Boundaries of the regions B_i, S_i and NT_i for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 20\%)$, $\sigma = (30\%, 50\%)$, $\nu = (1\%, 1\%)$.*

7. Theoretical analysis of the optimal strategy.

7.1. No transaction costs : the Merton problem. When the transaction costs are equal to zero, the optimal investment strategy is to keep a constant fraction of total wealth in each risky asset (see Merton [24], Sethi and Taksar [27], Karatzas, Lehoczky, Sethi and Shreve [17], Davis and Norman [10]). Indeed, set $\lambda = \mu = 0$ in equation (71). We obtain :

$$\max(\tilde{A}W + u^*(BW),\ \max_{1\le i\le n} \frac{\partial W}{\partial y_i},\ \max_{1\le i\le n,\, y_i>0} (-\frac{\partial W}{\partial y_i})) = 0 \quad \text{in } (\mathbb{R}^+)^n \tag{87}$$

which is equivalent to

$$\begin{cases} W = \text{ constant}, \\ -\beta(y)W + u^*(\gamma W) \le 0 \quad \forall y \in (\mathbb{R}^+)^n, \end{cases} \tag{88}$$

with $\beta(y)$ defined in (68). Uniqueness of the solution of (88) is not guaranteed since assumption (A.2) is not satisfied, but the function W defined in (58) is the minimal solution of VI (87). Hence, we have

$$\max_{y\in(\mathbb{R}^+)^n} \{-\beta(y)W + u^*(\gamma W)\} = 0. \tag{89}$$

Equation (89) coincides with the Bellman equation of the problem where the proportion y_i is considered as a control variable (see [10]). Under assumption (A.1), the optimal proportion denoted by π_i^* and called the "Merton proportion" is given by

$$\pi_i^* = \frac{\alpha_i - r}{\sigma_i^2(1-\gamma)}. \tag{90}$$

The optimal fraction of wealth dedicated to consumption is

$$C^* = \frac{1}{1-\gamma}\left(\delta - \gamma\left(r + \frac{1}{2(1-\gamma)}\sum_{i=1}^n \left(\frac{\alpha_i - r}{\sigma_i}\right)^2\right)\right),$$

and the value function W is equal to

$$W = \frac{C^{*(\gamma-1)}}{\gamma}.$$

The regions "sell i" and "buy i" are characterized by

$$\begin{aligned} B_i &= \{y \in (\mathbb{R}^+)^n,\ y_i \le \pi_i^*\}, \\ S_i &= \{y \in (\mathbb{R}^+)^n,\ y_i \ge \pi_i^*\}. \end{aligned}$$

Note that these regions are not obtained by merely setting $\lambda = \mu = 0$ in (86) but by taking the limit of these expressions when λ and μ tend to 0.

7.2. A general shape of the transaction regions. In this section, we derive formally from VI (71), without numerical computation, the general shape of the transaction regions, given in Figure 12. To that purpose, we assume the function W to be $\mathcal{C}^2$ in the interior of $(\mathbb{R}^+)^n$. Although this is not true in general, what is done below can be adapted by using the theory of viscosity solutions. This approach is

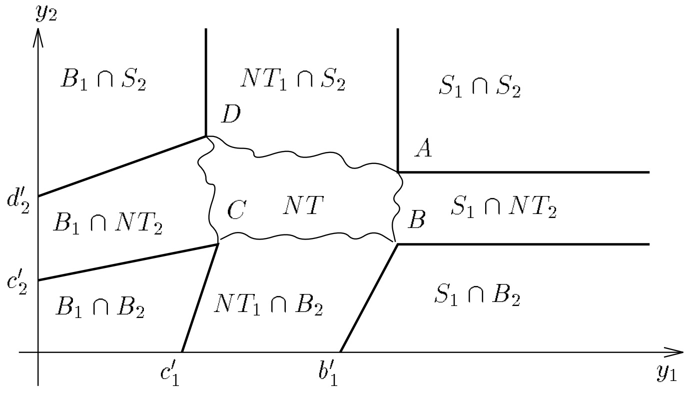

FIG. 12. *General shape of the transaction regions*

used for example in Fleming and Soner [13] to obtain regularity results for the value function V and general properties of the transaction regions for $n = 1$.

From (71), we have $\tilde{M}_i W \leq 0$; besides, the concavity of W implies that $\tilde{M}_i W = -\frac{\partial W}{\partial y_i}$ is non decreasing with respect to y_i. Consequently, the region S_i defined in (86) can be written as

$$S_i = \{y \in (\mathbb{R}^+)^n,\ y_i \geq \pi_i^+(\hat{y})\}$$

where π_i^+ is some mapping of

$$\hat{y} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n).$$

To obtain a similar characterization for B_i, we consider another change of variables (ρ', y') obtained by substituting $-\lambda_i$ for μ_i in (55) for some fixed $i \in \{1, \ldots, n\}$. Proceeding as above, and using Remark 6.1, we obtain :

$$B_i = \{y \in (\mathbb{R}^+)^n,\ y_i' \leq \pi_i^{'-}(\hat{y}')\} \tag{91}$$

with

$$y_i' = \frac{(1+\nu_i)y_i}{1+\nu_i y_i} \quad \text{and} \quad \hat{y}' = \frac{1}{1+\nu_i y_i}\hat{y}. \tag{92}$$

Since y_i' is non decreasing with respect to y_i, we get

$$B_i = \{y \in (\mathbb{R}^+)^n,\ y_i \leq \pi_i^-(\frac{1}{1+\nu_i y_i}\hat{y})\}.$$

Suppose $\pi_i^+ < +\infty$ and $\pi_i^- > 0$. This implies that π_i^+ and π_i^- are continuous functions and that the regions S_i and B_i are connected.

We restrict now to the case $n = 2$, but what is done below can easily be generalized to $n > 2$.

In S_1, $\tilde{M}_1 W = -\frac{\partial W}{\partial y_1} = 0$. The function W is thus constant with respect to y_1 in S_1. Consequently the parts of the boundaries ∂B_2 and ∂S_2 included in S_1 are straight lines of equation $y_2 =$ constant. Similarly, using the change of variables (92) with $i = 1$, we infer that the parts of the boundaries ∂B_2 and ∂S_2 included in B_1 are straight lines of equation

$$y_2' = \frac{y_2}{1 + \nu_1 y_1} = \text{ constant.}$$

By symmetry, we get similar properties for the boundaries ∂B_1 and ∂S_1 as displayed in Figure 12. No other property has been obtained for the boundary of NT.

A question which arises now is how is located the "Merton proportion" π^*. In general, π^* is not necessarily in the region NT. Nevertheless, we have

PROPOSITION 7.1. *We use the notation of Figure* 12 :

$$A = (a_1, a_2) = \partial S_1 \cap \partial S_2, \qquad B = (b_1, b_2) = \partial S_1 \cap \partial B_2,$$

$$C = (c_1, c_2) = \partial B_1 \cap \partial B_2, \qquad D = (d_1, d_2) = \partial B_1 \cap \partial S_2,$$

$$c_1' = \frac{c_1}{1 + \nu_2 c_2}, \quad b_1' = \frac{b_1}{1 + \nu_2 b_2}, \quad c_2' = \frac{c_2}{1 + \nu_1 c_1}, \quad d_2' = \frac{d_2}{1 + \nu_1 d_1}.$$

and set

$$\tilde{\pi}_i^* = \begin{cases} \dfrac{\pi_i^*}{1 + \nu_i - \nu_i \pi_i^*} & \text{if} \quad \pi_i^* < 1 + \dfrac{1}{\nu_i}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then,

$$\pi_1^* \leq a_1, b_1', \quad \pi_2^* \leq a_2, d_2'$$

and

$$d_1, c_1' \leq \tilde{\pi}_1^*, \quad b_2, c_2' \leq \tilde{\pi}_2^*.$$

Proof. We prove $\pi_i^* \leq a_i$, $i = 1, 2$. The other inequalities are obtained similarly by using the change of variables (92). In $S_1 \cap S_2$, the function W is equal to a constant W_0 and satisfies (71) which reduces to :

$$-\beta(y) W_0 + u^*(\gamma W_0) \leq 0$$

with $\beta(y)$ given in (68).

Hence,

$$-\beta(y) + (1-\gamma)\gamma^{\frac{1}{\gamma-1}} W_0^{\frac{1}{\gamma-1}} \leq 0 \quad \forall y \in S_1 \cap S_2.$$

On the other hand, the point A is in $S_1 \cap S_2 \cap \overline{NT}$. Assuming that W is $\mathcal{C}^2$ at point A, we obtain

$$\tilde{A}W + u^*(BW) = 0 \tag{93}$$

and

$$-\beta(A) + (1-\gamma)\gamma^{1/\gamma - 1} W_0^{\frac{1}{\gamma-1}} = 0.$$

Consequently

$$\beta(y) \geq \beta(A) \quad \forall y \in S_1 \cap S_2 = [a_1, +\infty) \times [a_2, +\infty).$$

As the function $\beta(y)$ is of the form $\beta_1(y_1) + \beta_2(y_2)$ with quadratic functions β_i, we get

$$\beta_i(y_i) \geq \beta_i(a_i) \quad \forall y_i \geq a_i.$$

Consequently, $a_i \geq \operatorname{Argmin} \beta_i = \pi_i^*$. □

7.3. Special case of no transaction cost for one of the risky assets. We suppose here $n = 2$, $\nu_1 = 0$, $\nu_2 > 0$. The VI (71) then reduces to

$$\max(\tilde{A}W + u^*(BW), \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2} - \nu_2 BW, \max_{i=1,2,\, y_i>0}(-\frac{\partial W}{\partial y_i}) = 0 \tag{94}$$

which implies that the function W is independent of y_1. Consequently the boundaries of B_2 and S_2 are horizontal straight lines of equation $y_2 = \pi_2^-$ and $y_2 = \pi_2^+$ respectively. Since equation (94) holds for all $y_1 \geq 0$, and W is the minimal solution of (94), we have

$$\max(\max_{y_1 \geq 0}(\tilde{A}W + u^*(BW)), \frac{\partial W}{\partial y_2} - \nu_2 BW, -\frac{\partial W}{\partial y_2}) = 0 \quad \text{for } y_2 > 0.$$

The regions B_1 and S_1 are delimited by the curve of equation $y_1 = \pi_1(y_2)$ where

$$\pi_1(y_2) = \operatorname*{Argmax}_{y_1 \geq 0}(\tilde{A}W + u^*(BW))$$

is the solution of

$$y_1 \sigma_1^2 y_2^2 \frac{\partial^2 W}{\partial y_2^2} - (2(\gamma - 1)\sigma_1^2 y_1 + (\alpha_1 - r)) y_2 \frac{\partial W}{\partial y_2} + \gamma((\alpha_1 - r) + (\gamma - 1)\sigma_1^2 y_1) W = 0.$$

Consequently

$$\pi_1(y_2) = \frac{\pi_1^* BW}{BW + \dfrac{y_2}{1-\gamma}\dfrac{\partial BW}{\partial y_2}}.$$

In particular $\pi_1(0) = \pi_1^*$ and $\pi_1(y_2) = (1 + \nu_2 y_2)\pi_1^*$ in B_2. In S_2, W is constant and $\pi_1(y_2) = \pi_1^*$ (see Figure 13). Moreover, by using the concavity of W, we obtain the estimate

$$0 < \pi_1(y_2) \leq \frac{\pi_1^*}{1 - \nu_2 y_2}.$$

REFERENCES

[1] M. Akian, *Analyse de l'algorithme multigrille FMGH de résolution d'équations d'Hamilton-Jacobi-Bellman*, in Analysis and Optimization of Systems, A. Bensoussan and J.L. Lions, eds., Lecture Notes in Control and Information Sciences, Vol. 144, Springer Verlag, 1990, pp 113–122.

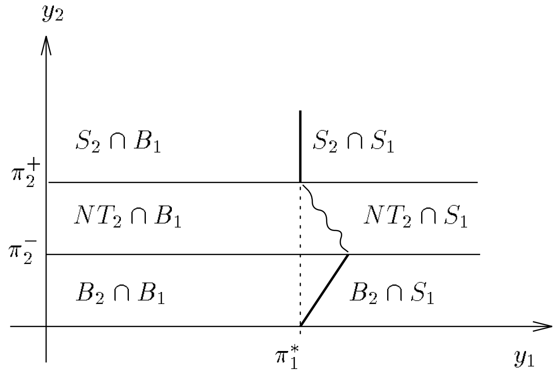

FIG. 13. *Boundaries of the transaction regions in the case of no transaction cost for the first risky asset.*

[2] ———, *Méthodes multigrilles en contrôle stochastique*, Thèse de l'Université Paris IX-Dauphine, Paris, France, 1990.
[3] G. BARLES AND P. E. SOUGANIDIS, *Convergence of approximation schemes for fully nonlinear second-order equations*, Asymptotic Analysis, 4 (1991), pp 271–283.
[4] R. BELLMAN, *Dynamic programming*, Princeton University Press, 1957.
[5] ———, *Introduction to the mathematical theory of control processes*, Academic Press, 1971.
[6] J. PH. CHANCELIER, C. GOMEZ, J. P. QUADRAT AND A. SULEM, *Automatic study in stochastic control*, in Stochastic Differential Systems, Stochastic Control Theory and Applications, W. Fleming and P. L. Lions, eds., IMA Volumes in Mathematics and its Applications, Vol. 10, Springer Verlag, 1987, pp 79–86.
[7] P. L. CHOW, J. L. MENALDI AND M. ROBIN, *Additive Control of Stochastic Linear Systems with Finite Horizon*, SIAM J. Control and Optim., 23 (1985), No. 6, pp 858–899.
[8] G. M. CONSTANTINIDES, *Capital market equilibrium with transaction costs*, J. of Political Economy, 94 (1986), No. 4, pp 842–862.
[9] M. G. CRANDALL, H. ISHII AND P. L. LIONS, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., 27 (1992), pp 1–67.
[10] M. DAVIS AND A. NORMAN, *Portfolio selection with transaction costs*, Math. Oper. Res., 15 (1990), pp 676–713.
[11] N. EL KAROUI, *Les aspects probabilistes du Contrôle stochastique*, in Lectures Notes in Mathematics, Vol. 876, Springer Verlag, 1981, pp 513–537.
[12] B. FITZPATRICK AND W. H. FLEMING, *Numerical Methods for an Optimal Investment-Consumption Model*, Math. Oper. Res., 16 (1991), pp 823–841.
[13] W. H FLEMING AND H. M. SONER, *Controlled Markov Processes and Viscosity Solutions*, Springer Verlag, New York, 1993.
[14] W. HACKBUSCH, *Multigrid methods and applications*, Springer Verlag, Berlin, Heidelberg, 1985.
[15] W. HACKBUSCH AND U. TROTTENBERG EDS., *Multigrid methods*, Lecture Notes in Mathematics, Vol. 960, Springer Verlag, 1981.
[16] R. A. HOWARD, *Dynamic programming and Markov process*, MIT, 1960.
[17] I. KARATZAS, J. LEHOCZKY, S. SETHI AND S. SHREVE, *Explicit solution of a general consumption/investment problem*, Math. Oper. Res., 11 (1986), No. 2, pp 261–294.
[18] H. J. KUSHNER *Probability methods in stochastic control and for elliptic equations*, Academic press, New York, 1977.
[19] P. L. LIONS *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations*, Part 1 : *The dynamic programming principle and applications*, Comm. in Partial Diff. Equ., 8 (1983), No. 10, pp 1101–1174.
[20] ———, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations*, Part 2 : *Viscosity solutions and uniqueness*, Comm. in Partial Diff. Equ., 8 (1983), No. 11, pp 1229–1276.
[21] M. J. P. MAGILL AND G. M. CONSTANTINIDES, *Portfolio selection with transaction costs*, J. Econ. Theory, 13 (1976), pp 245–263.
[22] S. F. MCCORMICK ED., *Multigrid methods*, SIAM frontiers in applied mathematics, Vol. 5, 1987.

[23] J. L. Menaldi and M. Robin, *On some Cheap Control Problems for Diffusion Processes*, Trans. Am. Math. Soc., 278 (1983), No. 2, pp 771–802.
[24] R. C. Merton, *Optimum consumption and portfolio rules in a continuous time model*, J. Economic Theory, 3 (1971), pp 373–413.
[25] P. A. Meyer, *Un cours sur les intégrales stochastiques*, Séminaire de Probabilités. Lectures Notes in Mathematics, Vol. 511, Springer-Verlag, Berlin, 1976, pp 245–400.
[26] M. Nisio, *On non linear semigroup attached to stochastic optimal control*, Publ. Res. Ins. Math. Sci., 12 (1976), pp 513–537.
[27] S. Sethi and M. Taksar, *A note on Merton's "Optimum consumption and portfolio rules in continuous-time model"*, J. Econ. Theory, 46 (1988), No. 2.
[28] S. E. Shreve and H. M. Soner, *Optimal investment and consumption with transaction costs*, Annals of Applied Probability, Vol. 4 (1994), No. 3, pp 909–692.
[29] S. E. Shreve, H. M. Soner and V. Xu, *Optimal investment and consumption with two bonds and transaction costs*, Mathematical Finance, 1 (1991), No. 3, pp 53–84.
[30] A. Sulem, *Application of stochastic control to portfolio selection with transaction costs*, Rapport de recherche INRIA, 1062 (1989).
[31] M. Taksar, M. J. Klass and D. Assaf, *A diffusion model for optimal portfolio selection in the presence of Brokerage fees*, Math. Oper. Res., 13 (1988), pp 277–294.
[32] T. Zariphopoulou, *Investment-Consumption model with transaction fees and Markov chain parameters*, SIAM J. of Control and Optim, 30 (1992), pp 613–636.