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A finite horizon multidimensional portfolio selection problem with singular transactions

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Abstract

This paper considers the optimal investment policy for an investor who has available one bank account paying a fixed interest rate r and n risky assets whose prices are correlated log-normal diffusions. We suppose that transactions between the assets incur a cost proportional to the size of the transaction. The problem is to maximize a function of the total net wealth on a finite horizon. Dynamic programming leads to a parabolic variational inequality for the value function which is solved by using a numerical algorithm based on policies iterations and multigrid methods. Numerical results are presented dealing with the issue of domestic asset allocation, that is the optimal split between cash, long bonds and equities. The impact of the transaction costs on the risk return characteristics of the optimal policies is analyzed.

Key words. Portfolio selection, transaction costs, viscosity solution, variational inequality, multigrid methods.

Introduction

Most of the methods used in portfolio management are derived from the works of H. Markowitz [9]. One of the problems is that the optimization proposed by Markowitz is made under a static framework. This paper aims to study dynamic optimization when the market is not frictionless. Previous studies in this direction have been done, in particular in [5], [6], [1], [2], [12].

The paper is organized as follows. Section 1 of the paper presents the model that we have adopted [5], [1]. It makes explicit the dynamics which the risky assets are supposed to follow, the utility function of the investor and how the transaction costs are taken into account. In section 2, using Dynamic Programming methodology, we establish the partial differential equations whose solution lead to the optimal strategy. The third part deals with the numerical methods. In the last section, numerical tests are performed in the case of two correlated risky assets and one riskless account. This example has been chosen to apply these techniques to the domestic asset allocation issue and to illustrate the decision that a manager must make to split his portfolio between cash, bonds and stocks. The impact of the transaction costs on the optimal policy and on the risk return characteristics of the optimal policies is studied.

I. The model

Consider an investor who has available one riskless bank account paying a fixed rate of interest r and n risky assets modeled by log-normal diffusions with expected rates of return $\alpha_i > r$ and rates of return variation σ_i^2. Any movement of money between the assets incurs a transaction cost proportional to the size of the transaction, paid from the bank account.

Let $s_0(t)$ (resp. $s_i(t)$ for $i = 1 \dots n$) be the amount of money in the bank account (resp. in the i-th risky asset) at time t and refer as $s(t) = (s_i(t))_{i=0,\dots,n}$ the investor position at time t. The evolution equations (in a given probability space $(\Omega, \mathcal{F}, P)$) of the investor holdings are

$$\begin{aligned} ds_0(t) &= rs_0(t)dt \\ &\quad + \sum_{i=1}^{n}((1-\mu_i)d\mathcal{M}_i(t) - (1+\lambda_i)d\mathcal{L}_i(t)), \\ ds_i(t) &= \alpha_i s_i(t)dt + \sigma_i s_i(t)dW_i(t) \\ &\quad + d\mathcal{L}_i(t) - d\mathcal{M}_i(t), \quad i = 1, \dots, n, \end{aligned}$$

with initial values $s_i(0^-) = x_i, \quad i = 0, \dots, n$, where $W_i(t), i = 1, \dots, n$, are correlated Wiener processes. Denote by ρ_{ij} the correlation rate between the processes W_i and W_j defined as :

$$E(W_i(t)W_j(t)) = \rho_{ij}\, t \text{ for } i \neq j.$$

An investment policy is a set $\mathcal{P} = (\mathcal{L}_i(t), \mathcal{M}_i(t))_{i=1,\dots,n})$ of adapted processes which represent cumulative purchase and sale of stock i on $[0, t]$ respectively such that $\mathcal{L}_i(t)$, $\mathcal{M}_i(t)$ are right-continuous, non-decreasing and $\mathcal{L}_i(0^-) = \mathcal{M}_i(0^-) = 0$. The process $s(t)$ is thus right continuous with left-hand limit, $s(t^-)$ denotes the left hand limit of the process s at time t. The coefficients λ_i and μ_i represent the proportional transaction costs. We suppose $\lambda_i \geq 0, \quad 0 \leq \mu_i \leq 1, \quad i = 1 \dots n$. We consider the admissible region $\mathcal{S} = \mathbb{R}_+^{n+1}$ which means that short position in any of the holdings is forbidden. A policy is admissible if the bankruptcy time $\bar{\tau} = \inf\{t \geq 0, s(t) \notin \mathcal{S}\}$ is larger than some fixed finite horizon T. We denote by $\mathcal{P}$ the set of admissible policies. Define the net wealth of the investor as the amount of money in the bank account resulting from the sale of the risky assets at time t :

$$\rho(t) = s_0(t) + \sum_{i=1}^{n}(1-\mu_i)s_i(t).$$

This definition of the wealth represents what is actually available for either consumption or investment. The investor's objective is to maximize over all admissible policies the quantity

$$E[\frac{1}{1-\gamma}\rho(T)^{1-\gamma}|s(0)=x]$$

where E denotes expectation and $\gamma \geq 0, \gamma \neq 1$. The coefficient γ is the relative risk aversion. The risk is maximal for $\gamma = 0$ (all the money is put in the asset with the largest rates of return α_i and return variation σ_i) and decreases as γ goes to $+\infty$. Define the value function as

$$V(t,x) = \sup_{\mathcal{P}\in\mathcal{P}} E[\frac{1}{1-\gamma}\rho(T)^{1-\gamma}|s(t)=x] \qquad (1)$$

Given a policy, define the random return of the portfolio at time t as $R(t) = \frac{\rho(t)-\rho}{\rho}$ where $\rho = \rho(0)$ is the initial wealth. We are interested in computing the expectation of the return $E(R(T) \mid s(0)=x)$ and the risk measured by the variance of the return $\mathrm{Var}(R(T) \mid s(0)=x)$ for the optimal policy. Given the optimal feedback, define

$$\begin{aligned} \mathcal{R}(t,x) &= E(\rho(T) \mid s(t)=x) && (2)\\ \mathcal{V}(t,x) &= E(\rho(T)^2 \mid s(t)=x). && (3) \end{aligned}$$

We can write:

$$\begin{aligned} E(R(T) \mid s(0)=x) &= \frac{\mathcal{R}(0,x)}{\rho} - 1,\\ \mathrm{Var}(R(T) \mid s(0)=x) &= \frac{\mathcal{V}(0,x)}{\rho^2} - \frac{\mathcal{R}(0,x)^2}{\rho^2}. \end{aligned}$$

II. Dynamic Programming Method

Theorem : The value function $V(t,x)$ defined in (1) is concave, continuous in x_i for $i = 0,\dots,n$ and in t. Moreover V is a viscosity solution of the parabolic variational inequality (VI) :

$$\begin{cases} \max\{\frac{\partial V}{\partial t} + AV, \max\limits_{1\leq i\leq n} L_iV, \max\limits_{\substack{1\leq i\leq n\\ x_i>0}} M_iV\} = 0 \\ \quad \text{in } \mathcal{S}\setminus\{x_0=0\}\times[0,T[, \\ \max\{\frac{\partial V}{\partial t} + AV, \max\limits_{\substack{1\leq i\leq n\\ x_i>0}} M_iV\} = 0 \\ \quad \text{in } \{x_0=0\}\times[0,T[, \\ V(T,x) = \frac{1}{1-\gamma}(x_0 + \sum_{i=1}^n (1-\mu_i)x_i)^{1-\gamma}, \end{cases} \qquad (4)$$

$$\begin{aligned} \text{where } AV &= \tfrac{1}{2}\textstyle\sum_{i=1}^n a_{ij}x_ix_j\frac{\partial^2 V}{\partial x_i \partial x_j} \\ &\quad + \textstyle\sum_{i=1}^n \alpha_i x_i \frac{\partial V}{\partial x_i} + rx_0\frac{\partial V}{\partial x_0}, \\ L_iV &= -(1+\lambda_i)\tfrac{\partial V}{\partial x_0} + \tfrac{\partial V}{\partial x_i}, \\ M_iV &= (1-\mu_i)\tfrac{\partial V}{\partial x_0} - \tfrac{\partial V}{\partial x_i}, \end{aligned}$$

$$a_{ii} = {\sigma_i}^2, \text{ and } a_{ij} = \rho_{ij}\sigma_i\sigma_j \text{ for } i \neq j.$$

Moreover, if the condition $\lambda_i + \mu_i > 0$ is satisfied, the solution is unique in the class of continuous functions in $\mathcal{S}\times[0,T]$ which satisfy

$$|V(x,t)| \leq C(1+\|x\|^{1-\gamma}), \quad \forall x \in \mathcal{S}, t\in[0,T]$$

for some constant $C > 0$. ∎**Sketch of the proof.** Concavity of V implies continuity in the interior of $\mathcal{S}$. Continuity on the boundary is obtained by proving the following: V is $(1-\gamma)$-continuous when $\gamma < 1$ (see [1]). When $\gamma > 1$, V is locally lipschitz-continuous with respect to x in $\mathcal{S}\setminus\{0\}$ and tends to ∞ when x goes to 0.

We can then state a weak Dynamic Programming Principle:

$$\forall t \in [0,T], \text{ for any stopping time } \theta \in [t,T],$$

$$V(t,x) = \sup_{\mathcal{P}\in\mathcal{P}^t} E_{tx}V(\theta, s(\theta^-;t,x,P))$$

where $\eta \in [t,T[\to s(\eta;t,x,P)$ is the unique solution of

$$\begin{aligned} ds_0(\eta) &= rs_0(\eta)d\eta + \\ &\quad \sum_{i=1}^n(-(1+\lambda_i)d\mathcal{L}_i(\eta) + (1-\mu_i)d\mathcal{M}_i(\theta)), \\ ds_i(\eta) &= \alpha_i s_i(\eta)d\eta + \sigma_i s_i(\eta)dW_i(\eta - t) + \\ &\qquad d\mathcal{L}_i(\eta) - d\mathcal{M}_i(\eta), \quad i = 1,\dots,n, \end{aligned}$$

$$s_i(t^-) = x_i, \quad i = 0,\dots,n,$$

and $\mathcal{P}^t$ is the set of admissible controls defined as

$$\begin{aligned} \{P: \quad [t,T]\times\Omega &\to \mathbb{R}^n\times\mathbb{R}^n, \\ (\theta,\omega) &\to (\mathcal{L}_i,\mathcal{M}_i)_{i=1,\dots n}(\theta,\omega)\}, \end{aligned}$$

- $\forall\theta \in [0,T], \mathcal{L}_i(\theta)$ and $\mathcal{M}_i(\theta)$ are $\mathcal{F}_{\theta-t}$measurable,
- the mappings $\theta \to \mathcal{L}_i(\theta), \mathcal{M}_i(\theta)$ are continuous, non decreasing and $\mathcal{L}_i(t^-) = \mathcal{M}_i(t^-) = 0$,
- $\inf\{\theta \geq t, s(\theta) \notin \mathcal{S}\} > T$.

Then, following Fleming and Soner [6, chapter 8, Theorem 5.1], we prove that V is a subsolution and a supersolution of the parabolic VI.

Uniqueness of the viscosity solution is proven by using the Ishii technique (see [1], [4]). ∎

At time t, the admissible region $\mathcal{S}$ is divided as follows :

$$\begin{aligned} {B_i}^t &= \{x\in\mathcal{S}, \ L_iV(t,x) = 0\}, \\ {S_i}^t &= \{x\in\mathcal{S}, \ M_iV(t,x) = 0\}, \\ {NT_i}^t &= \mathcal{S}\setminus({B_i}^t \cup {S_i}^t), \\ NT^t &= \bigcap_{i=1}^n {NT_i}^t. \end{aligned}$$

NT^t is the no transaction region at time t. Outside NT^t, an instantaneous transaction brings the position to the boundary of NT^t : buy stock i in ${B_i}^t$, sell stock i in ${S_i}^t$.

After the initial transaction, the agent position remains in $\overline{NT}^t = \{x \in \mathcal{S}, (\frac{\partial V}{\partial t} + AV)(t,x) = 0\}$, and further transactions occur only at the boundary (see [5]). The discrete control which selects the equation which satisfies the maximum in (4) indicates the optimal policy.

Given the optimal policy, we can state the Kolmogorov equations satisfied by the functions $\mathcal{R}(t,x)$ and $\mathcal{V}(t,x)$ defined in (2) and (3).

$\forall t \in [0,T[$, $\mathcal{R}(t,x)$ satisfies

$$\begin{cases} (\frac{\partial \mathcal{R}}{\partial t} + A\mathcal{R})(t,x) = 0 & \text{in } NT^t, \\ L_i\mathcal{R}(t,x) = 0 & \text{in } {B_i}^t, \\ M_i\mathcal{R}(t,x) = 0 & \text{in } {S_i}^t, \end{cases} \quad (5)$$

with $\mathcal{R}(T,x) = \rho$ and $\mathcal{V}(t,x)$ satisfies (5) with $\mathcal{V}(T,x) = \rho^2$.

Reduction of the state dimension: The value function V defined by (1) has the homothetic property [5]

$$\forall \rho > 0,\ V(t,\rho x) = \rho^{1-\gamma} V(t,x). \quad (6)$$

Consequently, the $(n+1)$-dimensional VI (4) satisfied by V can be reduced to a n-dimensional VI by considering the new state variables:

$$\begin{cases} \rho &=& x_0 + \sum_{i=1}^n (1-\mu_i)x_i \\ y_i &=& \frac{(1-\mu_i)x_i}{\rho}, i = 1\ldots n \end{cases}$$

where ρ represents the net wealth and y_i the fraction invested in stock i. We have

$$V(t,x) = \rho^{1-\gamma} W(t,y) \quad \text{where}$$

$$W(t,y) = V(t, 1 - \sum_{i=1}^n y_i, \frac{y_1}{1-\mu_1}, \ldots, \frac{y_n}{1-\mu_n}). \quad (7)$$

The function W satisfies

$$\begin{cases} \max\{\frac{\partial W}{\partial t} + \tilde{A}^\gamma W, \max_{1\le i\le n} \tilde{L}_i^\gamma W, \max_{\substack{1\le i\le n \\ y_i>0}} \tilde{M}_i W\} \\ \quad = 0 \quad \text{in } \Delta_n \setminus \{y_1 + \ldots + y_n = 1\} \times [0,T[, \\ \max\{\frac{\partial W}{\partial t} + \tilde{A}^\gamma W, \max_{\substack{1\le i\le n \\ y_i>0}} \tilde{M}_i W\} = 0 \\ \quad \text{in}\{y_1 + \ldots y_n = 1\} \times [0,T[, \\ W(T,y) = \frac{1}{1-\gamma}, \end{cases} \quad (8)$$

where $\Delta_n = \{y \in \mathbb{R}^n, y_i \ge 0, \sum_{i=1}^n y_i \le 1\}$,

$$\begin{cases} \tilde{A}^\gamma W = \sum_{k,l=1}^n \tilde{a}_{kl} \frac{\partial^2 W}{\partial y_k \partial y_l} + \sum_{k=1}^n b_k^\gamma \frac{\partial W}{\partial y_k} + \beta^\gamma W, \\ \tilde{L}_i^\gamma W = \frac{\partial W}{\partial y_i} - \nu_i((1-\gamma)W - \sum_{j=1}^n y_j \frac{\partial W}{\partial y_j}), \\ \tilde{M}_i W = -\frac{\partial W}{\partial y_i}, \end{cases} \quad (9)$$

$$\nu_i = \frac{\lambda_i + \mu_i}{1-\mu_i},$$

$$\tilde{a}_{kl} = \frac{y_k y_l}{2} \sum_{i,j=1}^n a_{ij}(\delta_{ki} - y_i)(\delta_{jl} - y_j),$$

$$b_k^\gamma = y_k \sum_{i=1}^n [-\gamma \sum_{j=1}^n a_{ij} y_j + \alpha_i - r](\delta_{ik} - y_i),$$

$$\beta^\gamma = (1-\gamma)(r + \sum_{i=1}^n [(\alpha_i - r)y_i - \frac{\gamma}{2} \sum_{i,j=1}^n a_{ij} y_i y_j]).$$

The symbol δ_{ij} denotes the Kronecker index which is equal to 0 when $i \ne j$ and equal to 1 when $i = j$.

Remark: The function W only depends on $\nu = (\nu_i)_{i=1\ldots n}$. Let us denote by $V_{\lambda,\mu}$ the value function in order to express explicitly the dependency of V on the transaction costs and by $W_{\lambda,\mu}$ the solution of (8). We have :

$$\begin{aligned} W_{\lambda,\mu}(t,y) &= W_{\nu,0}(t,y) \\ &= V_{\nu,0}(t, 1 - \sum_{i=1}^n y_i, y_1, \ldots, y_n). \end{aligned}$$

Using (6),we get

$$V_{\lambda,\mu}(t,x) = V_{\nu,0}(t, x_0, (1-\mu_1)x_1, \ldots (1-\mu_n)x_n).$$

Let us outline an explanation for this observation: a purchase of an amount of Δs_i of asset i corresponds to an effective monetary value of $(1-\mu_i)\Delta s_i = \bar{\Delta} s_i$. The investor pays $(1+\lambda_i)\Delta s_i = (1+\nu_i)\bar{\Delta} s_i$. The problem can thus be reformulated as if the transaction costs (equal to ν_i) would appear only on purchase and not on sale. ■Similar homothetic property is valid for $\mathcal{R}(t,x)$ and $\mathcal{V}(t,x)$. We have $\mathcal{R}(t,x) = \rho\overline{\mathcal{R}}(t,y)$ where $\overline{\mathcal{R}}(t,y)$ satisfies

$$\begin{cases} (\frac{\partial \overline{\mathcal{R}}}{\partial t} + \tilde{A}^\gamma \overline{\mathcal{R}})(t,y) = 0 & \text{in } NT^t, \\ \tilde{L}_i^\gamma \overline{\mathcal{R}}(t,y) = 0 & \text{in } {B_i}^t, \\ \tilde{M}_i \overline{\mathcal{R}}(t,y) = 0 & \text{in } {S_i}^t, \\ \overline{\mathcal{R}}(T,y) = 1 \end{cases} \quad (10)$$

with $\tilde{A}^\gamma, \tilde{L}_i^\gamma, \tilde{M}_i$ defined by (9) with $\gamma = 0$. Similarly, $\mathcal{V}(t,x) = \rho^2 \overline{\mathcal{V}}(t,y)$ where $\overline{\mathcal{V}}(t,y)$ satisfies (10) with $\gamma = -1$. Consequently, the expectation and the variance of the return of the portfolio are

$$E(R(T) \mid s(0) = x) = \overline{\mathcal{R}}(0,y) - 1$$

$$\text{Var}(R(T) \mid s(0) = x) = \overline{\mathcal{V}}(0,y) - \overline{\mathcal{R}}(0,y)^2.$$

The problem consists in solving first equation (8) to obtain the value function and the optimal strategy, and second, equation (10) with $\gamma = 0$ and $\gamma = -1$ to determine the mean return and the risk of the optimal portfolio.

III. Numerical study

We proceed with a technical change of variables which transforms the simplex Δ_n into $[0,1]^n$:

$$\begin{cases} z_1 = y_1 + \ldots + y_n, \\ z_i = \dfrac{y_i + \ldots + y_n}{y_{i-1} + \ldots + y_n} \quad i = 2, \ldots, n. \end{cases}$$

Equation (8) is transformed into an equation of the form :

$$\max(\frac{\partial \phi}{dt} + B^0 \phi, \max_{P \in \mathcal{P}(z)} B^P \phi) = 0 \qquad (11)$$

in $[0,1]^n \times [0, T[$ and $\phi(T, z) = \frac{1}{1-\gamma}$ where B^0 is a second order operator, B^P are first order operators and $\mathcal{P}(z) \subset \mathcal{P} = \{1, 2, \ldots, 2n\}$.

The solution of equation (11) leads to the optimal feedback $P(t,z) \in \mathcal{P}_{ad} \cup \{0\}$. $P(t,z) = 0$ means that $z \in NT^t$, otherwise $z \in B_i^t$ or S_i^t according to the value of $P(t,z)$. Equation (10) is then equivalent to

$$\begin{cases} \dfrac{\partial \phi}{dt} + B^0 W = 0 & \text{if } P(t,z) = 0 \\ B^P(t,z)\phi(t,z) = 0 & \text{otherwise} \\ \phi(T,z) = 1. \end{cases} \qquad (12)$$

The return and variance of the portfolio is obtained by solving equation (12) for $\gamma = 0$ and $\gamma = -1$ respectively.

For the numerical study, equations (11) and (12) are discretized and then solved by means of an iterative method. We use a Cranck-Nicholson scheme, for the time discretization, with time discretization step $\Delta t = \frac{T}{N}$, that is we make the following approximation:

$$(\frac{\partial \phi}{\partial t} + B^0 \phi)(t,x) \sim \frac{\phi(t + \Delta t, x) - \phi(t,x)}{\Delta t} + \frac{1}{2}(B^0 \phi(t + \Delta t, x) + B^0 \phi(t,x))$$

This discretisation leads to N elliptic equations with unknowns $\phi(t,.)$ in terms of $\phi(t + \Delta t, .)$, $t = k\Delta t, k = 0, \ldots, N$, which are solved backward starting from $\phi(T,.)$. Each equation is discretized in space by using classical finite difference approximation [8] and then solved by the Multigrid-Howard algorithm [1] based on the "Howard algorithm" (policy iteration) [7] and the multigrid method [10]. This procedure and the computer implementation is automatized by the expert system *Pandore* [3].

IV. The application to the domestic asset allocation issue

As an example for our numerical study, we focus on the domestic asset allocation issue. The riskless rate r is fixed at 6% . The drift α_1 of the first asset (long bond portfolio) is fixed at a level of 9% and its standard deviation σ_1 equal to 7%. The parameters of the other risky asset have been chosen to simulate the equity market : drift α_2 equal to 11% and volatility σ_2 equal to 20%, which is representative of the French stock market on the long run. The correlation coefficient ρ between the two risky assets is set at 40% and the time horizon T is equal to 1 year.

The Merton problem:
When the transaction costs are equal to zero, the optimal investment strategy is to keep a constant fraction of total wealth in each risky asset (see [11], [5], [6]). Indeed, set $\lambda = \mu = 0$ in equation (8). We obtain :

$$\begin{cases} \max(\dfrac{\partial W}{\partial t} + \tilde{A}W, \max\limits_{1 \le i \le n} \dfrac{\partial W}{\partial y_i}, \max\limits_{1 \le i \le n} (-\dfrac{\partial W}{\partial y_i})) \\ \quad = 0 \quad \text{in } \overset{\circ}{\Delta}_n \times [0, T[, \\ W(T,y) = \frac{1}{1-\gamma} \end{cases} \qquad (13)$$

which is equivalent to

$$\begin{cases} W(t,y) = W(t) \quad \text{constant with respect to y}, \\ \dfrac{\partial W}{\partial t} + (1-\gamma)(r + (\alpha - r, y) - \dfrac{\gamma}{2}(ay,y))W \\ \quad \le 0 \quad \text{in } \overset{\circ}{\Delta}_n, \\ W(T) = 1. \end{cases}$$

The value function W defined by (7)-(1) is the minimal solution of VI (13) and $(1-\gamma)W$ is positive. Hence, W satisfies

$$\frac{\partial W}{\partial t} + (1-\gamma) \max_{y \in \Delta_n} \{r + (\alpha - r, y) - \frac{\gamma}{2}(ay,y)\} W = 0$$

which coincides with the Bellman equation of the problem where the proportion y_i is considered as a control variable [5]. The optimal proportion π^* is given by

$$\pi^* = \frac{1}{\gamma} a^{-1}(\alpha - r) \;\text{ if } \pi^* \in \Delta_n,$$

$$\pi^* \in \operatorname*{Argmax}_{y \in \Delta_n}((\alpha - r, y) + \frac{\gamma - 1}{2}(ay,y)) \;\text{ otherwise.}$$

Denote

$$\bar{\beta} = (1-\gamma) \max_{y \in \Delta_n} \{r + (\alpha - r, y) - \frac{\gamma}{2}(ay,y)\}.$$

$$\text{If } \pi^* \in \Delta_n,\; \bar{\beta} = (1-\gamma)(r + \frac{(\alpha - r, a^{-1}(\alpha - r))}{2\gamma}).$$

We have $\frac{\partial W}{\partial t} = -\bar{\beta} W$ and the value function W is thus equal to

$$W(t) = \frac{e^{\bar{\beta}(T-t)}}{1-\gamma}.$$

The regions "sell i" and "buy i" are characterized $\forall t \in [0,T]$ by $B_i^t = \{y \in \Delta_n, \; y_i < \pi_i^*\}, S_i^t = \{y \in$

$\Delta_n,\ y_i > \pi_i^*\}$. Set $\bar{\alpha} = r + (\alpha - r, \pi^*)$ and $\bar{\sigma} = (a\pi^*, \pi^*)^{1/2}$. We have

$$\overline{\mathcal{R}}(t,y) = e^{\bar{\alpha}(T-t)}, \quad \overline{\mathcal{V}}(t,y) = e^{(2\bar{\alpha}+\bar{\sigma}^2)(T-t)}.$$

Consequently, in the case of no transaction costs, the mean and the variance of the return are

$$\begin{cases} E(R(T) \mid s(0) = x) & = & e^{\bar{\alpha}T} - 1 \\ \mathrm{Var}(R(T) \mid s(0) = x) & = & e^{2\bar{\alpha}T}(e^{\bar{\sigma}^2 T} - 1). \end{cases}$$

In the presence of transaction costs, the investor can be led to keep his portfolio unchanged when the costs incurred by the transactions are larger than the benefit in terms of utility provided by the readjustment of the portfolio. The existence of the no transaction region is the main difference with the Merton problem. In the following, we show how this region is modified in function of the level of transaction costs and of the time remaining before the end of the investment.

Sensitivity to transaction costs at the beginning and at the end of the investment period

When the time remaining before the end of the investment period is important, investors have a strong incentive to trade on risky assets as long as the transaction costs are not too high. The numerical tests show (see table IV) that transaction costs as high as 1% on both assets should not prevent the rational investor to adjust his portfolio. This adjustment leads him to the closest point on the boundary of the no transaction region. The results are different

transaction costs on both assets	no transaction region for asset 1 and asset 2
0%	$y1 = 17.3\%$ $y2 = 82.7\%$
0.1%	$24.4\% < y1 < 37.5\%$ $96.9\% < y2 < 100\%$
0.5%	$18.8\% < y1 < 43.8\%$ $96.9\% < y2 < 100\%$
0.75%	$18.8\% < y1 < 56.3\%$ $96.9\% < y2 < 100\%$
1%	$18.8\% < y1 < 68.8\%$ $96.9\% < y2 < 100\%$
2%	$18.8\% < y1 < 100\%$ $3.1\% < y2 < 100\%$

Table 1. Sensitivity to transaction costs at the beginning of the investment period:

when the same calculations are performed nearer to the end of the investment period (see Table IV) : the time remaining is 20% of the initial period. At this stage of the process the only example in which it may be sensible to trade on risky assets is when the transaction costs are as low as 0.1% on both assets. As expected, the no transaction region tends to be larger when the transaction costs increase and when the end of the investment period is closer. Indeed, the investor faces a trade-off between the instantaneous cost of transacting and the expectation of a higher level of final utility if he trades. Given that, the adjustment is all the more efficient than the time remaining is long.

transaction costs	no transaction region for asset 1 and asset 2
0%	$y1 = 17.3\%$ $y2 = 82.7\%$
0.1%	$24.4\% < y1 < 37.5\%$ $96.9\% < y2 < 100\%$
0.5%	$21.8\% < y1 < 56.3\%$ $15.6\% < y2 < 100\%$
0.75%	$21.8\% < y1 < 78.1\%$ $6.3\% < y2 < 100\%$
1%	$18.8\% < y1 < 93.8\%$ $3.1\% < y2 < 100\%$
2%	$9.4\% < y1 < 100\%$ $0\% < y2 < 100\%$

Table 2. Sensitivity to transaction costs at the end of the investment period

Mean variance properties of the portfolios:

We analyze now the impact of the transaction costs on the financial characteristics of the investment presented through graphs in the mean-variance plan. Figure 1

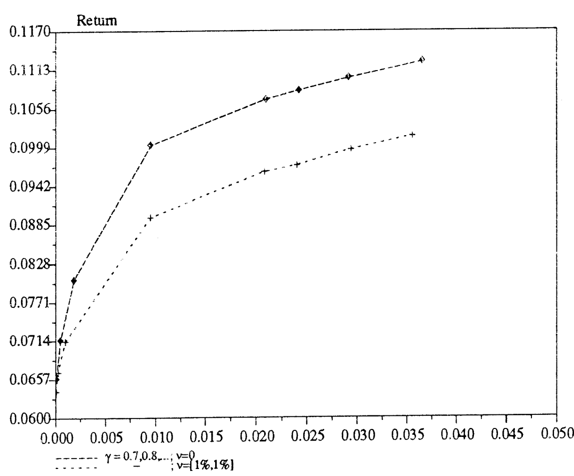

Fig. 1. Impact of uniformed transaction costs on the entire efficient frontier

represents the efficient frontiers when the transaction costs are equal respectively to 0 (Merton case) (upper curve) and 1% on both assets. The risk aversion γ varies from 0.7 to 50. We see that when the risk aversion is low enough, the only impact of the transaction costs is a decrease in returns of about 1%. This decrease occurs because the initial transaction involves the entire portfolio that must be split between the two risky assets. However, for an investor with such a behavior towards risk, the expected returns of the risky assets remain interesting so that he does not modify the risk level of his portfolio. At the left end of the graph, the conclusion is slightly different since risk adverse investors tend to be even less adventurous when they face transaction costs. As expected, the risk adverse investor demands high rewards to take risk. Figure 2 shows how the risk return of the

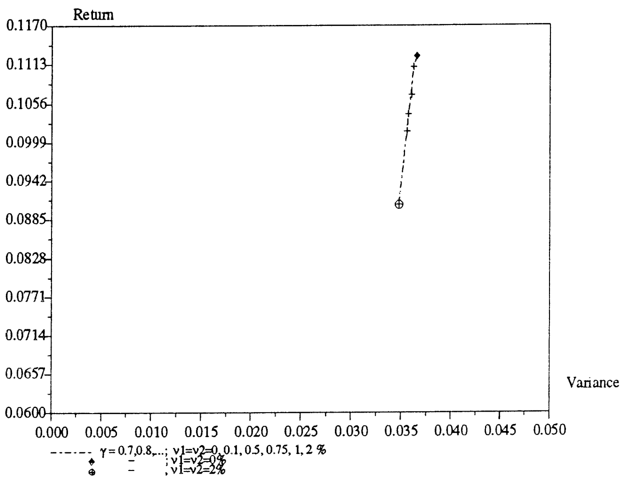

Fig. 2. Impact of increasing transaction costs on the strategy

optimal policy is modified when the transaction costs increase from 0 to 2%. This graph confirms that the main impact is observed on returns which decrease. The fixed risk aversion of the investor is quite low ($\gamma = 0.7$) in this example and then the expenses linked to the trades do no prevent him to invest on the risky assets. The

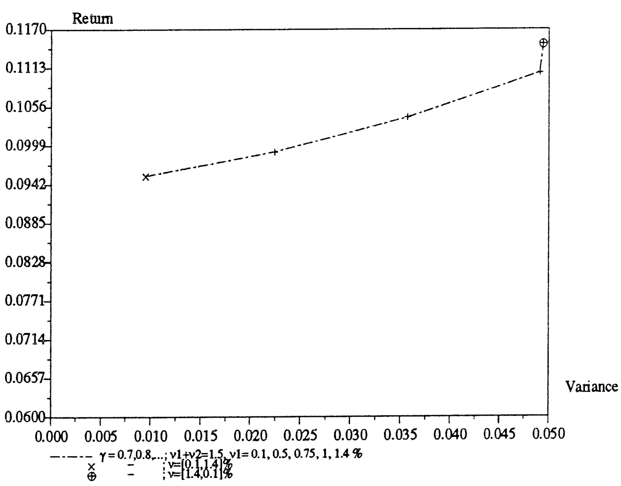

Fig. 3. Impact of different transaction costs on the strategy

impact of having two different transaction costs for the risky assets is displayed in Figure 3. This test has been performed with a rather low risk aversion ($\gamma = 0.7$) and with $\nu_1 + \nu_2 = 1.5$. Transaction cost on bond increases from 0.1 to 1.4 as cost on stock decreases from 1.4 to 0.1 from left to right on the graph. When bonds have much lower transaction costs compared to the stocks, the investor prefers portfolio with lower risk than the ones he would have implemented with comparable transaction costs on the two assets. On the other hand, if the stocks are much cheaper to buy and sell the investor increases the risk of his portfolio. Indeed, when the bonds are cheaper than the stocks, the excess return of the equities is diminished a lot and then the reward that they offer for the risk they entail is much less attractive. When the bonds incur high costs there is no point buying them since they do not provide any more a satisfactory risk premium over the riskless asset. Then, since the investor considered is not too risk adverse, he prefers being invested on the stock market.

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