TOPOLOGIES ON LATTICE ORDERED GROUPS, SEPARATION FROM CLOSED DOWNWARD SETS AND CONJUGATIONS OF TYPE LAU

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ABSTRACT. We extend the results of Rubinov and Glover on the separation of normal subsets of $(0,+\infty)^n$, and those of Martínez-Legaz, Rubinov and Singer on the separation of downward subsets of $\mathbb{R}^n$, to the unifying framework of subsets of A^n, where A is a continuous conditionally complete lattice ordered group. We also extend earlier results of the second author on the characterizations of conjugations of type Lau of functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, to the framework of functions $f : A^n \to \overline{A}$, where $\overline{A}$ is the canonical enlargement of A. These extensions are based on the notions and results of the theory of continuous lattices, generalized in an earlier work of the first author and here, to the conditionally complete case, in particular the way below order relation $\ll$, and the Scott and Lawson topologies. We also introduce the "way above order relation $\gg$", and consider the "bi-Scott topology" on A, introduced previously by the first author, which under our continuity assumption on A, turns out to be equal to the order topology; moreover, we show that for this topology, $(A, \otimes)$ is a topological group and $(A, \leq)$ is a topological lattice.

INTRODUCTION

In [12], Martínez-Legaz, Rubinov and Singer have characterized closed downward sets of $\mathbb{R}^n$ in terms of separation, as follows. Recall (see for instance [12]) that a subset G of $\mathbb{R}^n$ is

- *downward*, if along with each element it also contains all smaller elements (in the sense of the usual componentwise partial order of $\mathbb{R}^n$), that is, if
$$g \in G, x \in \mathbb{R}^n, x \leq g \Rightarrow x \in G \; ; \tag{0.1}$$
- *closed along diagonal lines*, if
$$(x \in \mathbb{R}^n, \lambda_k \in \mathbb{R}, x + \lambda_k \mathbf{1} \in G, k = 1, 2, ..., \lambda_k \to_{k\to\infty} \lambda \in \mathbb{R}) \Rightarrow x + \lambda \mathbf{1} \in G \; , \tag{0.2}$$

  where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

Downward sets arise as the set of all solutions of a system of inequalities $f_t(x) \leq 0$ $(t \in T)$, where T is an arbitrary index set and, for each $t \in T$, f_t is an increasing function defined on $\mathbb{R}^n$ (see [12] and the references therein). Convex downward subsets of $\mathbb{R}^n$ play an important role in mathematical economics and

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cooperative game theory, where they are called *comprehensive* sets. Also, not necessarily convex downward sets have already found applications in the theory of games with non-transferable utility.

Consider the "coupling function" $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(x, w) := \min_{1 \le i \le n} (x_i + w_i) \quad (x = (x_i), w = (w_i) \in \mathbb{R}^n) . \tag{0.3}$$

Theorem 0.1. [12, Theorem 8]*For a subset G of $\mathbb{R}^n$, the following statements are equivalent:*

1°. *G is a closed downward set.*

2°. *G is closed along diagonal lines and downward.*

3°. *For each $x \in \mathbb{R}^n \setminus G$, there exists $w \in \mathbb{R}^n$ such that*

$$\sup_{g \in G} \varphi(g, w) \le 0 < \varphi(x, w) . \tag{0.4}$$

4°. *For each $x \in \mathbb{R}^n \setminus G$, there exists $w \in \mathbb{R}^n$ such that*

$$\sup_{g \in G} \varphi(g, w) < \varphi(x, w) . \tag{0.5}$$

The separation properties (0.4) and (0.5) should be compared with the separation of convex sets. In particular, let us note that the coupling function φ of (0.3) is the canonical scalar product in the min-plus semifield denoted $\mathbb{R}_{\min}$, which is the set $\mathbb{R} \cup \{+\infty\}$ endowed with the addition $a \oplus b = \min(a, b)$ and the multiplication $a \otimes b = a + b$, with neutral elements $+\infty$ and 0 for the addition and multiplication respectively (see for instance [15]). Thus, Theorem 0.1 shows that downward sets are separated by min-plus linear forms.

Another interpretation of Theorem 0.1 is obtained by using abstract convexity notions that we next recall (see for instance [27] for all these notions). Let us fix two arbitrary sets X and W. A mapping $\Delta : 2^X \to 2^W$ is called a *duality* (2^X denotes the family of all subsets of X), if

$$\Delta(G) = \cap_{g \in G} \Delta(\{g\}) \quad (G \subseteq X) . \tag{0.6}$$

The *dual* of any mapping $\Delta : 2^X \to 2^W$ is the mapping $\Delta' : 2^W \to 2^X$ defined by

$$\Delta'(P) = \{x \in X | \, P \subseteq \Delta(\{x\})\} \quad (P \subseteq W) ; \tag{0.7}$$

it is well-known and easy to see that for any mapping $\Delta : 2^X \to 2^W, \Delta'$ is a duality and that Δ is the dual of Δ'.

A subset G of X is called $\Delta'\Delta$-*convex*, if $G = \Delta'\Delta(G) := \Delta'(\Delta(G))$, that is [27], if for each $x \notin G$ there exists $w \in W$ such that

$$G \subseteq \Delta'(\{w\}), \, x \in X \backslash \Delta'(\{w\}) . \tag{0.8}$$

If $\psi : X \times W \to \overline{\mathbb{R}}$ is a coupling function, the *associated duality* $\Delta_\psi : 2^X \to 2^W$, called ψ-*duality*, is defined [29] by

$$\Delta_\psi(G) := \{w \in W | \, \psi(g, w) \le 0 \; (g \in G)\} \quad (G \subseteq X) . \tag{0.9}$$

If $X = W$ and $\psi : X \times X \to \overline{\mathbb{R}}$ is a symmetric coupling function, i.e., satisfying

$$\psi(x, w) = \psi(w, x) \quad (x, w \in X) , \tag{0.10}$$

we have (see [29, Lemma 2.1])

$$\Delta'_\psi = \Delta_\psi , \tag{0.11}$$

where, for simplicity, we denote $(\Delta_\psi)'$ by Δ'_ψ.

For the ψ-duality Δ_ψ of (0.9), $\Delta'_\psi\Delta_\psi$-convexity (0.8) of a set $G \subseteq X$ means that for each $x \notin G$, there should exist $w \in W$ such that

$$\sup_{g\in G} \psi(g, w) \leq 0 < \psi(x, w) . \tag{0.12}$$

A weaker requirement is the (W, ψ)-*convexity* of G, which means that G should coincide with its (W, ψ)-convex hull, i.e., by [27, p. 88, formula (2.197)], that for each $x \notin G$ there should exist $w \in W$ such that

$$\sup_{g\in G} \psi(g, w) < \psi(x, w) . \tag{0.13}$$

In particular, for $X = W = \mathbb{R}^n$ and the coupling function $\psi = \varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of (0.3) (which is symmetric), formula (0.9) becomes

$$\Delta_\varphi(G) = \{w \in \mathbb{R}^n |\ \min_{i\in I} (g_i + w_i) \leq 0\ (g \in G)\} \quad (G \subseteq \mathbb{R}^n) , \tag{0.14}$$

and hence, by (0.11), for any set $G \subseteq \mathbb{R}^n$, there holds

$$\Delta'_\varphi\Delta_\varphi(G) = \Delta_\varphi(\Delta_\varphi(G)) = \{x \in \mathbb{R}^n |\ \min_{i\in I} (x_i + w_i) \leq 0\ (w \in \Delta_\varphi(G))\} . \tag{0.15}$$

Theorem 0.1 shows that $\Delta'_\varphi\Delta_\varphi$-convex sets, (W, φ)-convex sets and closed downward sets coincide. This fact has been used in [29] to obtain characterizations of $\Delta'_\varphi\Delta_\varphi$-quasi-convex functions (recall that for any property P of abstract convexity of subsets G of a set X, a function $f : X \to \overline{\mathbb{R}}$ is called P-*quasi-convex*, if all lower level sets

$$S_d(f) := \{x \in X \mid f(x) \leq d\} \quad (d \in \mathbb{R}) \tag{0.16}$$

are P-convex).

A useful tool in the study of quasi-convexity of functions is the conjugate of type Lau. Recall that, if $\Delta : 2^X \to 2^W$ is a duality, and $f : X \to \overline{\mathbb{R}}$ is a function, *the conjugate of type Lau of* f, *with respect to* Δ, is the function $f^{L(\Delta)} : W \to \overline{\mathbb{R}}$ defined by

$$f^{L(\Delta)}(w) := - \inf_{\substack{x\in X \\ w\in W\setminus\Delta(\{x\})}} f(x) \quad (w \in W) . \tag{0.17}$$

If $\psi : X \times W \to \overline{\mathbb{R}}$ is a coupling function, then for the conjugate of type Lau $f^{L(\Delta_\psi)}$ with respect to the ψ-duality $\Delta_\psi : 2^X \to 2^W$ of (0.9), which is also called the ψ-*conjugate of type Lau of* f, and denoted by $f^{L(\psi)}$, we have

$$f^{L(\psi)}(w) := f^{L(\Delta_\psi)}(w) = - \inf_{\substack{x\in X \\ \psi(x,w)>0}} f(x) \quad (w \in W) . \tag{0.18}$$

In particular, for $X = W = \mathbb{R}^n$ and the coupling function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of (0.3), we have $\varphi(x,w) > 0$ if, and only if, $x_i > -w_i$ for all $i \in I$, which we shall denote, briefly, by $x \gg -w$. Hence

$$f^{L(\varphi)}(w) = -\inf_{\substack{x\in\mathbb{R}^n\\ \varphi(x,w)>0}} f(x) = -\inf_{\substack{x\in\mathbb{R}^n\\ x\gg -w}} f(x) \quad (w \in \mathbb{R}^n)\ . \tag{0.19}$$

By [29, Lemma 3.1], for any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the φ-conjugate of type Lau $f^{L(\varphi)} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is an isotone lower semi-continuous function.

The *dual* of any mapping $u : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^W$ is the mapping $u' : \overline{\mathbb{R}}^W \to \overline{\mathbb{R}}^X$ defined by

$$h^{u'} := \inf_{\substack{f\in\overline{\mathbb{R}}^X\\ f^u\le h}} f \quad (h \in \overline{\mathbb{R}}^W)\ , \tag{0.20}$$

where, as above, for any mapping $u : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^W$ and any $f \in \overline{\mathbb{R}}^X$ we write f^u instead of $u(f)$. By [29, Lemma 3.3], if $X = W$ and $\psi : X \times X \to \overline{\mathbb{R}}$ is a symmetric coupling function (i.e., satisfying (0.10)), then the conjugation of type Lau $L(\psi) : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^X$ of (0.18) is self-dual, i.e., for the dual (0.20) of $L(\psi)$, we have

$$L(\psi)' = L(\psi)\ . \tag{0.21}$$

If $\Delta : 2^X \to 2^W$ is any duality, then the $\Delta'\Delta$-quasi-convex hull $f_{\mathrm{q}(\Delta'\Delta)}$ of any function $f : X \to \overline{\mathbb{R}}$, i.e. the greatest $\Delta'\Delta$-quasi-convex minorant of f, satisfies (see e.g. [27, p. 301, Theorem 8.21])

$$f_{\mathrm{q}(\Delta'\Delta)} = f^{L(\Delta)L(\Delta)'} \quad (f \in \overline{\mathbb{R}}^X)\ , \tag{0.22}$$

where $L(\Delta)' : \overline{\mathbb{R}}^W \to \overline{\mathbb{R}}^X$ is the dual (0.20) of $L(\Delta) : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^W$ defined by (0.17), and for any mappings $u : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^Y$ and $v : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ and any $f \in \overline{\mathbb{R}}^X$, we write f^{uv} instead of $(f^u)^v$. Hence, f is $\Delta'\Delta$-quasi-convex if, and only if,

$$f = f^{L(\Delta)L(\Delta)'}\ . \tag{0.23}$$

Since, for all $f \in \overline{\mathbb{R}}^W$, we have $f^{L(\Delta)'} = f^{L(\Delta)'L(\Delta)L(\Delta)'}$, $f^{L(\Delta)'}$ is $\Delta'\Delta$-quasi-convex.

In particular, for $X = W = \mathbb{R}^n$ and the φ-duality $\Delta = \Delta_\varphi : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ of (0.14), observing that a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is isotone if and only if its lower level sets (0.16) are downward for all $d \in \mathbb{R}$, and using also Theorem 0.1 (applied to $G = S_d(f), d \in \mathbb{R}$), (0.18), (0.21), and (0.23) one obtains the following result :

Theorem 0.2. [29, Theorem 2.2 and Proposition 3.1]*For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ the following statements are equivalent:*

1°. *f is isotone and lower semi-continuous.*

2°. *f is $\Delta'_\varphi\Delta_\varphi$-quasi-convex (i.e., all lower level sets $S_d(f)$, $d \in \mathbb{R}$, are $\Delta'_\varphi\Delta_\varphi$-convex).*

3°. *f is $(\mathbb{R}^n, \varphi)$-quasi-convex (i.e., all lower level sets $S_d(f), d \in \mathbb{R}$, are $(\mathbb{R}^n, \varphi)$-convex).*

4°. *We have*

$$f = f^{L(\varphi)L(\varphi)} . \tag{0.24}$$

Hence, for a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, all lower level sets $S_d(f)$ $(d \in \mathbb{R})$ are closed and downward if, and only if, we have (0.24). Also, for any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^{L(\varphi)}$ *is* $\Delta'_\varphi \Delta_\varphi$-quasi-convex.

Moreover, the following result describes a class of functions whose φ-conjugates of type Lau have a particularly simple form:

Theorem 0.3. [29, Theorem 3.1] *For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ the following statements are equivalent:*

1°. *f is isotone and upper semi-continuous.*

2°. *There holds*

$$f^{L(\varphi)}(w) = -f(-w) \quad (w \in \mathbb{R}^n) . \tag{0.25}$$

The above notions have also been studied in the literature under the multiplicative setting, which can be obtained for instance by applying the exponential map. Indeed, denote

$$\mathbb{R}_{++} := (0, +\infty), \quad \mathbb{R}^n_{++} := (\mathbb{R}_{++})^n = \{x = (x_i) \in \mathbb{R}^n \mid x_i > 0 \ (i = 1, ..., n)\} . \tag{0.26}$$

A subset G of $\mathbb{R}^n_{++}$ is said to be

- *normal*, if

$$g \in G, x \in \mathbb{R}^n_{++}, x \leq g \Rightarrow x \in G ; \tag{0.27}$$

- *closed along rays* [21, 20] if

$$(x \in \mathbb{R}^n_{++}, \lambda_k > 0, \lambda_k x \in G, k = 1, 2, ..., \lambda_k \to_{k \to \infty} \lambda > 0) \Rightarrow \lambda x \in G . \tag{0.28}$$

Normal sets arise as the set of all solutions of a system of inequalities $f_t(x) \leq 0$ $(t \in T)$, where T is an arbitrary index set and, for each $t \in T$, f_t is an increasing function defined on $\mathbb{R}^n_{++}$ (see [25] and the references therein). Normal sets have found many applications, e.g. in mathematical economics, where they are usually called *sets with free disposal.* Also, non-convex normal sets are applied to the so-called von Neumann dynamics.

Considering the "coupling function"

$$\varphi_0(x, w) := \min_{1 \leq i \leq n} x_i w_i \qquad (x = (x_i), w = (w_i) \in \mathbb{R}^n_{++}) , \tag{0.29}$$

part of the following characterizations of closed normal sets have been proved by Rubinov, Glover and Jeyakumar:

Theorem 0.4. [22, Proposition 5.15] *For a subset G of $\mathbb{R}^n_{++}$, the following statements are equivalent:*

1°. *G is a closed normal set.*

2°. *G is closed along rays and normal.*

3°. *For each* $x \in \mathbb{R}^n_{++} \setminus G$*, there exists* $w \in \mathbb{R}^n_{++}$ *such that*

$$\sup_{g \in G} \varphi_0(g, w) \leq 1 < \varphi_0(x, w) . \tag{0.30}$$

4°. *For each* $x \in \mathbb{R}^n_{++} \setminus G$*, there exists* $w \in \mathbb{R}^n_{++}$ *such that*

$$\sup_{g \in G} \varphi_0(g, w) < \varphi_0(x, w) . \tag{0.31}$$

Also, the *conjugate of type Lau* of a function $f : \mathbb{R}^n_{++} \to \overline{\mathbb{R}}_+$ with respect to the coupling function φ_0 of (0.29) is the function $f^{L(\varphi_0)} : \mathbb{R}^n_{++} \to \overline{\mathbb{R}}_+$ defined [23, 24, 21, 29] by

$$f^{L(\varphi_0)}(w) := \frac{1}{\inf_{\substack{x \in \mathbb{R}^n_{++} \\ \varphi_0(x,w)>1}} f(x)} = \sup_{\substack{x \in \mathbb{R}^n_{++} \\ x_i w_i > 1 \ (i=1,\dots,n)}} \frac{1}{f(x)} \quad (f \in \overline{\mathbb{R}}_+^{\mathbb{R}^n_{++}}, w \in \mathbb{R}^n_{++}) , \tag{0.32}$$

that is, by replacing in (0.19) $\mathbb{R}^n$, subtraction and 0 by $\mathbb{R}^n_{++}$, division and 1, respectively. Part of the following results have been proved by Rubinov and Glover:

Theorem 0.5. [21, Proposition 5.7]*For a function* $f : \mathbb{R}^n_{++} \to \overline{\mathbb{R}}_+$ *the following statements are equivalent:*

1°. f *is isotone and lower semi-continuous.*

2°. *We have*

$$f = f^{L(\varphi_0)L(\varphi_0)} . \tag{0.33}$$

Theorem 0.6. [21, property 3, p. 119]*If* $f : \mathbb{R}^n_{++} \to \overline{\mathbb{R}}_+$ *is isotone and continuous at* $\frac{1}{w}$*, where* $w \in \mathbb{R}^n_{++}$*, then*

$$f^{L(\varphi_0)}(w) = \frac{1}{f(\frac{1}{w})}. \tag{0.34}$$

Let us now consider the following unifying framework. Let $A = (A, \leq, \otimes)$ be a conditionally complete lattice ordered group, where $\otimes$ denotes the group operation, and define a coupling function $\varphi : A^n \times A^n \to A$ by

$$\varphi(x, w) := \inf_{1 \leq i \leq n} (x_i \otimes w_i) \quad (x = (x_i), w = (w_i) \in A^n) ; \tag{0.35}$$

note that, since A is not assumed to be totally ordered (A is totally ordered if, for any pair of elements x, y of A, we have either $x \leq y$ or $y \leq x$), the inf in (0.35) need not be attained. For $A = (\mathbb{R}, \leq, +)$, φ becomes the coupling function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by (0.3), and for $A = (\mathbb{R}_{++}, \leq, \times)$, φ becomes the coupling function $\varphi_0 : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ defined by (0.29). In [12] and [29], there have been raised the problems to generalize Theorems 0.1–0.6 to the case of subsets G of A and functions $f : A^n \to \overline{A}$, where $\overline{A}$ is the "canonical enlargement" of A (i.e. the enlargement obtained by adding a smallest and a largest element and extending the group operation and the order $\leq$ to this larger set, see Definition 1.7).

The aim of the present paper is to solve these problems and to develop the necessary machinery (which may have interest for other applications as well). We shall also give some related results on conditionally complete lattices and lattice ordered groups.

One of the difficulties of the generalization has been observed above: while $\mathbb{R}$ and $\mathbb{R}_{++}$ are totally ordered, the lattice A is not assumed to be totally ordered, and therefore we have to use inf instead of min in (0.35).

A further difficulty is that while $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}_{++}} = \overline{\mathbb{R}_{+}}$ are complete lattices (that is, each subset has an infimum and a supremum in the lattice), $\mathbb{R}$ and $\mathbb{R}_{++}$ are not complete, but only conditionally complete (that is, each nonempty upper bounded subset has a supremum and each nonempty lower bounded subset has an infimum, in the lattice; throughout this paper, we use the term "bounded" in the sense of the partial order $\leq$); therefore, we shall need to work in conditionally complete lattices A. An extension of some of the known notions and results on complete lattices (see e.g. [9]), to the case of conditionally complete lattices, has been given in [1]. Since we need here many results from [9] but in the context of conditionally complete lattices, we shall recall the definitions of [1] and we shall state and prove all the needed results for conditionally complete lattices, while mentioning whenever they extend a result of [9]. Let us note that our techniques of proofs are necessarily different from those of [9] (since we do not assume completeness of the lattice, nor the existence of a bottom and a top element), our main tool being the construction of suitable upper bounded directed sets D; for example, we do not use the "auxiliary orders" $\prec$ of [9]. Hence, our arguments give new proofs even in the case of complete lattices, for some results of [9]. Also, we shall give some results which are new even for complete lattices, such as the implications $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ$ of Theorem 4.2 on semi-continuity.

Another difficulty is that the relation $x = (x_i) < (y_i) = y$ on $\mathbb{R}^n$ (meaning that $x \leq y, x \neq y$) is not equivalent to $x_i < y_i$ $(i = 1, ..., n)$, occurring in (0.19) and (0.32); therefore, instead of $<$, we shall use on A and A^n the so-called "way below" relation $\ll$, extended from complete lattices (see e.g. [9] for the definition and properties of $\ll$ in complete lattices) to conditionally complete lattices, as defined in [1]; for $\mathbb{R}$ and $\mathbb{R}_{++}$, $\ll$ coincides with $<$, and for $\mathbb{R}^n$ and $\mathbb{R}^n_{++}$ it coincides with the usual $\ll$ (defined as componentwise $<$).

Also, since in Theorems 0.1–0.6 there occur topological conditions for subsets of $\mathbb{R}^n$ (resp. $\mathbb{R}^n_{++}$) and functions on $\mathbb{R}^n$, in order to ensure good closure and continuity properties of subsets of A^n and functions on A^n, we want to work with a topology on A for which $(A, \otimes)$ is a topological group, i.e. (see e.g. [6]), the mappings $(a, b) \mapsto a \otimes b$ and $a \mapsto a^{-1}$ are continuous, and $(A, \leq)$ is a topological lattice, i.e. (see e.g. [4, 10]) the mappings $(a, b) \mapsto a \vee b$ and $(a, b) \mapsto a \wedge b$ are continuous, and which on the product A^n reduces, for the particular cases of $A = (\mathbb{R}, \leq, +)$ and $A = (\mathbb{R}_{++}, \leq, \times)$, to the usual topology of $\mathbb{R}^n$ and $\mathbb{R}^n_{++}$ respectively. The so-called "order topology" on A (i.e., the topology defined with the aid of order convergence, see [4, Ch. X, Section 9] and Section 3), the "interval topology" on A, in the sense of Frink (which is defined by taking the sets $\{x \in A \mid a \leq x \leq b\}$ $(a \leq b \in A)$ as a pseudo-basis for the closed sets, see e.g. [4, Ch. X, Section 12]), and the "open-interval topology" on A (i.e., the topology generated by the sets $\{x \in A \mid x > a\}$ and $\{x \in A \mid x < a\}$ $(a \in A)$) do not satisfy these properties in general. More precisely, it is well-known (see e.g. [8, Chapter II, Section 8]) that for the order topology special conditions are necessary to ensure the continuity of the mappings $(a, b) \mapsto a \otimes b$. For the interval topology some important lattice ordered groups are not topological groups. Finally, the open-interval topology is Hausdorff only under an additional assumption of "density". Namely (see e.g. [8, Ch. II, Section 8, Theorem 11 and the remark after it]), if a lattice ordered group A is "dense", in the sense that for each pair $a, b \in A, a < b$, there exists $c \in A$ such that $a < c < b$, then the open-interval topology on A is Hausdorff, so A is a topological group (this result should be compared with Theorem 1.1 below, in which it is assumed, in addition, that A is continuous, that is, each $x \in A$ is the supremum of all elements of A that are $\ll x$).

We shall use here the so-called "Scott topology" and "Lawson topology" (and the auxiliary "lower topology"), extended from complete lattices [9] to conditionally complete lattices, as in [1]. We shall show that these topologies are well adapted to conditionally complete lattices A that are continuous, a property which is satisfied for $A = \mathbb{R}$ and $A = \mathbb{R}_{++}$. Moreover, when A is continuous, the Scott and Lawson topologies satisfy good properties with respect to the way-below relation. They are thus suitable for the generalizations of Theorem 0.1–0.6. Although these topologies do not ensure themselves that $(A, \otimes)$ is a topological group or $(A, \leq)$ is a topological lattice, it will turn out that they ensure, when A is continuous, the continuity of the maps $w_\varphi : A^n \to A,\ x \mapsto \varphi(x, w)$ $(w \in A^n)$, where $\varphi : A^n \times A^n \to A$ is the coupling function of (0.35), which is sufficient for our purpose. We shall also use the dual Scott and dual Lawson topologies, that is, the Scott and Lawson topologies associated to the opposite order $\geq$, as well as the bi-Scott topology, that is, the common refinement of Scott and dual Scott topologies, which is equal to the common refinement of Lawson and dual Lawson topologies. We shall show (in Subsection 4.1) that, when A is a continuous conditionally complete lattice ordered group, the bi-Scott topology is equal to the order topology and it does ensure that $(A, \otimes)$ is a topological group and $(A, \leq)$ is a topological lattice.

The structure of the paper is as follows. In the first part of Section 1, after recalling from [1] their definitions, we give some preliminary results on (continuous) conditionally complete lattices. We also introduce the concept of the "way above" relation $\gg$ in a conditionally complete lattice, dual to the known "way below" relation $\ll$. In the second part of Section 1, we recall briefly some known definitions and results concerning conditionally complete lattice ordered groups. In Section 2, we obtain some relations between the group operation $\otimes$ and the way below and way above orders, $\ll$ and $\gg$, in conditionally complete lattice ordered groups. Also, we show that the presence of the group operation $\otimes$ in a conditionally complete lattice $(A, \leq)$ ensures some pleasant properties; for example, it ensures the equivalence $a \ll b \Leftrightarrow b \gg a$ (while in general conditionally complete lattices, $a \ll b$ is not equivalent to $b \gg a$). In Section 3, after recalling from [1] their definitions, we give some results on the Scott, Lawson, dual Scott, dual Lawson and bi-Scott topologies on a continuous conditionally complete lattice, and we compare these topologies with the order and interval topologies. In Section 4, we study continuity and semi-continuity of functions with values in conditionally complete lattices, and, in particular, continuity of some mappings between conditionally complete lattice ordered groups. This allows us to show that for the bi-Scott topology, a continuous conditionally complete lattice ordered group is a topological group and a topological lattice. In Section 5, we give some characterizations of Scott, Lawson or bi-Scott closed downward sets in A^n, where A is a conditionally complete lattice ordered group, in terms of "separation" from outside points, which generalize Theorems 0.1 and 0.4 above. Finally, in Section 6, we introduce the "conjugate of type Lau" of a function $f : A^n \to \overline{A}$, with respect to the coupling function $\varphi : A^n \times A^n \to A$ defined by (0.35), and we give a generalization of Theorems 0.2, 0.3, 0.5 and 0.6 to such conjugations of type Lau.

1. Preliminaries

1.1. **Continuous conditionally complete lattices.** Let us first recall some classical terminology about lattices.

Definition 1.1. Let L be a set endowed with a partial order $\leq$.

- We say that $L = (L, \leq)$ is an *inf-semilattice* (resp. a *sup-semilattice*, resp. a *lattice*) if every nonempty finite set admits a greatest lower bound or infimum (resp. a least upper bound or supremum, resp. an infimum and a supremum).
- The supremum is denoted by sup or $\vee$ and the infimum by inf or $\wedge$.
- A subset M of a lattice (resp. inf-semilattice, resp. sup-semilattice) L is a *sublattice* (resp. *inf-subsemilattice*, resp. *sup-subsemilattice*) of L if $a \vee b$ and $a \wedge b \in M$ (resp. $a \wedge b \in M$, resp. $a \vee b \in M$), for all $a, b \in M$.
- We denote by $\leq^{\mathrm{op}}$ or $\geq$ the *opposite order* of L : $a \leq^{\mathrm{op}} b \Leftrightarrow b \leq a$. If $(L, \leq)$ is a partially ordered set, then L^{op} denotes $(L, \leq^{\mathrm{op}})$.
- L is *distributive* if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.
- If it exists in L, the *top element* or supremum of L is denoted by the symbol $\top$. Also, if it exists in L, the *bottom element* or infimum of L is denoted by the symbol $\perp$.
- L is a *complete lattice* if every (possibly empty) set admits an infimum (or equivalently if every set admits a supremum).

In the above definition and throughout the sequel, we use the convention that the infimum (resp. the supremum) of the empty set is the top element $\top$ (resp. the bottom element $\perp$) of the lattice L.

The following notion (see e.g. [4]) has been called "locally complete lattice" in [1], and the term "boundedly complete" has been also used in the literature.

Definition 1.2. A lattice $(L, \leq)$ is said to be *conditionally complete* if it satisfies one of the following equivalent conditions :

1°. every nonempty lower bounded set admits an infimum;
2°. every nonempty upper bounded set admits a supremum;

3°. there exists a complete lattice $\overline{L} := L \cup \{\bot, \top\}$, which we shall call the *minimal completion* of L, with bottom element $\bot$ and top element $\top$, such that L is a sublattice of $\overline{L}$, $\inf L = \bot$ and $\sup L = \top$.

Clearly, every complete lattice L is conditionally complete, since in a complete lattice every set is bounded (by $\bot$ and $\top$), and we have $\overline{L} = L$. Also, L is conditionally complete if, and only if, so is L^{op}.

Definition 1.3. (see e.g. [9]). Let $L = (L, \leq)$ be a lattice, and let L^{op} denote the lattice $(L, \geq)$, where $\geq$ denotes the opposite order.

- $D \subset L$ is a *directed set* if any finite subset of D has an upper bound in D.
- F is a *filtered subset* of L if F is a directed subset of L^{op}.
- $G \subset L$ is a *downward* set (or a *lower* set, or a *decreasing* set [17]) if $G = \downarrow G := \{x \in L \mid \exists y \in G,\ x \leq y\}$, or, equivalently, if the relations $y \in G, x \in L, x \leq y$ imply $x \in G$.
- $U \subset L$ is an *upward* set (or an *upper* set, or an *increasing* set [17]) if $U = \uparrow U := \{x \in L \mid \exists y \in U,\ y \leq x\}$, that is, if U is downward in L^{op}.

Definition 1.4. (see [9] and [1]). Let $L = (L, \leq)$ be a conditionally complete lattice.

- L is said to be *meet continuous* if

$$x \wedge \sup D = \sup(x \wedge D) \quad \text{for all upper bounded directed subsets } D \text{ of } L\,. \tag{1.1}$$

- The *way below* relation $\ll$ in L is defined by : $a \ll b$ if, and only if, for all upper bounded directed subsets D of L, such that $b \leq \sup D$, there exists $x \in D$ such that $a \leq x$.
- L is said to be *continuous* if

$$x = \sup\{y \in L \mid y \ll x\} \qquad \text{for all} \quad x \in L\,. \tag{1.2}$$

- L is said to be *dually continuous* if L^{op} is continuous.
- More generally, the term "dual" will mean: relative to the opposite order.

We introduce here the following dual concept to the way below relation:

Definition 1.5. Let $L = (L, \leq)$ be a conditionally complete lattice. The way below relation associated with the opposite order $\geq$, denoted $\ll^{\mathrm{op}}$ or $\gg$, is called the *way above* relation. It is defined by : $a \gg b$ if, and only if, for all lower bounded filtered subsets F of L, such that $b \geq \inf F$, there exists $x \in F$ such that $a \geq x$.

Let us give now some examples. The first one shows that, in general, $a \ll b \not\Leftrightarrow b \gg a$.

Example 1.1. In a complete totally ordered lattice L (i.e., such that for any pair $a, b \in L$ we have either $a \leq b$ or $a \geq b$), $a < b$ or $a = \bot$ implies $a \ll b$ (by definition $a < b \Leftrightarrow (a \leq b$ and $a \neq b)$). If $L = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the usual order $\leq$, then $a \ll b \;\Leftrightarrow\; (a < b$ or $a = b = -\infty)$, and L is continuous (see [9, Example I.1.3, p.39]). Similarly, $b \gg a \;\Leftrightarrow\; (b > a$ or $a = b = +\infty)$, and L is dually continuous. However, $a \ll b \;\not\Leftrightarrow\; b \gg a$ (indeed, e.g. for $a = b = -\infty$, we have $a \ll b$, but $b \not\gg a$). The way below relations $\ll$ and $\gg$ defined on $\mathbb{R}$ are exactly the restrictions of those of $\overline{\mathbb{R}}$. Hence, $\mathbb{R}$ is continuous and dually continuous and there $a \ll b \Leftrightarrow b \gg a$.

Example 1.2. If $L = \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, then $a \ll b \Leftrightarrow (a \leq b \text{ and } a \neq +\infty)$, and $b \gg a \Leftrightarrow (b \geq a \text{ and } b \neq -\infty)$. Hence, L is continuous (and dually continuous) [1, Example 1.6], and $a \ll b \not\Leftrightarrow b \gg a$.

Example 1.3. (see [9, Example I.1.3, p.39 and Theorem I.2.7.i, p. 60]). Let $(L_i)_{i\in I}$ be a family of complete lattices, and consider $L = \prod_{i\in I} L_i$ endowed with the componentwise order relation. Then, L is a complete lattice and $a = (a_i)_{i\in I} \ll b = (b_i)_{i\in I}$ in L if, and only if, $a_i \ll b_i$ for all $i \in I$ and $a_i = \perp_i$ the bottom element of L_i for all but a finite number of $i \in I$. Therefore, if all the L_i are continuous, L is continuous.

Example 1.4. Assume now that $(L_i)_{i\in I}$ is a family of conditionally complete lattices with no bottom element, and consider again $L = \prod_{i\in I} L_i$ endowed with the componentwise order relation. Then, L is a conditionally complete lattice, and $a \ll b$ in L if, and only if, $a_i \ll b_i$ for all $i \in I$ and I is finite. This means that L cannot be continuous if I is not finite. If I is finite, L is continuous, when all the L_i are continuous. In particular $((\overline{\mathbb{R}})^n, \leq)$ and $(\mathbb{R}^n, \leq)$ are continuous and dually continuous (respectively complete and conditionally complete) lattices. However, for $n \geq 2$, $\overline{\mathbb{R}^n}$ (the minimal completion of the lattice ordered group $(\mathbb{R}^n, \leq, +)$) is neither continuous nor dually continuous [1, Example 1.7], since in $\overline{\mathbb{R}^n}$, $a \ll b$ implies $a = \perp$, and $a \gg b$ implies $a = \top$. Note also that in the complete lattice of all closed subsets of $\mathbb{R}$ (with inclusion as order relation), $a \ll b$ implies $a = \perp$.

Remark 1.1. In this subsection, and in part of Sections 3 and 4.1, we give extensions of some results of [9] from the case of a complete lattice to that of a conditionally complete lattice. As has been mentioned in the Introduction, our techniques of proofs are necessarily different from those of [9] (since we do not assume completeness of the lattice, nor the existence of a bottom and a top element), our main tool being the construction of suitable upper bounded directed sets D. Another proof of these extensions could have been to apply the corresponding results of [9] to all complete lattices $[\perp, d] := \{\perp\} \cup \downarrow d = \{x \in L \cup \{\perp\} \mid x \leq d\}$ and use for instance the fact, noticed in [1], that $a \ll b$ in L if, and only if, $a \ll b$ in $[\perp, d]$ for all $d \in L$ such that $a, b \leq d$. This suggests, in particular, that many other results of [9] may be extended to conditionally complete lattices, with this method.

The following result is an obvious consequence of Definition 1.4 of the way below relation $\ll$, and we omit the proof.

Proposition 1.1. (extends [9, Proposition I.1.2, p. 39]). *Let L be a conditionally complete lattice. For all $x, y, z, t \in L$, we have:*

1°. *$x \ll y$ implies $x \leq y$.*
2°. *$x \leq y \ll z \leq t$ implies $x \ll t$.*
3°. *$x \ll z$ and $y \ll z$ together imply $x \vee y \ll z$.*
4°. *$\perp \ll x$ when $\perp \in L$.*

Theorem 1.1. (extends [9, Theorem I.1.18, p. 47]). *If L is a continuous conditionally complete lattice, then for all $a, b \in L$, such that $a \ll b$, there exists $c \in L$ such that $a \ll c \ll b$.*

Proof. Assume $a, b \in L, a \ll b$. Let

$$D = \cup_{c \ll b}\{x \in L \mid x \ll c\} = \{x \in L \mid \exists c \in L,\, x \ll c \ll b\}\ . \tag{1.3}$$

We claim that $D \neq \emptyset$. Indeed, if $\perp \in L$, then, by $a \ll b$ and Proposition 1.1,4°, we have $\perp \in D$. Assume now that $\perp \notin L$ and $D = \emptyset$. Then, for each $c \ll b, \{x \in L \mid x \ll c\} = \emptyset$, whence, since L is continuous, $c = \sup_{x \ll c} x = \sup\ \emptyset = \perp$; in particular, since $a \ll b$, it follows that $a = \perp \in L$, a contradiction which proves the claim.

The set D is upper bounded (by b) and we claim that it is a sup-subsemilattice of L, hence a directed set. Indeed, let $x, x' \in D$. Then, by (1.3), there exist $c, c' \in L$, such that $x \ll c \ll b$ and $x' \ll c' \ll b$. Using Proposition 1.1, we get $c \vee c' \ll b$, and $x \ll c \vee c'$, $x' \ll c \vee c'$, whence $x \vee x' \ll c \vee c' \ll b$. Thus, by (1.3), $x \vee x' \in D$, which proves the claim. We also have $\sup D = \sup_{c \ll b} \sup\{x \in L \mid x \ll c\} = \sup_{c \ll b} c = b$, since L is continuous. Then, since $a \ll b$, there exists $x \in D$ such that $a \leq x$. This means that there exists $c \in L$ such that $a \leq x \ll c \ll b$, and by Proposition 1.1, 2°, we get $a \ll c \ll b$. ∎

Example 1.5. There exist noncontinuous complete lattices L which satisfy the conclusion of Theorem 1.1. For instance, in the complete lattice $\overline{\mathbb{R}^2}$ or in the complete lattice of closed subsets of $\mathbb{R}$ (with inclusion as order relation), we have $a \ll b \Rightarrow a = \perp$ (see Example 1.4), hence $a \ll b \Rightarrow a \ll a \ll b$.

Let us give an example of a complete lattice L which does not satisfy the conclusion of Theorem 1.1. This example will be useful in building a counter-example in Section 4.2 (see Remark 4.2, (d)). Consider the set $L = (0, 1]\times(0, 1]\cup \{(0, 0), (1, 0)\}$ endowed with the following order: for any (x, y), $(x', y') \in L$,

$$(x,y) \leq (x',y') \iff \begin{array}{l} \text{either } x = x' \text{ and } y \leq y'\ , \\ \text{or } x \leq x' \text{ and } y' = 1\ , \\ \text{or } (x,y) = (1,0) \text{ and } y' = 1\ , \\ \text{or } (x,y) = (0,0)\ . \end{array} \tag{1.4}$$

We claim that L is a complete lattice. Indeed, let G be a subset of L. Denote:

$$\begin{array}{ll} G_\alpha & = \{y \in (0,1] \mid (\alpha, y) \in G\} \quad (\alpha \in (0,1])\ , \\ G^+ & = \{\alpha \in (0,1] \mid G_\alpha \neq \emptyset\}\ . \end{array} \tag{1.5}$$

The point $(x, y) \in L$ is an upper bound of G if, and only if, $(x, y) \geq (\alpha, \sup G_\alpha)$ for all $\alpha \in G^+$ and, when $(1, 0) \in G$, $(x, y) \geq (1, 0)$. Considering the cases where the cardinality of G^+ is 0, 1 or ≥ 2, and either $(1, 0)$ belongs or does not belong to G, we obtain that $\sup G$ exists and is given by:

$$\sup G = \begin{cases} (0,0) & \text{if } G \subseteq \{(0,0)\} \\ (1,0) & \text{if } G \setminus \{(0,0)\} = \{(1,0)\} \\ (\alpha, \sup G_\alpha) & \text{if } G^+ = \{\alpha\} \text{ and } ((1,0) \notin G \text{ or } \alpha = 1) \\ (\sup G^+, 1) & \text{otherwise;} \end{cases} \tag{1.6}$$

hence L is a complete lattice.

Next, we show that

$$a \ll b \text{ in } L \Longleftrightarrow a = (0,0) \text{ or } (a = (1,0) \text{ and } b \in \{1\} \times (0,1]) . \tag{1.7}$$

Assume first, that $a \ll b$ in L. The sets $D = \{1\} \times (0,1)$ and $D' = (0,1) \times \{1\}$ are directed and satisfy $\sup D = \sup D' = (1,1) \geq b$. Hence, by the definition of $\ll$, there exist $d \in D$ and $d' \in D'$ such that $a \leq d$ and $a \leq d'$. By (1.4), this yields $a \leq d \wedge d' = (1,0)$, whence either $a = (0,0)$ or $a = (1,0)$. Consider the case where $a = (1,0)$, and let us show that

$$b \in \{1\} \times (0,1] . \tag{1.8}$$

Since $a \ll b$, we have $a \leq b$, whence, by $a = (1,0)$ and (1.4),

$$b \in (\{1\} \times [0,1]) \cup ((0,1) \times \{1\}) . \tag{1.9}$$

If (1.8) does not hold, then, by (1.9), we have either $b = (1,0)$ or $b = (\alpha,1)$, with $0 < \alpha < 1$. In both cases, taking the directed set $D'' = \{\alpha\} \times (0,1)$ with $0 < \alpha < 1$, we obtain that $b \leq (\alpha,1) \leq \sup D''$, whence $b \leq \sup D''$. Hence, by $(1,0) = a \ll b$ and the definition of $\ll$, there exists $d = (\alpha,y) \in D''$, so $0 < y < 1$, such that $(1,0) \leq (\alpha,y)$, in contradiction with (1.4). Thus, we must have (1.8), which completes the proof of the "$\Rightarrow$" part of (1.7).

Conversely, the bottom element $(0,0)$ of L clearly satisfies $(0,0) \ll b$ for all $b \in L$. Let us show that $(1,0) \ll (1,\beta)$ for all $\beta \in (0,1]$. Let D be a directed subset of L such that $\sup D \geq (1,\beta)$. Then, $\sup D = (1,\gamma)$ for some $\gamma \geq \beta > 0$. By (1.6), this implies that, using the notations (1.5), either (i) $D^+ = \{1\}$ and $\gamma = \sup D_1$ or (ii) $\gamma = 1$, D^+ contains at least two elements and $\sup D^+ = 1$. In case (i), there exists $y \in D_1$, with $y > 0$, hence $(1,y) \in D$ and $(1,y) \geq (1,0)$. In case (ii), there exist (x,y), $(x',y') \in D$ with y, $y' > 0$ and $0 < x < x'$. Since D is directed, there exists $(x'',y'') \in D$ such that $(x'',y'') \geq (x,y) \vee (x',y')$. Since $(x,y) \vee (x',y') = (x',1) \geq (1,0)$, we obtain $(x'',y'') \geq (1,0)$, which completes the proof of $(1,0) \ll (1,\beta)$, and of (1.7).

From (1.7), one deduces that L is not continuous. Moreover, by (1.7), we have $(1,0) \ll (1,1)$, but there is no $c \in L$ such that $(1,0) \ll c \ll (1,1)$.

Proposition 1.2. (extends [9, Corollary I.1.19, p. 47]). *If L is a conditionally complete lattice in which, for all $a, b \in L$, such that $a \ll b$, there exists $c \in L$ such that $a \ll c \ll b$ (in particular, by Theorem 1.1, this holds if L is continuous), then for any $a, b \in L$, we have $a \ll b$ if, and only if, for all upper bounded directed sets D of L, such that $b \leq \sup D$, there exists $d \in D$ such that $a \ll d$.*

Proof. The "if" part is obvious since $a \ll d$ implies $a \leq d$, by Proposition 1.1, 1°. Let us prove the "only if" part. Let $a, b \in L$ such that $a \ll b$. By our assumption, there exists $c \in L$ such that $a \ll c \ll b$. Hence for all upper bounded directed subsets D of L such that $b \leq \sup D$, there exists $d \in D$ such that $c \leq d$. Since $a \ll c$, we get $a \ll d$, by Proposition 1.1, 2°. ∎

Proposition 1.3. (extends [9, Corollary I.2.2, p. 57]). *Every continuous conditionally complete lattice L is meet continuous.*

Proof. Let D be an upper bounded directed subset of L and $x \in L$. The set $x \wedge D$ is directed and upper bounded, and we have $x \wedge \sup D \geq \sup(x \wedge D)$. On the other hand, let $a \ll x \wedge \sup D$. Then, by Proposition 1.1, 2°, we get $a \ll \sup D$. Hence, there exists $d \in D$ such that $a \leq d$. Also, by $a \ll x \wedge \sup D$ and Proposition 1.1, 1°, we have $a \leq x$, whence $a \leq x \wedge d \leq \sup(x \wedge D)$. Taking the supremum over all $a \ll x \wedge \sup D$, we obtain $x \wedge \sup D \leq \sup(x \wedge D)$ (since L is continuous). Therefore, $x \wedge \sup D = \sup(x \wedge D)$ and L is meet continuous. ∎

1.2. **Lattice ordered groups.** In this subsection, we give a short presentation of some definitions, results and examples concerning lattice ordered groups, which we shall need in the sequel. Proofs can be found in [4, Chapter XIII], [3] and [13, 14].

Definition 1.6. A *lattice ordered group* $A = (A, \leq, \otimes)$ is a set A endowed with a partial order $\leq$ such that $(A, \leq)$ is a lattice, with a binary operation $\otimes$ such that $(A, \otimes)$ is a group, and such that all group translations are isotone, that is, for all $a, b, c \in A$, $a \leq b \Rightarrow a \otimes c \leq b \otimes c,\ c \otimes a \leq c \otimes b$. The neutral element for $\otimes$ is denoted by e. The inverse of a in A is denoted by a^{-1}.

Any lattice ordered group is a distributive lattice (see for instance [4, p. 294, Theorem 4]). A lattice ordered group $(A, \leq, \otimes)$ such that $(A, \leq)$ is conditionally complete will be called a *conditionally complete lattice ordered group* or, briefly, a *c.c.l.o.g.* (note that in [4], such a group is called a "complete l-group"). By [4, Chapter XIII, Theorem 25], every conditionally complete lattice ordered group is *infinitely distributive*, which means that it is distributive, meet continuous and join continuous (a lattice L is *join continuous* if L^{op} is meet continuous). By a result of Iwasawa (see for instance [4, Chapter XIII, Theorem 28]), every conditionally complete lattice ordered group is commutative. Except when A is the trivial group (that is, reduced to $\{e\}$), $(A, \leq)$ has no bottom and no top element, hence the minimal completion $\overline{A}$ of A (see Definition 1.2) is the disjoint union of A and $\{\bot, \top\}$. The complete lattice $\overline{A}$ is distributive. Following [13, §1] and [14, §1], we use the following operations on $\overline{A}$:

Definition 1.7. Let $(A, \leq, \otimes)$ be a c.c.l.o.g., and let $(\overline{A}, \leq)$ be the complete lattice given in Definition 1.2. Let $\underset{\cdot}{\otimes}$ and $\dot{\otimes}$ be the following two extensions of $\otimes$ to the set $\overline{A}$:

$$a \underset{\cdot}{\otimes} b = a \dot{\otimes} b = a \otimes b \quad \forall a,\ b \in A$$

$$a \underset{\cdot}{\otimes} \bot = \bot \underset{\cdot}{\otimes} a = \bot \quad \forall a \in \overline{A}$$

$$a \underset{\cdot}{\otimes} \top = \top \underset{\cdot}{\otimes} a = \top \quad \forall a \in A \cup \{\top\}$$

$$a \dot{\otimes} \top = \top \dot{\otimes} a = \top \quad \forall a \in \overline{A}$$

$$a \dot{\otimes} \bot = \bot \dot{\otimes} a = \bot \quad \forall a \in A \cup \{\bot\}.$$

The quadruple $(\overline{A}, \leq, \dot{\otimes}, \underset{\cdot}{\otimes})$ is called the *canonical enlargement* of $(A, \leq, \otimes)$. In $\overline{A}$, we extend the inverse operation by:

$$\top^{-1} = \bot, \quad \bot^{-1} = \top\ .$$

Theorem 1.2. *Let $(\overline{A}, \leq, \dot{\otimes}, \underset{\cdot}{\otimes})$ be the canonical enlargement of a c.c.l.o.g. $(A, \leq, \otimes)$. Let $a, b, c \in \overline{A}$ and $B \subset \overline{A}$. Denoting $B^{-1} = \{b \in \overline{A} \mid b^{-1} \in B\}$, $a \underset{\cdot}{\otimes} B = \{a \underset{\cdot}{\otimes} b \mid b \in B\}$, $a \dot{\otimes} B = \{a \dot{\otimes} b \mid b \in B\}$, we get*

$$\sup(a \underset{\cdot}{\otimes} B) = a \underset{\cdot}{\otimes} \sup B, \quad \inf(a \dot{\otimes} B) = a \dot{\otimes} \inf B \tag{1.10}$$

$$(\sup B)^{-1} = \inf B^{-1}, \quad (\inf B)^{-1} = \sup B^{-1} \tag{1.11}$$

$$(a \dot{\otimes} b)^{-1} = a^{-1} \underset{\cdot}{\otimes} b^{-1}, \quad (a \underset{\cdot}{\otimes} b)^{-1} = a^{-1} \dot{\otimes} b^{-1} \tag{1.12}$$

$$b \leq c \;\Rightarrow\; a \underset{\cdot}{\otimes} b \leq a \underset{\cdot}{\otimes} c,\; a \dot{\otimes} b \leq a \dot{\otimes} c \tag{1.13}$$

$$c \leq a \dot{\otimes} b \;\Leftrightarrow\; c \underset{\cdot}{\otimes} b^{-1} \leq a. \tag{1.14}$$

Since $\overline{A}$ is a complete lattice, so is $\overline{A}^{\mathrm{op}}$. Hence, $(\overline{A}, \vee)$ and $(\overline{A}, \wedge)$ are commutative idempotent monoids with neutral elements $\bot$ and $\top$ respectively (see for instance [4, §I.5 and II.2]). Theorem 1.2 above shows in particular that $(\overline{A}, \vee, \underset{\cdot}{\otimes})$ and $(\overline{A}, \wedge, \dot{\otimes})$ are commutative idempotent semirings. The respective subsemirings $(A \cup \{\bot\}, \vee, \underset{\cdot}{\otimes})$ and $(A \cup \{\top\}, \wedge, \dot{\otimes})$ are idempotent semifields. In the particular case when $A = \mathbb{R}$, endowed with the $+$ group law and the usual order $\leq$, the above semirings $(\overline{A}, \vee, \underset{\cdot}{\otimes})$, $(\overline{A}, \wedge, \dot{\otimes})$, and subsemirings $(A \cup \{\bot\}, \vee, \underset{\cdot}{\otimes})$ and $(A \cup \{\top\}, \wedge, \dot{\otimes})$ become the so called complete max-plus algebra $\overline{\mathbb{R}}_{\max}$, complete min-plus algebra $\overline{\mathbb{R}}_{\min}$ and max-plus algebra $\mathbb{R}_{\max}$, min-plus algebra $\mathbb{R}_{\min}$, respectively. See for instance [4, 3] for the proof of Theorem 1.2 restricted to A, [13, 14] for its generalization to $\overline{A}$, [28] for the idempotent semiring interpretation, and [2] for the theory and applications of idempotent semirings and, in particular, of max-plus algebras and min-plus algebras.

If n is a positive integer, we shall endow the n-th power A^n of A with the laws $\otimes$ and $\leq$ defined componentwise: for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in A^n$, $x \otimes y := (x_1 \otimes y_1, \ldots, x_n \otimes y_n)$, and $x \leq y \Leftrightarrow x_i \leq y_i$ for $i = 1, \ldots, n$. Then, A^n becomes a conditionally complete lattice ordered group, since $(A^n, \leq)$ is a conditionally complete lattice (see Example 1.4), $(A^n, \otimes)$ is a group, and for all nonempty upper bounded subsets B of A^n, we have

$$\sup B = (\sup B_1, \ldots, \sup B_n), \tag{1.15}$$

where $B_i = \{a \in A \mid \exists b \in B, a = b_i\}$. Note also that

$$x^{-1} = (x_i^{-1}) \quad (x = (x_i) \in A^n). \tag{1.16}$$

2. Some relations between $\otimes$, $\ll$, and $\gg$ in conditionally complete lattice ordered groups

In the sequel, we consider a conditionally complete lattice ordered group $A = (A, \leq, \otimes)$. We shall first give some relations between the group operation $\otimes$ and the way below relation $\ll$ (see Definition 1.3), which we shall need in the sequel, but which may be of interest also for other applications.

Lemma 2.1. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. For any $a, b, c \in A$, we have*

$$a \ll b \Rightarrow c \otimes a \ll c \otimes b \ . \tag{2.1}$$

Proof. Assume that $a \ll b$, i.e., that for any upper bounded directed set $D' \subseteq A$ with $b \leq \sup D'$, there exists $d' \in D'$ such that $a \leq d'$, and let D be an upper bounded directed set in A with $c \otimes b \leq \sup D$. Then $D' := c^{-1} \otimes D$ is an upper bounded directed set (by isotony of $\otimes$). Moreover,

$$b = c^{-1} \otimes c \otimes b \leq c^{-1} \otimes \sup D = \sup (c^{-1} \otimes D) = \sup D'$$

(by isotony of $\otimes$ and (1.10)) and hence, by our assumption, there exists $d' \in D'$ such that $a \leq d'$. Since $d' = c^{-1} \otimes d$ for some $d \in D$ and $\otimes$ is isotone, we obtain $c \otimes a \leq c \otimes d' = d$, which proves that $c \otimes a \ll c \otimes b$. ∎

Corollary 2.1. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. For any $a, b \in A$, we have*

$$a \ll b \Leftrightarrow a \otimes b^{-1} \ll e \Leftrightarrow b^{-1} \ll a^{-1} \Leftrightarrow e \ll b \otimes a^{-1} \ . \tag{2.2}$$

Proof. Let $a \ll b$. Then, by Lemma 2.1, we have $a \otimes b^{-1} \ll b \otimes b^{-1} = e$. Again by Lemma 2.1, $a \otimes b^{-1} \ll e$ implies $b^{-1} = a^{-1} \otimes a \otimes b^{-1} \ll a^{-1} \otimes e = a^{-1}$. Also, $b^{-1} \ll a^{-1}$ implies $e = b \otimes b^{-1} \ll b \otimes a^{-1}$. Finally, $e \ll b \otimes a^{-1}$ implies $a = e \otimes a \ll b \otimes a^{-1} \otimes a = b$. ∎

In the sequel we shall often use that, by the isotony of $\otimes$, in any lattice ordered group we have (see e.g. [4, Chapter XIII, Section 2, Lemma 2])

$$a \leq b \Leftrightarrow a^{-1} \geq b^{-1} \ . \tag{2.3}$$

Lemma 2.2. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. For any $a, b \in A$, we have*

$$a \ll b \Leftrightarrow a^{-1} \gg b^{-1} \ . \tag{2.4}$$

Proof. Let $a \ll b$, and let F be a lower bounded filtered subset of A such that $b^{-1} \geq \inf F$. By (2.3), this implies $b = (b^{-1})^{-1} \leq (\inf F)^{-1}$, which, by (1.11), implies $b \leq \sup(F^{-1})$. The set F^{-1} is necessarily upper bounded. Let us verify that it is directed. If $x, y \in F^{-1}$, then x^{-1} and $y^{-1} \in F$. Since F is a filtered set, there exists $z \in F$ such that $z \leq x^{-1}$ and $z \leq y^{-1}$. By (2.3), we get $x \leq z^{-1}$, and $y \leq z^{-1}$, hence $z^{-1} \in F^{-1}$ is an upper bound of x and y, which shows that F^{-1} is a directed subset of A. By the assumption $a \ll b$, and the property $b \leq \sup(F^{-1})$, one deduces that there exists $x \in F^{-1}$ such that $a \leq x$. Again, by (2.3), this yields $a^{-1} \geq x^{-1}$ with $x^{-1} \in F$, which shows (by Definition 1.5) that $a^{-1} \gg b^{-1}$. This proves the implication $\Rightarrow$ in (2.4). Applying this implication to the opposite order $\geq$ and to the inverses of a and b respectively, one obtains the reverse implication. ∎

Theorem 2.1. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. Then, A is continuous if, and only if, it is dually continuous.*

Proof. Assume that $(A, \leq, \otimes)$ is a c.c.l.o.g. and that $(A, \leq)$ is continuous. This means that for all $b \in A$, $b = \sup\{a \in A \mid a \ll b\}$. Let $b \in A$ and $F = \{a \in A \mid a \gg b\}$. Then, $F^{-1} = \{a \in A \mid a^{-1} \gg b\}$, and by Lemma 2.2, $F^{-1} = \{a \in A \mid a \ll b^{-1}\}$. Since A is continuous, $b^{-1} = \sup(F^{-1})$. Hence, by (1.11), $b = (\sup(F^{-1}))^{-1} = ((\inf F)^{-1})^{-1} = \inf F$, which shows that A is dually continuous. This proves the "only if" part of the theorem. Applying this partial result to the opposite order $\geq$, one obtains the "if" part of the theorem. ∎

In Example 1.1 we have seen that in a complete lattice L, in general, $a \ll b \Leftrightarrow b \not\gg a$ (see also [1, Example 1.8]). Now we shall show that in conditionally complete lattice ordered groups the situation is different.

Theorem 2.2. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. For any $a, b \in A$ we have*

$$a \ll b \Leftrightarrow b \gg a \; . \tag{2.5}$$

Proof. Let $a \ll b$. Then, by Corollary 2.1, $b^{-1} \ll a^{-1}$. Hence, by Lemma 2.2, we obtain $b = (b^{-1})^{-1} \gg (a^{-1})^{-1} = a$. This proves the implication $\Rightarrow$ in (2.5). Applying this implication to the opposite order $\geq$ one obtains the reverse implication. ∎

Corollary 2.2. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. For any $a, b \in A$ we have*

$$\sup{(a, b)} \ll e \Leftrightarrow a, b \ll e, \tag{2.6}$$

$$e \ll \inf{(a, b)} \Leftrightarrow e \ll a, b \; . \tag{2.7}$$

Proof. Assume that $\sup{(a, b)} \ll e$. Then $a, b \leq \sup(a, b) \ll e$, whence, by Proposition 1.1, 2°, it follows that $a, b \ll e$. The reverse implication holds by Proposition 1.1, 3°.

Applying (2.6) to the opposite order, we get that $\inf{(a, b)} \gg e$ if, and only if, $a, b \gg e$. By Theorem 2.2, this yields (2.7). ∎

Corollary 2.3. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. Let φ be the coupling function of* (0.35)*. For any $x, w \in A^n$, we have*

$$e \ll \varphi(x, w) \Leftrightarrow w_i^{-1} \ll x_i \; (1 \leq i \leq n) \Leftrightarrow w^{-1} \ll x \text{ in } A^n \; . \tag{2.8}$$

Proof. From (2.7), one can prove, by induction on n, that

$$e \ll \varphi(x, w) = \inf_{1 \leq i \leq n} (x_i \otimes w_i) \Leftrightarrow e \ll x_i \otimes w_i \; (i = 1, ..., n),$$

which is equivalent to $w_i^{-1} \ll x_i$ $(1 \leq i \leq n)$ (by Corollary 2.1). Finally, the last equivalence in (2.8) follows from (1.16) and Example 1.4. ∎

3. Topologies on continuous conditionally complete lattices

We recall here the definitions of the Scott and Lawson topologies introduced in [9] and generalized in [1]. We also introduce the dual Scott, dual Lawson, and bi-Scott topologies, and compare them with the interval and order topologies.

The Scott topology defined below is related to liminf convergence (defined below) and thus to the lower semi-continuity of functions (see Section 4.2 and [9, Theorem II.1.8, p.104]).

Definition 3.1. [1, Definition 1.12]. Let L be a conditionally complete lattice. We say that $U \subset L$ is *Scott open* (open for the *Scott topology* of L) if it satisfies the following two conditions:

1°. U is an upward set.

2°. $\sup D \in U$ implies $D \cap U \neq \emptyset$, for all upper bounded directed sets $D \subset L$.

The Scott topology of L is denoted by $\sigma(L)$, and the topological space $(L, \sigma(L))$ is denoted by ΣL.

We shall call *dual Scott topology* of L the Scott topology of L^{op}, and *bi-Scott topology* of L the common refinement of the Scott and dual Scott topologies of L. The bi-Scott topology of L will be denoted by $\sigma^{\mathrm{b}}(L)$, and the topological space $(L, \sigma^{\mathrm{b}}(L))$ will be denoted by $\Sigma^{\mathrm{b}} L$.

Proposition 3.1. (extends [9, Remark II.1.4.i, p. 100]). *Let L be a conditionally complete lattice. A subset G of L is closed for the Scott topology of L if, and only if, G is a downward set and G is closed under upper bounded directed sups, that is,* $\sup D \in G$ *for all upper bounded directed subsets $D \subset G$.*

Proof. A subset G of L is downward if, and only if, its complement $U = L \setminus G$ is upward. Moreover, a set G is closed under upper bounded directed sups if, and only if, its complement $U = L \setminus G$ satisfies Condition 2° of Definition 3.1. Indeed, if there exists an upper bounded directed set $D \subset L$ with $\sup D \in L \setminus G$, such that $D \cap (L \setminus G) = \emptyset$, then D is an upper bounded directed subset of G, with $\sup D \notin G$; conversely, if there exists an upper bounded directed subset D of G, with $\sup D \notin G$, then $D \cap (L \setminus G) = \emptyset$ and $\sup D \in L \setminus G$. ∎

Corollary 3.1. *Let L be a conditionally complete lattice. Then, all sets*

$$\downarrow a := \{y \in L \mid y \leq a\} \quad (a \in L) \tag{3.1}$$

are Scott closed in L.

Proof. They are downward and closed under arbitrary upper bounded sups. ∎

Proposition 3.2. (extends [9, Proposition II.1.10(i), p. 104]). *Let L be a continuous conditionally complete lattice. Then, the sets*

$$V_a := \{x \in L \mid a \ll x\} \quad (a \in L) \tag{3.2}$$

are a basis of open neighborhoods for the Scott topology of L.

Proof. ¿From Proposition 1.1, 2° and Proposition 1.2, it follows that the sets V_a are Scott open. Let U be a Scott open subset of L and let $x \in U$. The set

$$D_x := \{d \in L \mid d \ll x\} \tag{3.3}$$

is a sup-subsemilattice of L (by Proposition 1.1, 3°), whence a directed set, and satisfies $\sup D_x = x$, since L is continuous. Hence, since $\sup D_x = x \in U$ and U is Scott open, there exists $d_0 \in D_x \cap U$. Thus, d_0 satisfies $x \in V_{d_0}$. Moreover, $d_0 \in U$, and since U is upward, $V_{d_0} \subset U$. ∎

Example 3.1. If L is not continuous, one may be in a situation where the sets V_a of (3.2) are trivial, but the Scott open sets are not. Let us give an example in this direction, which will be useful in building a counter-example in Section 4.2 (see Remark 4.2 (c)). Consider the complete lattice $L = \overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\bot, \top\}$. Then, $V_a = L$ when $a = \bot$ and $V_a = \emptyset$ otherwise (see Example 1.4). We shall prove that a subset F of L is Scott closed if, and only if,

$$\text{either } F = L \text{ or } F = \emptyset \text{ or } F = F^{(h)} := \{\bot\} \cup \operatorname{Hypo} h\ , \tag{3.4}$$

where

$$\text{Hypo } h := \{(x,y) \in \mathbb{R}^2 \mid y \leq h(x)\} , \tag{3.5}$$

and

$$\begin{aligned} &h : \mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{-\infty\},\ x \mapsto h(x) \\ &\text{is nonincreasing and left continuous, with } h(+\infty) = -\infty . \end{aligned} \tag{3.6}$$

Let us prove the "only if" part. Let F be a nonempty Scott closed subset of L. If $\top \in F$, then $F = L$ since, by Proposition 3.1, F is downward. Assume that $\top \notin F$. Then $F \subseteq \{\bot\} \cup \mathbb{R}^2$, and since F is nonempty and downward, $\bot \in F$.

Denote

$$F_x := \{y \in \mathbb{R} \mid (x,y) \in F\} \quad (x \in \mathbb{R}) . \tag{3.7}$$

We have

$$\{x\} \times F_x = F \cap (\{x\} \times \mathbb{R}) \quad (x \in \mathbb{R}) . \tag{3.8}$$

Let $x \in \mathbb{R}$. We claim that

$$F_x = (-\infty, h(x)] , \tag{3.9}$$

for some $h(x) \in \mathbb{R} \cup \{-\infty\}$, with the convention that $(-\infty, -\infty] = \emptyset$. This claim is obvious when F_x is empty (take $h(x) = -\infty$). Assume now that F_x is nonempty. Since $\{x\} \times \mathbb{R}$ is totally ordered, the set $\{x\} \times F_x$ is also totally ordered, hence, by (3.8), it is a directed subset of F. Since F is closed under directed sups (note that L is complete, hence all sets are upper bounded in L), we deduce that $\sup(\{x\} \times F_x) \in F$. Then, $\sup F_x \in \mathbb{R}$, since otherwise $\sup(\{x\} \times F_x) = \top \in F$, in contradiction with our assumption. Denoting $h(x) = \sup F_x$, we thus get that $h(x) \in \mathbb{R}$ and $\sup(\{x\} \times F_x) = (x, h(x)) \in F$, hence $h(x) \in F_x$. Moreover, since F is downward, F_x is downward in $\mathbb{R}$, so we have (3.9), which proves our claim.

This, together with $\bot \in F$, shows that $F = F^{(h)}$. Putting $h(+\infty) = -\infty$, we get that the map $h : \mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{-\infty\},\ x \mapsto h(x)$ is nonincreasing, since F is downward. By symmetry between x and y, we also get that, for all $y \in \mathbb{R}$,

$$F^y := \{x \in \mathbb{R} \mid (x,y) \in F\} = (-\infty, g(y)] \quad \text{ with } g(y) \in \mathbb{R} \cup \{-\infty\} . \tag{3.10}$$

Since h is nonincreasing, $h(x^-) := \lim_{x' \to x,\, x' < x} h(x')$ exists for all $x \in \mathbb{R} \cup \{+\infty\}$ and $h(x^-) \geq h(x)$. If $h(+\infty^-) > h(+\infty) = -\infty$, then $F^{h(+\infty^-)} = \mathbb{R}$, a contradiction with (3.10). Similarly, if $h(x^-) > h(x)$ for some $x \in \mathbb{R}$, then $F^{h(x^-)} = (-\infty, x)$, a contradiction with (3.10). It follows that h is left continuous, hence h satisfies (3.6).

Conversely, let us prove that the set $F^{(h)}$ defined by (3.4)–(3.6), is Scott closed. Firstly, $F^{(h)}$ is downward, since $\bot \in F^{(h)}$ and if $(x,y) \leq (x',y')$ with $(x',y') \in F^{(h)}$, then $y \leq y' \leq h(x') \leq h(x)$, so $(x,y) \in F^{(h)}$. Secondly, let D be a directed subset of $F^{(h)}$. If $D \subseteq \{\bot\}$, then $\sup D = \bot \in F^{(h)}$. Assume now that $D \not\subseteq \{\bot\}$, and consider the set

$$\downarrow D := \{z \in L \mid \exists d \in D,\, z \leq d\}.$$

The set $\downarrow D$ is downward (in fact, it is the smallest downward set containing D), satisfies $\downarrow D \subset F^{(h)}$ (since $F^{(h)}$ is downward), and it is a sup-subsemilattice of L. Indeed, if $z_1, z_2 \in \downarrow D$, then there exist $d_1, d_2 \in D$ with $z_1 \leq d_1, z_2 \leq d_2$; then, since D is directed, there exists $d \in D$ such that $z_1 \vee z_2 \leq d_1 \vee d_2 \leq d$, so $z_1 \vee z_2 \in \downarrow D$. Denote

$$a = \sup\{x \in \mathbb{R} \mid \exists y \in \mathbb{R},\ (x, y) \in \downarrow D\} \in \mathbb{R} \cup \{+\infty\} \ ,$$
$$b = \sup\{y \in \mathbb{R} \mid \exists x \in \mathbb{R},\ (x, y) \in \downarrow D\} \in \mathbb{R} \cup \{+\infty\} \ .$$

We claim that $(-\infty, a) \times (-\infty, b) \subset \downarrow D$. Indeed, if $a' < a$ and $b' < b$, then there exist (x_1, y_1) and $(x_2, y_2) \in \downarrow D$ such that $a' \leq x_1$ and $b' \leq y_2$, whence $(a', b') \leq (x_1, y_2) \leq (x_1, y_1) \vee (x_2, y_2)$. Since $\downarrow D$ is a sup-subsemilattice, $(x_1, y_1) \vee (x_2, y_2) \in \downarrow D$, and since $\downarrow D$ is downward, $(a', b') \in \downarrow D$, which proves our claim. Since $\downarrow D \subset F^{(h)}$, we obtain $(-\infty, a) \times (-\infty, b) \subset F^{(h)}$, that is, $b' \leq h(a')$ for all $a' < a$ and $b' < b$. Passing to the limit when $b' \to b^-$ and $a' \to a^-$, and using the left-continuity of h and $h(+\infty) = -\infty$, we deduce that $a, b \in \mathbb{R}$ and $b = \liminf_{b' \to b^-} b' \leq \liminf_{a' \to a^-} h(a') = h(a)$, whence $(a, b) \in F^{(h)}$. Consequently, since $\sup D \leq \sup \downarrow D \leq (a, b)$ and $F^{(h)}$ is downward, we obtain $\sup D \in F^{(h)}$, which shows that $F^{(h)}$ is closed under directed sups.

Remark 3.1. It is natural to ask whether the Scott topology on a conditionally complete lattice L_0 is equal to the topology induced by the Scott topology of the minimal completion $L = \overline{L_0}$ of L_0. In other words: Is a subset of L_0 Scott closed if, and only if, it is the intersection with L_0 of some Scott closed subset of the minimal completion of L_0 ? The answer is negative. Indeed, observe that if $L_0 = \mathbb{R}^2$, then $L = \overline{L_0}$ is the complete lattice of Example 3.1, and hence the restrictions to L_0 of the Scott closed subsets of $L = \overline{L_0}$ are either $\emptyset$ or L_0 or the sets Hypo h, with $h : \mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{-\infty\}$ of the form (3.6). Now, since L_0 is continuous, by Proposition 3.2 the set

$$M := L_0 \backslash V_{(0,0)} = \{x \in L_0 \mid (0, 0) \not\ll x\}$$

is Scott closed. On the other hand, we have $M = \text{Hypo } h$, where

$$h(y) := \begin{cases} 0 & \text{if } y > 0 \\ +\infty & \text{if } y \leq 0 \, . \end{cases}$$

Thus, h takes the value $+\infty$, so h is not of the form (3.6), and hence, by the above observation, M is not the restriction to L_0 of any Scott closed subset of $L = \overline{L_0}$.

The Scott topology is related to the liminf convergence which is, in some sense, one half of the order convergence. For a net $(x_k)_{k \in K}$ with values in a conditionally complete lattice L (that is, a map $k \in K \mapsto x_k \in L$, where K is a directed set), we define:

$$\liminf_{k \in K} x_k := \sup_{k \in K} \inf_{l \geq k} x_l \ , \tag{3.11}$$

$$\limsup_{k \in K} x_k := \inf_{k \in K} \sup_{l \geq k} x_l \ , \tag{3.12}$$

where infs and sups are taken in the minimal completion $\overline{L}$ of L.

Definition 3.2. Let L be a conditionally complete lattice, and let $(x_k)_{k\in K}$ be a net with values in L. We say that $(x_k)_{k\in K}$ *liminf converges* to $x \in L$ if $x \le \liminf_{k\in K} x_k$. We say that $(x_k)_{k\in K}$ *order converges* to $x \in L$ if $\limsup_{k\in K} x_k = x = \liminf_{k\in K} x_k$.

Remark 3.2. Definition 3.2 of order convergence coincides with that of [4, Ch. X, Section 9] in the case where L is complete and is approximately the same when L is only conditionally complete. Namely, for a general conditionally complete lattice L, $(x_k)_{k\in K}$ order converges to $x \in L$ in our sense if, and only if, there exist a nondecreasing net $(b_k)_{k\in K}$ and a nonincreasing net $(t_k)_{k\in K}$ with values in $\overline{L}$ such that $b_k \le x_k \le t_k$ $(k \in K)$ and $\sup_{k\in K} b_k = x = \inf_{k\in K} t_k$ (for the "only if" part, take $b_k = \inf_{l\ge k} x_l$ and $t_k = \sup_{l\ge k} x_l$). In that case, there exists $k_0 \in K$ such that $b_k \in L$ for $k \ge k_0$ (indeed, otherwise, for each $k_0 \in K$ there would exist $k \ge k_0$ such that $b_k \notin L$, hence, since $b_k \le \sup_{l\in K} b_l = x \in L$, it would follow that $b_k = \perp$ and that $\perp \notin L$, and using that $(b_k)_{k\in K}$ is nondecreasing, this would yield $b_{k_0} = \perp$; whence $b_k = \perp$ for all $k \in K$ and $x = \sup_{k\in K} b_k = \perp$, hence $x = \perp \notin L$ in contradiction with our assumption). Similarly, there exists $k_1 \in K$ such that $t_k \in L$ for $k \ge k_1$. Taking $k_2 \in K$ such that $k_2 \ge k_0$ and $k_2 \ge k_1$, we get that $(x_k)_{k\ge k_2}$ order converges (in L) to x in the sense given in [4, Ch. X, Section 9] for a general partially ordered set L.

In a complete lattice, the topology of the liminf convergence coincides with the Scott topology [9, Lemma 1.2, p. 99]. For a general conditionally complete lattice, we prove the following result:

Lemma 3.1. (extends [9, Lemma 1.2, p. 99]). *Let L be a conditionally complete lattice. A subset U of L is Scott open if, and only if, for all nets $(x_k)_{k\in K}$ liminf converging to some $x \in U$ and such that $\liminf_{k\in K} x_k \in L$, there exists $k \in K$ such that $x_l \in U$ for all $l \ge k$.*

Proof. Assume first that U is Scott open and consider $x \in U$ and a net $(x_k)_{k\in K}$ such that $x \le \liminf_{k\in K} x_k$ and $\liminf_{k\in K} x_k \in L$. By the same arguments as in Remark 3.2, there exists $k_0 \in K$ such that $\inf_{l\ge k} x_l \in L$ for all $k \ge k_0$. Let $D = \{\inf_{l\ge k} x_l \mid k \ge k_0\}$. The set D is included in L, upper bounded by $\liminf_{k\in K} x_k$, directed and such that $x \le \sup D$. Since U is upward (by Definition 3.1) and $x \in U$, we have $\sup D \in U$; therefore, by Definition 3.1, there exists $d \in D \cap U$, that is, there exists $k \ge k_0$ such that $\inf_{l\ge k} x_l \in U$. Hence, again since U is upward, it follows that $x_l \in U$ for all $l \ge k$.

Conversely, assume that U satisfies the condition of the lemma. Let $x \in U$, and let $y \in L$ be such that $y \ge x$. Considering the constant net $x_k = y$ for $k \in K = \{0\}$, which liminf converges to x, we deduce that $y \in U$, hence U is upward. Let D be an upper bounded directed set such that $\sup D \in U$. Taking $K = D$ and the net $x_k = k$ for $k \in K$, we get that $\liminf_{k\in K} x_k = \sup_{k\in K} \inf_{l\ge k} l = \sup D \in U$. Hence, by our assumption, there exists $k \in K$ such that $x_l \in U$ for all $l \ge k$. In particular, $D \cap U \ne \emptyset$, and thus U is Scott open. ∎

Lemma 3.1 shows that if $(x_k)_{k\in K}$ liminf converges to $x \in L$ and $\liminf_{k\in K} x_k \in L$, then x_k converges to x for the Scott topology.

In a complete lattice the liminf convergence is equivalent to the convergence for the Scott topology, that is, the liminf convergence is "topological", if, and only if, L is continuous [9, Theorem 1.8, p. 104]. For a general conditionally complete lattice, we prove the following result:

Proposition 3.3. (extends partially [9, Theorem 1.8, p. 104]). *Let L be a continuous conditionally complete lattice. Then, the convergence for the Scott topology implies the liminf convergence.*

Proof. Let $(x_k)_{k\in K}$ be a net with values in L converging to $x \in L$ for the Scott topology of L. Consider $a \ll x$. Since L is continuous, the set $V_a = \{x \in L \mid a \ll x\}$ is Scott open (by Proposition 3.2). Since $x \in V_a$, there exists $k \in K$ such that $x_l \in V_a$ for all $l \geq k$. Hence, by Proposition 1.1, 1°, $a \leq x_l$ for all $l \geq k$, which implies that $a \leq \inf_{l\geq k} x_l \leq \liminf_{k\in K} x_k$. Therefore, since L is continuous, $x = \sup_{a\ll x} a \leq \liminf_{k\in K} x_k$, that is, $(x_k)_{k\in K}$ liminf converges to x. ∎

Definition 3.3. Let L be a conditionally complete lattice. We say that a set $U \subset L$ is *order open*, if for all nets $(x_k)_{k\in K}$ order converging to some $x \in U$, there exists $k \in K$ such that $x_l \in U$ for all $l \geq k$. The *order topology* is denoted $o(L)$.

Remark 3.3. By the arguments of [11, Ch. 2, pp. 73-75], for a complete lattice L the above definition of the order topology is equivalent to the classical one [4, Ch. 10, Section 9], according to which a set $G \subset L$ is said to be order closed if for all nets $(x_k)_{k\in K} \subset G$ order converging to some $x \in L$, we have $x \in G$.

Corollary 3.2. *Let L be a conditionally complete lattice. Then, $\sigma^b(L) \subseteq o(L)$.*

Proof. Let U be a Scott open subset of L. If the net $(x_k)_{k\in K}$ order converges to $x \in U$, then $x = \liminf_{k\in K} x_k$, whence $\liminf_{k\in K} x_k \in L$, and x_k liminf converges towards $x \in U$. By Lemma 3.1, it follows that there exists $k \in K$ such that $x_l \in U$ for all $l \geq k$. Therefore, U is order open, which shows that $\sigma(L) \subseteq o(L)$. Applying this last result to L^{op}, we obtain that $\sigma(L^{\mathrm{op}}) \subseteq o(L^{\mathrm{op}}) = o(L)$, whence $\sigma^{\mathrm{b}}(L) \subseteq o(L)$. ∎

In general, even for complete lattices, the convergence for the order topology is not equivalent to the order convergence, that is, the order convergence is not "topological" (indeed, this follows e. g. from [4, Ch. X, Section 9] and [11, p. 74]). The following result gives a sufficient condition for this property to hold.

Corollary 3.3. *Let L be a conditionally complete lattice. If L is both continuous and dually continuous, then, the convergence for the bi-Scott topology is equivalent to the order convergence, and $\sigma^b(L) = o(L)$.*

Proof. Let $(x_k)_{k\in K}$ be a net with values in L. If $(x_k)_{k\in K}$ converges to $x \in L$ for the bi-Scott topology, it converges necessarily for the Scott topology and by Proposition 3.3, it liminf converges to x. Similarly, $(x_k)_{k\in K}$ converges to $x \in L$ for the Scott topology of L^{op}, and thus it liminf converges to x in

L^{op}, that is, $\limsup_{k\in K} x_k \leq x$. This shows that x_k order converges to x. Conversely, assume that $(x_k)_{k\in K}$ order converges to $x \in L$, and let U be a bi-Scott open subset of L containing x. Then, there exist a Scott open subset V and a dually Scott open subset W of L such that $x \in V \cap W \subseteq U$. Since $(x_k)_{k\in K}$ order converges to x, we get that $(x_k)_{k\in K}$ liminf converges to x and that $\liminf_{k\in K} x_k \in L$, hence by Lemma 3.1, there exists $k_0 \in K$ such that $x_l \in V$ for all $l \geq k_0$. Similarly, replacing L by L^{op}, we obtain that there exists $k_1 \in K$ such that $x_l \in W$ for all $l \geq k_1$. Taking $k_2 \in K$ such that $k_2 \geq k_0$ and $k_2 \geq k_1$, we obtain that $x_l \in U$ for all $l \geq k_2$. This shows that $(x_k)_{k\in K}$ converges for the bi-Scott topology of L. Hence, the convergence for the bi-Scott topology is equivalent to the order convergence. In view of the definition of the order topology $o(L)$ (Definition 3.2), this implies that $\sigma^{\mathrm{b}}(L) = o(L)$. ∎

The bi-Scott topology is thus well adapted to conditionally complete lattices that are both continuous and dually continuous. In general, the Scott topology is T_0 and the bi-Scott topology is T_1. The Lawson topology, defined below, is well adapted to conditionally complete lattices that are only continuous. It is in general T_1, and it is Hausdorff if L is continuous (see [9, Theorem III.1.10, p. 146] and [1]).

Definition 3.4. [1, Definition 1.14]. Let L be a conditionally complete lattice. The *lower topology* of L, denoted $\omega(L)$, is the topology generated by the sets

$$L \backslash \uparrow a \quad \text{with } \uparrow a := \{x \in L \mid a \leq x\} \quad (a \in L) . \tag{3.13}$$

The *Lawson topology* of L, denoted $\lambda(L)$, is defined as the common refinement of the lower topology $\omega(L)$ and the Scott topology $\sigma(L)$. The topological space $(L, \lambda(L))$ is denoted $\Lambda(L)$.

We shall call *dual Lawson topology* of L the Lawson topology of L^{op} and *bi-Lawson topology* of L, denoted $\lambda^{\mathrm{b}}(L)$, the common refinement of the Lawson and dual Lawson topologies of L.

If we define, similarly, the *dual lower topology* as the lower topology of L^{op}, and the *bi-lower topology*, denoted $\omega^{\mathrm{b}}(L)$, as the common refinement of the lower and dual lower topologies of L, we get that the bi-lower topology is nothing else than the interval topology in the sense of Frink, see e.g. [4, Ch. X, Section 12].

Lemma 3.2. *Let L be a conditionally complete lattice. We have*

$$\omega^b(L) \subseteq \lambda(L) \subseteq \lambda^b(L) = \sigma^b(L) . \tag{3.14}$$

Proof. By Corollary 3.1 and Definition 3.4, $\omega(L^{\mathrm{op}}) \subseteq \sigma(L) \subseteq \lambda(L)$, whence, by Definition 3.4, $\omega^{\mathrm{b}}(L) \subseteq \lambda(L)$. By Corollary 3.1 and Definition 3.4 again, we get that $\omega(L) \subseteq \sigma(L^{\mathrm{op}}) \subseteq \sigma^{\mathrm{b}}(L)$, whence $\lambda(L) \subseteq \sigma^{\mathrm{b}}(L)$. By symmetry, we get $\lambda(L^{\mathrm{op}}) \subseteq \sigma^{\mathrm{b}}(L)$, whence $\lambda^{\mathrm{b}}(L) \subseteq \sigma^{\mathrm{b}}(L)$. Since, by Definition 3.4, $\sigma(L) \subseteq \lambda(L)$ for all conditionally complete lattices L, we deduce $\sigma^{\mathrm{b}}(L) \subseteq \lambda^{\mathrm{b}}(L)$, and hence the equality. ∎

Proposition 3.4. (extends [9, Proposition III.1.6, p. 144]). *Let L be a conditionally complete lattice, and let G be a downward subset of L. The following statements are equivalent:*

1°. *G is closed for the bi-Scott topology of L;*
2°. *G is closed for the Lawson topology of L;*
3°. *G is closed for the Scott topology of L;*
4°. *G is closed under upper bounded directed sups.*

Proof. The equivalence between 3° and 4° is nothing else than Proposition 3.1. The implications 3° $\Rightarrow$ 2° $\Rightarrow$ 1° follow from Definition 3.4 and Lemma 3.2 respectively.

Let us show 1° $\Rightarrow$ 4°. Let G be a downward subset of L such that G is bi-Scott closed. Then, G can be written as $G = \cap_{i\in I}(G_i \cup H_i)$, where I is some index set, G_i are Scott closed and H_i are dual Scott closed. Let D be an upper bounded directed subset of G. Then, for all $i \in I$, $D \subset G_i \cup H_i$. But, any dual Scott closed set is downward for the opposite order, hence is upward. This implies that if $D \cap H_i \neq \emptyset$, then $\sup D \in H_i \subset G_i \cup H_i$. On the other hand, if $D \cap H_i = \emptyset$, then $D \subset G_i$ and since G_i is Scott closed, $\sup D \in G_i \subset G_i \cup H_i$. We have shown that $\sup D \in G$, hence G is closed under upper bounded directed sups. ∎

The following results will allow us to characterize the continuity of functions with multiple arguments. First note that, even when L is a complete lattice, in general $\Sigma(L \times L) \neq \Sigma L \times \Sigma L$ (see the end of p. 106 and Theorem II.4.11, p. 133 in [9]).

Proposition 3.5. (extends [9, Corollary II.4.12, p. 133]). *Let L be a continuous conditionally complete lattice. Then $\Sigma(L^n) = (\Sigma L)^n$, that is, the Scott topology on L^n coincides with the product of the Scott topologies on each set L.*

Proof. Let L be a continuous conditionally complete lattice. By Proposition 3.2, $\sigma(L)$ is generated by the sets $V_a = \{x \in L \mid a \ll x\}$ $(a \in L)$, and hence the product topology $\sigma(L)^n$ is generated by the sets $V_{a_1} \times \cdots \times V_{a_n}$, for $a = (a_1, \ldots, a_n) \in L^n$. On the other hand, L^n is continuous (see Example 1.4), and hence, again by Proposition 3.2, $\sigma(L^n)$ is generated by the sets $V'_a := \{x \in L^n \mid a \ll x\}$ $(a \in L^n)$. Consequently, since $V'_a = V_{a_1} \times \cdots \times V_{a_n}$, for $a = (a_1, \ldots, a_n) \in L^n$ (see Example 1.4), the proposition follows. ∎

Corollary 3.4. *Let L be a conditionally complete lattice which is both continuous and dually continuous. Then $\Sigma^b(L^n) = (\Sigma^b L)^n$, that is, the bi-Scott topology on L^n coincides with the product of the bi-Scott topologies on each set L.*

Proposition 3.6. (extends [9, Proposition III.2.2, p. 153]). *Let L be a continuous conditionally complete lattice. Then $\Lambda(L^n) = (\Lambda L)^n$, that is, the Lawson topology on L^n coincides with the product of the Lawson topologies on each set L.*

Proof. It is easy to show that the product of the lower topologies on L coincides with the lower topology on L^n (for both topologies the set of closed sets is generated by the sets $\uparrow a = \uparrow a_1 \times \cdots \times \uparrow a_n$ with $a = (a_1, \ldots, a_n) \in L^n$). The result thus follows by using Proposition 3.5. ∎

4. Functions with values in conditionally complete lattices and lattice ordered groups

4.1. **Continuity.**

Proposition 4.1. (extends [9, Proposition II.2.1, p. 112]). *Let M and L be conditionally complete lattices. A function $f : M \to L$ is continuous with respect to the Scott topologies of M and L if, and only if, f preserves upper bounded directed sups, that is,*

$$f(\sup D) = \sup f(D) \tag{4.1}$$

for all upper bounded directed subsets D of M. In particular, every Scott continuous function is isotone.

Proof. Assume first that f is Scott continuous, that is, for all Scott closed subsets F of L, the set $f^{-1}(F) = \{x \in M \mid f(x) \in F\}$ is Scott closed. In particular, since by Corollary 3.1 all sets $\downarrow a$ $(a \in L)$ of (3.1) are Scott closed in L, the lower level sets

$$S_a(f) := f^{-1}(\downarrow a) = \{x \in M \mid f(x) \leq a\} \quad (a \in L) \tag{4.2}$$

are Scott closed in M. Taking $a = f(y)$, where $y \in M$, we get that $S_{f(y)}(f) = \{x \in M \mid f(x) \leq f(y)\}$ is Scott closed. Therefore, $S_{f(y)}(f)$ is downward, hence f is isotone.

Let D be an upper bounded directed subset of M. Since f is isotone, we have $\sup f(D) \leq f(\sup D)$. On the other hand, taking $a = \sup f(D)$ in (4.2), we obtain that $G := \{x \in M \mid f(x) \leq \sup f(D)\}$ is Scott closed. Hence, since $D \subseteq G$ and, by Proposition 3.1, G is closed under upper bounded directed sups, we obtain $\sup D \in G$, that is, $f(\sup D) \leq \sup f(D)$, which yields $f(\sup D) = \sup f(D)$.

Conversely, assume now that f preserves upper bounded directed sups. Taking $D = \{a, b\}$, where $a \leq b \in L$, we deduce that f is isotone. Let F be a Scott closed subset of L, and denote by $G = f^{-1}(F) = \{x \in M \mid f(x) \in F\}$ its preimage. The set G is downward, since F is downward and f is isotone. If $D \subseteq G$ is an upper bounded directed subset of G, then the set $f(D)$ is directed (since f is isotone), upper bounded (by $f(\sup D)$) and contained in F. Therefore, since F is Scott closed, by Proposition 3.1 we obtain $\sup f(D) \in F$, whence $f(\sup D) = \sup f(D) \in F$, which yields $\sup D \in G$. This shows, by Proposition 3.1, that $G = f^{-1}(F)$ is Scott closed, so f is Scott continuous. ∎

Corollary 4.1. *Let M and L be conditionally complete lattices, and $f : M \to L$ a function. Assume that f preserves upper bounded directed sups and lower bounded filtered infs. Then, f is continuous with respect to the bi-Scott topologies of M and L.*

Proposition 4.2. (extends [9, Theorem III.1.8, p. 145]). *Let M and L be conditionally complete lattices, and $f : M \to L$ a function. Assume that f preserves upper bounded directed sups and arbitrary lower bounded nonempty infs, and that for all $B \subseteq M$, the set $f(B)$ is lower bounded (if and) only if B is lower bounded. Then, f is continuous with respect to the Lawson topologies of M and L.*

Proof. Let $f : M \to L$ be as in the proposition. Then, f preserves upper bounded directed sups, whence, by Proposition 4.1, f is Scott continuous, and thus isotone. Let us prove that f is also continuous for the lower topologies of M and L, which will yield the Lawson continuity of f. Consider the set $F :=\uparrow a$ of (3.13), where $a \in L$, and denote by $G = f^{-1}(F) = \{x \in M \mid a \leq f(x)\}$ its preimage. Assume that G is nonempty. Since $f(G) \subseteq F$, $f(G)$ is lower bounded by a. Hence, by our assumption on f, G is lower bounded, that is, $\inf G$ exists in M. Since f preserves arbitrary lower bounded nonempty infs, we get $f(\inf G) = \inf f(G) \geq a$, that is, $\inf G \in G$. Since f is isotone, this implies that $G =\uparrow \inf G$; indeed, we have $G \subseteq\uparrow \inf G$ for all subsets G of M, and $x \in\uparrow \inf G$ implies $f(x) \geq f(\inf G) \geq a$, whence $x \in G$. Thus, G is closed for the lower topology of M, and f is continuous for the lower topologies of M and L. ∎

Lemma 4.1. (extends [9, Lemma II.2.9, p.116]). *Let R, S, L be conditionally complete lattices, let T be the product lattice $R \times S$, and consider a function $f : T \to L$. Then, f is Scott continuous if, and only if, the functions $g_s : R \to L$, $r \mapsto f(r,s)$ and $h_r : S \to L$, $s \mapsto f(r,s)$ are Scott continuous for all $s \in S$ and $r \in R$.*

Proof. The "only if" part is obvious since the functions $r \in R \mapsto (r,s) \in T$ and $s \in S \mapsto (r,s) \in T$ are Scott continuous (by Proposition 4.1). Let us prove the "if" part. Assume that all partial functions g_s and h_r are Scott continuous. Then, they are all isotone, hence f is also isotone. Let D be an upper bounded directed subset of T. Then, the sets

$$D_1 := \{r \in R \mid \exists s \in S, (r,s) \in D\}\,, \; D_2 := \{s \in S \mid \exists r \in R, (r,s) \in D\} \tag{4.3}$$

are upper bounded and directed in R and S respectively. We have

$$\sup D = \sup_{(r,s) \in D} (r,s) = (\sup D_1, \sup D_2),$$

and

$$\begin{aligned} f(\sup D) = f(\sup D_1, \sup D_2) &= g_{\sup D_2}(\sup D_1) = \sup_{r \in D_1} g_{\sup D_2}(r) \\ = \sup_{r \in D_1} f(r, \sup D_2) &= \sup_{r \in D_1} h_r(\sup D_2) = \sup_{r \in D_1} \sup h_r(D_2) \\ &= \sup_{r \in D_1} \sup_{s \in D_2} f(r,s), \end{aligned} \tag{4.4}$$

since, by Proposition 4.1, the Scott continuous functions $g_{\sup D_2}$ and h_r ($r \in D_1$) preserve upper bounded directed sups. But, by (4.3), for all $r \in D_1$ and $s \in D_2$ there exist $s' \in S$ and $r' \in R$ such that $(r,s') \in D$ and $(r',s) \in D$. Then, since D is directed, there exists $(r'',s'') \in D$ such that $(r,s') \leq (r'',s'')$ (hence $r \leq r''$) and $(r',s) \leq (r'',s'')$ (hence $s \leq s''$). Since $(r,s) \leq (r'',s'')$ and f is isotone, we obtain $f(r,s) \leq f(r'',s'') \leq \sup f(D)$, whence, using (4.4), it follows that $f(\sup D) \leq \sup f(D)$. The reverse inequality is also true, for any isotone function f. Hence, f preserves upper bounded directed sups, and therefore, by Proposition 4.1, f is Scott continuous. ∎

By induction, one deduces:

Corollary 4.2. *Let $T_1, \ldots, T_n$, L be conditionally complete lattices, T the conditionally complete product lattice $T_1 \times \cdots \times T_n$ and $f : T \to L$. Then, f is Scott continuous if, and only if, for all $i = 1, \ldots, n$ and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in T_1 \times \cdots \times T_{i-1} \times T_{i+1} \times \cdots \times T_n$, the function*

$$f_i : x_i \in T_i \mapsto f(x_1, \ldots, x_n) \in L \tag{4.5}$$

is Scott continuous.

Corollary 4.3. *Let L be a conditionally complete lattice. If L is meet continous, then the mapping*

$$i_n : L^n \to L,\ x \mapsto x_1 \wedge \cdots \wedge x_n \tag{4.6}$$

is continuous from $\Sigma(L^n)$ to ΣL, from $\Sigma^b(L^n)$ to $\Sigma^b L$, and from $\Lambda(L^n)$ to ΛL. If L is continuous, then i_n is continuous from $(\Sigma L)^n$ to ΣL and from $(\Lambda L)^n$ to ΛL. If L is also dually continuous, then i_n is continuous from $(\Sigma^b L)^n$ to $\Sigma^b L$.

Proof. By Corollary 4.2, i_n is Scott continuous from L^n to L (if and) only if for all $i = 1, \ldots, n$ and for all $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in L^{n-1}$, the function $f_i : L \to L,\ x \mapsto x \wedge y_i$ with $y_i = x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_n \in L$, is Scott continuous. By Proposition 4.1, f_i is Scott continuous if, and only if, it preserves upper bounded directed sups, that is, $\sup D \wedge y_i = \sup(D \wedge y_i)$. This property holds if L is meet continuous, which yields the Scott continuity of i_n. Furthermore, i_n clearly preserves arbitrary lower bounded infs, and for any $B \subseteq L^n$, $i_n(B)$ is lower bounded if, and only if, B is lower bounded. Hence, by Corollary 4.1 and Proposition 4.2, i_n is also bi-Scott and Lawson continuous. Finally, the last assertions follow from Propositions 3.5, 3.6, 1.3 and Corollary 3.4. ∎

Next, we prove the Scott, Lawson or bi-Scott continuity of some mappings built with the binary operations $\otimes$, $\wedge$ and $\vee$ of a conditionally complete lattice ordered group $A = (A, \leq, \otimes)$. Also, we prove that for a continuous c.c.l.o.g. A the bi-Scott topology and the order topology on A are equal and that, under this topology, $(A, \otimes)$ is a topological group and $(A, \leq)$ is a topological lattice.

Lemma 4.2. *Let A be a c.c.l.o.g. For any $c = (c_1, \ldots, c_n) \in A^n$, the mapping*

$$\pi_c : A^n \to A^n,\ x = (x_1, \ldots, x_n) \mapsto c \otimes x = (c_1 \otimes x_1, \ldots, c_n \otimes x_n) \tag{4.7}$$

is continuous for the Scott, the bi-Scott, or the Lawson topology of A^n, that is, from $\Sigma(A^n)$ to $\Sigma(A^n)$, from $\Sigma^b(A^n)$ to $\Sigma^b(A^n)$, and from $\Lambda(A^n)$ to $\Lambda(A^n)$. If A is continuous, then π_c is continuous for the Scott, the bi-Scott, or the Lawson topology of A (with the product topology on A^n).

Proof. Let us first consider the case when $n = 1$, that is, let us prove that for any $c \in A$, the mapping

$$\pi_c : A \to A,\ x \mapsto c \otimes x \tag{4.8}$$

is Scott, bi-Scott and Lawson continuous. By Proposition 4.1, π_c is Scott continuous if, and only if, $\pi_c(\sup D) = \sup \pi_c(D)$ for all upper bounded directed subsets D of A, which follows clearly from $\pi_c(x) = c \underset{\cdot}{\otimes} x = c \dot{\otimes} x$ (c, $x \in A$) and (1.10). Furthermore, by (1.10), π_c preserves arbitrary lower bounded nonempty infs. Hence, by Corollary 4.1, π_c is bi-Scott continuous. Also, if $B \subseteq A$ and $\pi_c(B)$ is lower bounded, say, $\pi_c(x) = c \otimes x \geq a$ $(x \in B)$, then $x = c^{-1} \otimes c \otimes x \geq c^{-1} \otimes a$ $(x \in B)$, so B is lower bounded. Hence, by Proposition 4.2, π_c is Lawson continuous.

Now let n be arbitrary. Since the set A^n endowed with the componentwise order and the componentwise $\otimes$ operation is a c.c.l.o.g. (see the end of Section 1.2), by the first case of this proof, applied to A^n, π_c is continuous from $\Sigma(A^n)$ to $\Sigma(A^n)$, from $\Sigma^{\mathrm{b}}(A^n)$ to $\Sigma^{\mathrm{b}}(A^n)$, and from $\Lambda(A^n)$ to $\Lambda(A^n)$. The last assertion follows from Propositions 3.5 and 3.6, and Corollary 3.4 together with Theorem 2.1. ∎

Corollary 4.4. *Let A be a c.c.l.o.g. The functions $w_\lambda : A^n \to A^n$ and $w_x : A \to A^n$ defined by*

$$w_\lambda(x) := \lambda \otimes x \quad (x \in A^n, \lambda \in A) , \tag{4.9}$$

$$w_x(\lambda) := \lambda \otimes x \quad (x \in A^n, \lambda \in A) , \tag{4.10}$$

are continuous for the Scott, the bi-Scott, or the Lawson topology of A^n and A. If A is continuous, then w_λ and w_x are continuous for the Scott, the bi-Scott, or the Lawson topology of A (with the product topology on A^n).

Proof. The statement for the mapping w_λ of (4.9) follows as a particular case of Lemma 4.2, since w_λ is nothing else than π_c with $c = (\lambda, \dots, \lambda) \in A^n$.

The mapping w_x of (4.10) is the composition $w_x = \pi_x \circ i^n$ of the mapping

$$i^n : A \to A^n,\ \lambda \mapsto (\lambda, \dots, \lambda) \tag{4.11}$$

with the mapping $\pi_x : A^n \to A^n$ of (4.7) (mutatis mutandis). The mapping i^n is trivially continuous from A endowed with some topology to A^n endowed with the product topology. Since i^n clearly preserves arbitrary upper bounded sups and lower bounded infs and $i^n(B)$ is lower bounded if, and only if, B is lower bounded, i^n is also Scott, bi-Scott, or Lawson continuous from A to the product lattice A^n, by Propositions 4.1, 4.2 and Corollary 4.1. Hence, the statement for w_x follows from this property and Lemma 4.2. The last assertion follows from Propositions 3.5 and 3.6, and Corollary 3.4 together with Theorem 2.1. ∎

Corollary 4.5. *Let A be a c.c.l.o.g. Then, the mapping*

$$\pi : A^2 \to A,\ (x, y) \mapsto x \otimes y \tag{4.12}$$

is continuous for the Scott or the bi-Scott topology of A^2 and A. If A is continuous, then π is continuous from $(\Sigma A)^2$ to ΣA and from $(\Sigma^b A)^2$ to $\Sigma^b A$.

Proof. Applying Lemma 4.2 for $n = 1$, we get that the maps $\pi_x : y \in A \mapsto \pi(x, y) := x \otimes y$ $(x \in A)$ are Scott continuous, and hence, by Lemma 4.1, π is Scott continuous. Applying this result to the opposite order, that is, to the c.c.l.o.g. $(A, \geq, \otimes)$, we get that π is dual Scott continuous, and hence it is bi-Scott continuous. The last assertion follows from Proposition 3.5 and Corollary 3.4 together with Theorem 2.1. ∎

Corollary 4.6. *Let A be a c.c.l.o.g. Then, for each $w \in A^n$, the function $w_\varphi : A^n \to A$ defined by*

$$w_\varphi(x) = \varphi(x,w) = \inf_{1\leq i\leq n} (x_i \otimes w_i) \quad (x \in A^n)\ , \tag{4.13}$$

is continuous for the Scott, the bi-Scott, or the Lawson topology of A^n and A. If A is continuous, then w_φ is continuous for the Scott, the bi-Scott, or the Lawson topology of A (with the product topology on A^n).

Proof. The first statement follows from Corollary 4.3 and Lemma 4.2, since w_φ is the composition $w_\varphi = i_n \circ \pi_w$ of the mapping $\pi_w : A^n \to A^n$ of (4.7) and the mapping $i_n : A^n \to A$ of (4.6), and since A is meet continuous, by [4, Chapter XIII, Theorem 25]. The last assertion follows from Propositions 3.5 and 3.6, and Corollary 3.4 together with Theorem 2.1. ∎

Lemma 4.3. *Let A be a c.c.l.o.g. The mapping*

$$o : A \to A,\ x \mapsto x^{-1}\ , \tag{4.14}$$

(respectively, the mapping

$$\overline{o} : \overline{A} \to \overline{A},\ x \mapsto x^{-1})\ , \tag{4.15}$$

is Scott and Lawson bicontinuous from A to A^{op} (resp. $\overline{A}$ to $\overline{A}^{op}$), that is, from A (resp. $\overline{A}$) endowed with the Scott (resp. Lawson) topology to A (resp. $\overline{A}$) endowed with the dual Scott (resp. dual Lawson) topology.

Proof. By Propositions 4.1 and 4.2, the continuity of o (resp. $\bar{o}$) from A to A^{op} (resp. $\overline{A}$ to $\overline{A}^{\mathrm{op}}$) follows from (1.11). Replacing A by A^{op} (resp. $\overline{A}$ by $\overline{A}^{\mathrm{op}}$), we deduce the continuity of $o^{-1} = o$ (resp. $\bar{o}^{-1} = \bar{o}$) from A^{op} to A (resp. $\overline{A}^{\mathrm{op}}$ to $\overline{A}$), hence the bicontinuity of o and $\bar{o}$. ∎

Theorem 4.1. *Let $A = (A, \leq, \otimes)$ be a c.c.l.o.g. Assume that A is continuous. Then, the bi-Scott topology and the order topology on A are equal. Moreover, under this topology, $(A, \otimes)$ is a topological group and $(A, \leq)$ is a topological lattice.*

Proof. Since A is continuous, A is also dually continuous, by Theorem 2.1. Then, by Corollary 3.3, $\sigma^{\mathrm{b}}(A) = o(A)$. Moreover, by Corollary 3.4, $\Sigma^{\mathrm{b}}(A^2) = (\Sigma^{\mathrm{b}} A)^2$.

By Lemma 4.3, the mapping $o : A \to A,\ x \mapsto x^{-1}$ is bicontinuous from ΣA to ΣA^{op} and, since it is equal to its inverse, it is continuous from ΣA^{op} to ΣA, hence o is continuous from $\Sigma^{\mathrm{b}} A$ to itself. Moreover, by Corollary 4.5, $\pi : (x,y) \in A^2 \mapsto x \otimes y \in A$ is continuous from $(\Sigma^{\mathrm{b}} A)^2$ to $\Sigma^{\mathrm{b}} A$. This shows that $(A, \otimes)$ is a topological group.

By Corollary 4.3, the map $i_2 : (x,y) \in A^2 \mapsto x \wedge y \in A$ is continuous from $(\Sigma^{\mathrm{b}} A)^2$ to $\Sigma^{\mathrm{b}} A$. Applying this property to the opposite order $\geq$, we obtain that $s_2 : (x,y) \in A^2 \mapsto x \vee y \in A$ is continuous from $(\Sigma^{\mathrm{b}} A)^2$ to $\Sigma^{\mathrm{b}} A$. This shows that $(A, \leq)$ is a topological lattice.

Since by Corollary 3.3, the convergence for the bi-Scott (or the order) topology is equivalent to the order convergence, the continuity of π, i_2 and s_2 from $(\Sigma^{\mathrm{b}} A)^2$ to $\Sigma^{\mathrm{b}} A$ may also be deduced from [4, Chapter XIII, Theorem 26], which states that for a general c.c.l.o.g. A, the maps π, i_2 and s_2 are continuous for the order convergence, in the sense that if $(x_k)_{k\in K}$ and $(y_\ell)_{\ell\in L}$ are nets with values in A, which order converge towards x and y respectively, then $\pi(x_k, y_\ell)$ (resp. $i_2(x_k, y_\ell)$, resp. $s_2(x_k, y_\ell)$) order converges towards $\pi(x,y)$ (resp. $i_2(x,y)$, resp. $s_2(x,y)$). ∎

4.2. **Semi-continuity.** First, we generalize the notion of lower semi-continuity as in [9] and [1].

Definition 4.1. Let X be a topological space and denote by $\mathcal{U}$ the set of its open subsets. Let L be a conditionally complete lattice and $\overline{L}$ be its minimal completion. A function $f : X \to L$ is said to be *lower semi-continuous*, or, briefly, *l.s.c.* if

$$f(x) \leq \liminf_{y\to x} f(y) := \sup_{U\in\mathcal{U},\, U\ni x} \inf_{y\in U} f(y) \quad \forall x \in X \ ,$$

where infs and sups are taken in $\overline{L}$. Dually, a function $f : X \to L$ is said to be *upper semi-continuous*, or, briefly, *u.s.c.* if

$$f(x) \geq \limsup_{y\to x} f(y) := \inf_{U\in\mathcal{U},\, U\ni x} \sup_{y\in U} f(y) \quad \forall x \in X \ .$$

Remark 4.1. For various other notions of semi-continuity of functions defined on a topological space, with values in a partially ordered space, see e.g. [16, 30, 18, 19, 5].

Theorem 4.2. *Let X be a topological space, L a conditionally complete lattice, and $f : X \to L$ a function. Consider the following statements:*

1°. *The function f is l.s.c.*
2°. *The function f is continuous for the Scott topology of L.*
3°. *The lower level sets*

$$S_a(f) := \{x \in X \mid f(x) \leq a\} \tag{4.16}$$

are closed for all $a \in L$.
4°. *The strict upper level sets*

$$U_a(f) := \{x \in X \mid a \ll f(x)\} \tag{4.17}$$

are open for all $a \in L$.

In general, we have $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ$.

If for all $a, b \in L$, such that $a \ll b$, there exists $c \in L$ such that $a \ll c \ll b$, then $1^\circ \Rightarrow 4^\circ$.

When L is continuous, we have $1^\circ \Leftrightarrow 2^\circ \Leftrightarrow 4^\circ$.

Proof. Assume that L is an arbitrary conditionally complete lattice, and let us prove the implications $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ$ (no such results are stated in [9], even in the case where L is complete).

$1^\circ \Rightarrow 2^\circ$: Let $\mathcal{U}$ denote the topology of X, and let V be a Scott open subset of L. We shall prove that if $f : X \to L$ is l.s.c., then $U = f^{-1}(V) := \{x \in$

$X \mid f(x) \in V\}$ is open in X. Let $x \in U$. Since f is l.s.c., we obtain $f(x) \leq \sup_{W\in\mathcal{U},\, W\ni x} \inf_{y\in W} f(y) \leq f(x)$, so

$$f(x) = \sup_{W\in\mathcal{U},\, W\ni x} \inf_{y\in W} f(y) \ . \tag{4.18}$$

Let

$$D := \{ \inf_{y\in W} f(y) \mid W \in \mathcal{U},\, x \in W,\, \inf_{y\in W} f(y) \in L\} \ . \tag{4.19}$$

The set D is non-empty, since $f(x) \in L$ (indeed, if $\inf_{y\in W} f(y) = \perp \notin L$ for all $W \in \mathcal{U}$ with $x \in W$, then, by (4.18), $f(x) = \perp \notin L$, a contradiction). The set D is upper bounded (by $f(x)$), directed, and it satisfies $\sup D = f(x) \in V$. By Definition 3.1, this implies $D \cap V \neq \emptyset$, hence there exists $W \in \mathcal{U}$ such that $x \in W$ and $\inf_{y\in W} f(y) \in V$. Since V is upward (by Definition 3.1), this shows that $f(y) \in V$ for all $y \in W$, hence $W \subseteq U$ and U is open.

$2^\circ \Rightarrow 3^\circ$: This is obvious, since by Corollary 3.1, all sets $\downarrow a = \{y \in L \mid y \leq a\}$ are Scott closed in L, and $S_a(f) = f^{-1}(\downarrow a) = \{x \in X \mid f(x) \in \downarrow a\}$.

Assume now that for all $a, b \in L$, such that $a \ll b$, there exists $c \in L$ such that $a \ll c \ll b$ and let us prove $1^\circ \Rightarrow 4^\circ$. Let $f : X \to L$ be l.s.c., let $x \in U_a(f)$, that is, $a \ll f(x)$, and consider the set D defined in (4.19). We get, as in the above proof of the implication $1^\circ \Rightarrow 2^\circ$, that D is an upper bounded directed set and that it satisfies $f(x) = \sup D$. Hence, by Proposition 1.2, there exists $d \in D$ such that $a \ll d$. By definition of D, there exists an open neighborhood W of x such that $d = \inf_{y\in W} f(y)$. Therefore $d \leq f(y)$ for all $y \in W$, and since $a \ll d$, we get $a \ll f(y)$ for all $y \in W$, that is, $W \subset U_a(f)$, which shows that $U_a(f)$ is open.

If L is a continuous conditionally complete lattice, then, from Proposition 3.2 one deduces the equivalence $2^\circ \Leftrightarrow 4^\circ$, since $U_a(f) = \{x \in X \mid f(x) \in V_a\} = f^{-1}(V_a)$, where $V_a = \{y \in L \mid a \ll y\}$.

Finally, let us prove the implication $4^\circ \Rightarrow 1^\circ$ when L is a continuous conditionally complete lattice. Assume that the sets $U_a(f)$ are open in X. Let $x_0 \in X$ and $a \in L$ be such that $a \ll f(x_0)$. Since $U_a(f)$ is open, $U_a(f)$ contains x_0 and $U_a(f) \subseteq \{x \in X \mid a \leq f(x)\}$ (by Proposition 1.1, 1°), we obtain $\liminf_{x\to x_0} f(x) \geq \inf_{x\in U_a(f)} f(x) \geq a$ (by Definition 4.1). Taking the supremum over $a \ll f(x_0)$, we get $\liminf_{x\to x_0} f(x) \geq f(x_0)$, since L is continuous. ∎

Remark 4.2. (a) Since in a continuous complete lattice, the Scott topology is equivalent to the topology of the liminf convergence (see Section 3), it is natural to obtain in that case, the equivalence $1^\circ \Leftrightarrow 2^\circ$ in Theorem 4.2.

(b) In general, in contrast with the case where $L = \mathbb{R}$, $3^\circ \not\Rightarrow 2^\circ$ (and thus $3^\circ \not\Rightarrow 1^\circ$) in Theorem 4.2, even when L is continuous. Take for instance $X = \mathbb{R}^2$ endowed with the usual topology (which is indeed equivalent to the Lawson or the bi-Scott topology) and $L = \mathbb{R}^2$, which is a continuous conditionally complete lattice (one can also replace L by $\overline{\mathbb{R}}^2$ which is a continuous complete lattice). Consider $f : X \to L$ defined by:

$$f(x) = \begin{cases} (0, -1/x_1) & \text{if } x_1 < 0 \\ (1, 0) & \text{otherwise.} \end{cases} \tag{4.20}$$

The function f is not l.s.c., since its first projection $x \mapsto (f(x))_1 \in \mathbb{R}$ is not l.s.c. (hence f is not continuous for the Scott topology of L, since L is continuous). However, all lower level sets $S_a(f)$ of f are closed since $S_a(f) = S_a^1 \cup S_a^2$, with cases

$$S_a^1 = \begin{cases} (-\infty, -1/a_2] \times \mathbb{R} & \text{if } a_1 \geq 0, a_2 > 0 \\ \emptyset & \text{otherwise,} \end{cases} \tag{4.21}$$

$$S_a^2 = \begin{cases} [0, +\infty) \times \mathbb{R} & \text{if } a \geq (1, 0) \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.22}$$

(c) When L is not continuous, $2° \not\Rightarrow 1°$ in Theorem 4.2 (this is natural, since by [9, Theorem II.1.8, p. 104], when L is complete, the Scott topology of L is equivalent to the topology of the liminf convergence if, and only if, L is continuous). Indeed, consider the function f defined in (4.20) as a function from $X = \mathbb{R}^2$ endowed with the usual topology, to $L = \overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\bot, \top\}$. Then, again (as in (b) above) f is not l.s.c. However, we shall show that f is continuous for the Scott topology of L. Indeed, using the characterization (3.4)–(3.6) of a Scott closed subset of L, we deduce that a function $f = (f_1, f_2) : X \to L$ with values in $\mathbb{R}^2$ is continuous for the Scott topology of L if, and only if, the sets

$$\begin{aligned} G^{(h)} := f^{-1}(F^{(h)}) &= \{x \in X \mid (f_1(x), f_2(x)) \in F^{(h)}\} \\ &= \{x \in X \mid f_2(x) \leq h(f_1(x))\} \end{aligned}$$

are closed for all functions h satisfying (3.6). For the function f of (4.20), one has $G^{(h)} = G_1^{(h)} \cup G_2^{(h)}$, where

$$G_1^{(h)} = \begin{cases} (-\infty, -\frac{1}{h(0)}] \times \mathbb{R} & \text{if } h(0) > 0 \\ \emptyset & \text{otherwise,} \end{cases} \tag{4.23}$$

$$G_2^{(h)} = \begin{cases} [0, +\infty) \times \mathbb{R} & \text{if } 0 \leq h(1) \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.24}$$

Hence, $G^{(h)}$ is closed for all h and f is continuous from X to $\Sigma(\overline{\mathbb{R}^2})$.

(d) One can also show that when L is not continuous, there are no implications between 1°–3° and 4° of Theorem 4.2. Indeed, $4° \not\Rightarrow 3°$ (and thus 1° or 2°), since for instance in $L = \overline{\mathbb{R}^2}$, one has $a \ll b \Leftrightarrow a = \bot$ (see Example 1.4), so any map $f : X \to L$ satisfies 4°, but it is easy to find a map f which does not satisfy 3°.

Conversely, one can construct a topological space X, a complete lattice L, and a l.s.c. function $f : X \to L$ for which some of the strict upper level sets (4.17) are not open, whence 1° (and thus 2° or 3°) $\not\Rightarrow$ 4°. Consider, for instance, $X = (0, 1]$ with its usual topology, the complete lattice $(L, \leq)$ of Example 1.5, and the map:

$$f : (0, 1] \to L, \ x \mapsto (x, 1) \ .$$

We get that f is isotone, l.s.c. and u.s.c. for the usual topology of $X := (0, 1]$ (hence the Lawson or the bi-Scott topology of X), but, by (1.7), the strict upper level set of f:

$$U_{(1,0)}(f) = \{x \in (0, 1] \mid (1, 0) \ll f(x) = (x, 1)\} = \{1\}$$

is not open in $X = (0, 1]$.

The following result shows that if X is the topological space ΣM, where M is a conditionally complete lattice, the behavior of Remarks 4.2 (b) and (c) no longer occurs. However, the examples of Remark 4.2 (d) show that if L is not continuous, then there are no implications between 1° –6° and 8° (or 9°) of Theorem 4.3, even when M is continuous.

Theorem 4.3. *Let L and M be conditionally complete lattices, $f : M \to L$ a function, and consider the following statements:*

1°. *The function f is l.s.c. for the Scott topology of M.*
2°. *The function f is isotone and l.s.c. for the Lawson topology of M.*
3°. *The function f is isotone and l.s.c. for the bi-Scott topology of M.*
4°. *The function f is Scott continuous, that is, continuous for the Scott topologies of M and L.*
5°. *The lower level sets $S_a(f)$ of (4.16) are Scott closed in M for all $a \in L$.*
6°. *The function f is isotone and the lower level sets $S_a(f)$ of (4.16) are Lawson closed in M for all $a \in L$.*
7°. *The function f is isotone and the lower level sets $S_a(f)$ of (4.16) are bi-Scott closed in M for all $a \in L$.*
8°. *The strict upper level sets $U_a(f)$ of (4.17) are Scott open in M for all $a \in L$.*
9°. *The function f is isotone and the strict upper level sets $U_a(f)$ of (4.17) are Lawson open in M for all $a \in L$.*
10°. *The function f is isotone and the strict upper level sets $U_a(f)$ of (4.17) are bi-Scott open in M for all $a \in L$.*

In general, we have $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ \Rightarrow 4^\circ \Leftrightarrow 5^\circ \Leftrightarrow 6^\circ \Leftrightarrow 7^\circ$.

If L is continuous, all statements 1° to 10° are equivalent.

If M is continuous (but not necessarily L), then $4^\circ \Rightarrow 1^\circ$ (and hence the statements 1° to 7° are equivalent).

Proof. $1^\circ \Rightarrow 2^\circ$: Assume that f is l.s.c. for the Scott topology of M. Applying Theorem 4.2 to $X = \Sigma M$, we deduce that f is continuous from ΣM to ΣL, that is, 4°. This implies in particular (by Proposition 4.1) that f is isotone. Moreover, since the Scott topology $\sigma(M)$ is included in the Lawson topology $\lambda(M)$ (by Definition 3.4), we get that for all $x \in M$, $f(x) \leq \sup_{U \in \sigma(M),\, U \ni x} \inf_{y \in U} f(y) \leq \sup_{U \in \lambda(M),\, U \ni x} \inf_{y \in U} f(y)$, hence f is l.s.c. for the Lawson topology of M.

$2^\circ \Rightarrow 3^\circ$: This follows from $\lambda(M) \subset \sigma^{\mathrm{b}}(M)$ (see (3.14)).

$3^\circ \Rightarrow 4^\circ$: Assume that f is isotone and l.s.c. for the bi-Scott topology of M. Applying Theorem 4.2 to $X = \Sigma^{\mathrm{b}} M$, we deduce that f is continuous from $\Sigma^{\mathrm{b}} M$ to ΣL. Hence, for all Scott closed subsets F of L, the set $G = f^{-1}(F) := \{x \in M \mid f(x) \in F\}$ is bi-Scott closed. Since f is isotone, and F is downward (by

Proposition 3.1), G is downward. Indeed, if $x \le y$ and $y \in G$, then, since f is isotone, $f(x) \le f(y)$, whence, since $f(y) \in F$ and F is downward, $f(x) \in F$, so $x \in G$. This implies, by Proposition 3.4, that G is Scott closed. Consequently, f is continuous from ΣM to ΣL.

$4° \Rightarrow 5°$: This is the implication $2° \Rightarrow 3°$ of Theorem 4.2 in the case $X = \Sigma M$.

$5° \Rightarrow 4°$: By the first part of the proof of Proposition 4.1, we see that if f satisfies 5°, then f preserves upper bounded directed sups, which implies, by (the second part of) Proposition 4.1, that f is Scott continuous.

$5° \Rightarrow 6°$: Assume that all lower level sets $S_a(f)$ are Scott closed. They are then Lawson closed (by Definition 3.4). Moreover, the lower level sets are downward, which yields the isotony of f.

$6° \Rightarrow 7°$: This follows from $\lambda(M) \subset \sigma^{\mathrm{b}}(M)$.

$7° \Rightarrow 5°$: If f is isotone, then all lower level sets $S_a(f)$ are downward. If in addition these lower level sets are bi-Scott closed, they are necessarily Scott closed (by Proposition 3.4).

When L is continuous, the equivalences $1° \Leftrightarrow 4° \Leftrightarrow 8°$, $2° \Leftrightarrow 9°$ and $3° \Leftrightarrow 10°$ are obtained by applying the equivalences $1° \Leftrightarrow 2° \Leftrightarrow 4°$ of Theorem 4.2 to $X = \Sigma M$, $X = \Lambda M$ and $X = \Sigma^{\mathrm{b}} M$ respectively.

Assume now that M is continuous (and that L is an arbitrary conditionally complete lattice) and let us prove the implication $4° \Rightarrow 1°$ in that case. Let f be a Scott continuous function from M to L. Consider $x \in M$ and $a \in M$ such that $a \ll x$. The set $V_a = \{y \in M \mid a \ll y\}$ contains x and by Proposition 3.2, it is Scott open in M (since M is continuous). This implies that $\liminf_{y \to x} f(y) := \sup_{U \in \sigma(M),\, U \ni x} \inf_{y \in U} f(y) \ge \inf_{y \in V_a} f(y)$. Since f is isotone (by Proposition 4.1), $f(y) \ge f(a)$ for all $y \in V_a$, hence $\liminf_{y \to x} f(y) \ge f(a)$. Since this holds for all $a \ll x$, we obtain $\liminf_{y \to x} f(y) \ge \sup f(D)$, where $D := \{a \in M \mid a \ll x\}$. The set D is clearly upper bounded by x and, since M is continuous, $\sup D = x$. Moreover, D is a sup-semilattice (by Proposition 1.1, 3°), hence it is an upper bounded directed set. This implies, by Proposition 4.1, that $\sup f(D) = f(\sup D) = f(x)$. Hence, $\liminf_{y \to x} f(y) \ge f(x)$, and thus f is l.s.c. ∎

5. Downward sets and separation in finite powers of conditionally complete lattice ordered groups

We have the following characterization of downward sets in terms of "separation" from outside points.

Proposition 5.1. *Let $A = (A, \le, \otimes)$ be a c.c.l.o.g. For a subset G of A^n, the following statements are equivalent:*

1°. *G is downward.*

2°. *For each $x \in A^n \setminus G$ there exists $w \in A^n$ such that*

$$e \le \varphi(x, w), \quad e \not\le \varphi(g, w) \quad \forall g \in G\,, \tag{5.1}$$

where φ is the coupling function (0.35).

3°. *For each $x \in A^n \setminus G$ we have*

$$e \not\le \varphi(g, x^{-1}) \quad \forall g \in G\,. \tag{5.2}$$

Proof. $1^\circ \Rightarrow 3^\circ$: Assume that 1° holds. If 3° does not hold, i.e., if there exist $x \in A^n \setminus G$ and $g \in G$ such that $\varphi(g, x^{-1}) = \inf_{1 \le i \le n}(g_i \otimes x_i^{-1}) \ge e$, then $g_i \otimes x_i^{-1} \ge e$ $(i = 1, ..., n)$, whence $g \ge x$. Since G is a downward set, we obtain $x \in G$, in contradiction with our assumption on x.

$3^\circ \Rightarrow 2^\circ$: If (5.2) holds, then for $w = x^{-1}$ we have (5.1).

$2^\circ \Rightarrow 1^\circ$: Assume that 2° holds and that G is not a downward set, so there exist $g \in G$ and $x \in A^n \setminus G$, with $x \le g$. Then, for any $w \in A^n$ we have $\varphi(x, w) \le \varphi(g, w)$ and hence, for any $w \in A^n$ as in 2° we obtain (by the first part of (5.1)) $\varphi(g, w) \ge \varphi(x, w) \ge e$, which contradicts the second part of (5.1). ∎

Theorem 5.1. *Let $A = (A, \le, \otimes)$ be a continuous c.c.l.o.g. For a subset G of A^n, the following statements are equivalent:*

1°. *G is Scott closed in A^n, that is, closed in $\Sigma(A^n) = (\Sigma A)^n$.*

2°. *G is downward, and for each $x \in A^n$ the set*

$$H := \{\lambda \in A \mid \lambda \otimes x \in G\} \tag{5.3}$$

is Scott closed in A.

3°. *For each $x \in A^n \setminus G$, there exist $w \in A^n$ and $a \in A$ such that*

$$a \ll \varphi(x, w),\ a \not\ll \varphi(g, w) \quad \forall g \in G\ . \tag{5.4}$$

4°. *For each $x \in A^n \setminus G$, there exists $w \in A^n$ such that*

$$e \ll \varphi(x, w),\ e \not\ll \varphi(g, w) \quad \forall g \in G\ . \tag{5.5}$$

5°. *G is downward and Lawson closed, that is, closed in $\Lambda(A^n) = (\Lambda A)^n$.*

6°. *G is downward, and for each $x \in A^n$ the set H defined in (5.3) is Lawson closed in A.*

7°. *G is downward and bi-Scott closed, that is, closed in $\Sigma^b(A^n) = (\Sigma^b A)^n$.*

8°. *G is downward, and for each $x \in A^n$ the set H defined in (5.3) is bi-Scott closed in A.*

Proof. $1^\circ \Rightarrow 2^\circ$: If 1° holds, then, by Proposition 3.1, G is downward. Consider $x \in A^n$ and define $w_x : A \to A^n$ as in (4.10); then, $H = w_x^{-1}(G)$. Since, by Corollary 4.4, w_x is continuous for the Scott topology, H is Scott closed in A.

$2^\circ \Rightarrow 3^\circ$: Assume 2°, and let $x \in A^n \setminus G$. Then, since $e \otimes x = x \notin G$, we have $e \in A \setminus H$. Hence, since $A \setminus H$ is Scott open in A (by 2°) and the sets $V_a := \{y \in A \mid a \ll y\}$ $(a \in A)$ are a basis of open neighborhoods for the Scott topology on A (by Proposition 3.2), there exists $a \in A$ such that $e \in V_a \subseteq A \setminus H$, so

$$e \in V_a \text{ and } H \subset A \setminus V_a = \{y \in A \mid a \not\ll y\}.$$

Thus,

$$a \ll e \text{ and } a \not\ll \lambda \quad \forall \lambda \in H\ . \tag{5.6}$$

Let

$$w = x^{-1}\ . \tag{5.7}$$

Then $a \ll e = \varphi(x, x^{-1}) = \varphi(x, w)$, so in order to show (5.4), it will suffice (by (5.6)) to show that $\varphi(g, w) \in H$ $(g \in G)$, i.e., by (5.3), that $\varphi(g, w) \otimes x \in G$ $(g \in G)$. Let $g = (g_i) \in G$. Then,

$$\varphi(g, w) \otimes x = \varphi(g, x^{-1}) \otimes x = (\inf_{1 \leq j \leq n} (g_j \otimes x_j^{-1})) \otimes x \leq g \ ; \tag{5.8}$$

indeed, for any $1 \leq i \leq n$ we have $(\inf_{1 \leq j \leq n}(g_j \otimes x_j^{-1})) \otimes x_i \leq g_i \otimes x_i^{-1} \otimes x_i = g_i$, which proves (5.8). Hence, since G is a downward set (by 2°), it follows that $\varphi(g, w) \otimes x \in G$.

3° $\Rightarrow$ 4°: Assume that 3° holds and let $x \in A^n \setminus G$. If $w \in A^n$ and $a \in A$ satisfy (5.4), let $w' = a^{-1} \otimes w$ $(\in A^n)$. Then, by Lemma 2.1 and (1.10), $e = a^{-1} \otimes a \ll a^{-1} \otimes \varphi(x, w) = \varphi(x, w')$. Similarly, $e \not\ll \varphi(g, w')$ $(g \in G)$.

4° $\Rightarrow$ 1°: Assume that 4° holds and let us denote

$$Q_w := \{y \in A^n \mid e \ll \varphi(y, w)\} \quad (w \in A^n) \ . \tag{5.9}$$

Then, by 4°, for each $x \in A^n \setminus G$ there exists $w_x \in A^n$ satisfying $x \in Q_{w_x}$ and $G \subseteq A^n \setminus Q_{w_x}$. Hence, $G = \cap_{x \in A^n \setminus G}(A^n \setminus Q_{w_x})$. By Corollary 2.3, $Q_{w_x} = \{y \in A^n \mid w^{-1} \ll y\}$, whence, by Proposition 3.2, Q_{w_x} is Scott open in A^n. Therefore, $A^n \setminus Q_{w_x}$ is Scott closed, and hence so is $G = \cap_{x \in A^n \setminus G}(A^n \setminus Q_{w_x})$.

1° $\Leftrightarrow$ 5° $\Leftrightarrow$ 7°: By Proposition 3.4, a downward set is Scott closed if, and only if, it is Lawson closed, and if, and only if, it is bi-Scott closed. Moreover, by Proposition 3.1, any Scott closed set is downward. Thus, 1° $\Leftrightarrow$ 5° $\Leftrightarrow$ 7°.

2° $\Leftrightarrow$ 6° $\Leftrightarrow$ 8°: Similarly to the above proof of the equivalences 1° $\Leftrightarrow$ 5° $\Leftrightarrow$ 7°, it will be enough to show that H of (5.3) is a downward subset of A as soon as G is a downward set of A^n. Let $\lambda \in H$, i.e., $\lambda \otimes x \in G$, and let $\mu \leq \lambda$. Then $\mu \otimes x \leq \lambda \otimes x$, and hence, since G is downward, $\mu \otimes x \in G$, i.e., $\mu \in H$. ∎

Remark 5.1. Considering, in particular, the totally ordered groups $A = \mathbb{R} = (\mathbb{R}, \leq, +)$ and $A = \mathbb{R}_{++} = (\mathbb{R}_{++}, \leq, \times)$, from Theorem 5.1 we obtain again Theorems 0.1 and 0.4 of the Introduction. Indeed, in these groups $a \ll b \Leftrightarrow a < b$ (see Example 1.1). Furthermore, for $A = (\mathbb{R}, \leq, +)$, we have $e = 0$, and φ of (0.35) becomes the coupling function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by (0.3), while for $A = (\mathbb{R}_{++}, \leq, \times)$, we have $e = 1$, and φ becomes the coupling function $\varphi_0 : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ defined by (0.29). Also, by Definitions 3.1 and 3.4, the bi-Scott and Lawson topologies on $\mathbb{R}$ and $\mathbb{R}_{++}$ coincide with the usual topology of $\mathbb{R}$ and $\mathbb{R}_{++}$, respectively, and it is clear that the usual topology of $\mathbb{R}^n$ and $\mathbb{R}^n_{++}$ is the product topology of the usual topology of $\mathbb{R}$ and $\mathbb{R}_{++}$, respectively. Finally, it remains to observe that in these particular groups, a set G is downward and for each $x \in A^n$ the set H defined in (5.3) is Lawson closed in A, if, and only if, G is downward and closed along diagonal lines, respectively normal and closed along rays.

6. Conjugations of type Lau for functions with values in extensions of ordered groups

In this section, we consider a conditionally complete lattice ordered group $A = (A, \leq, \otimes)$, and denote by $\overline{A} = (\overline{A}, \leq, \dot{\otimes}, \underset{\cdot}{\otimes})$ its canonical enlargement.

6.1. **Preliminaries about general dualities.** If A is continuous and $\psi : X \times W \to A$ is a coupling function between two sets X and W, Theorem 5.1 suggests to define the duality $\Delta_\psi : 2^X \to 2^W$ associated to ψ by

$$\Delta_\psi(G) := \{w \in W |\, e \not\ll \psi(g,w)\ (g \in G)\}\ ; \tag{6.1}$$

in particular, when $A = \mathbb{R}$, (6.1) reduces to (0.9). For the duality $\Delta = \Delta_\psi$ of (6.1), $\Delta'_\psi \Delta_\psi$-convexity (0.8) of a set $G \subseteq X$ means that for each $x \notin G$, there should exist $w \in W$ satisfying

$$e \ll \psi(x,w), \quad e \not\ll \psi(g,w) \quad \forall g \in G\ . \tag{6.2}$$

A weaker requirement is the (generalized) (W,ψ)-*convexity* of G, which we define by the requirement that for each $x \notin G$, there should exist $w \in W$ and $a \in A$ satisfying

$$a \ll \psi(x,w), \quad a \not\ll \psi(g,w) \quad \forall g \in G\ . \tag{6.3}$$

In particular, for $X = W = A^n$ and the coupling function $\psi = \varphi : A^n \times A^n \to A$ of (0.35), conditions (6.2) and (6.3) reduce to (5.5) and (5.4) respectively, and formula (6.1) becomes

$$\Delta_\varphi(G) = \{w \in \mathbb{R}^n |\, e \not\ll \inf_{1\leq i\leq n} (g_i \otimes w_i)\ (g \in G)\} \quad (G \subseteq \mathbb{R}^n)\ , \tag{6.4}$$

which, by Corollary 2.3, can also be written in the form

$$\Delta_\varphi(G) = \{w \in \mathbb{R}^n |\, w^{-1} \not\ll g\ (g \in G)\} \quad (G \subseteq \mathbb{R}^n)\ . \tag{6.5}$$

If X is a set and L is a lattice, then for any function $f : X \to L$ one defines the lower level sets as in (4.16):

$$S_d(f) := \{x \in X |\, f(x) \leq d\} \quad (d \in L)\ , \tag{6.6}$$

and for any property P of abstract convexity of subsets G of a set X, it is natural to call a function $f : X \to L$ (generalized) P-*quasi-convex,* if all lower level sets $S_d(f)$ $(d \in L)$ are P-convex.

In order to study $\Delta'\Delta$-quasi-convexity of functions $f : X \to \overline{A}$, we shall introduce the conjugate of type Lau with respect to Δ, as in the case $A = \mathbb{R}$. For any function $f : A^n \to \overline{A}$, we denote by f^{-1} the function $f^{-1} : A^n \to \overline{A}$, defined by

$$f^{-1}(x) := f(x)^{-1} \quad (x \in A^n) \tag{6.7}$$

(see Definition 1.7) and by f^- the function $f^- : A^n \to \overline{A}$, defined by

$$f^-(x) = f^{-1}(x^{-1}) = [f(x^{-1})]^{-1} \quad (x \in A^n)\ . \tag{6.8}$$

Definition 6.1. Let A be a c.c.l.o.g., with canonical enlargement $\overline{A}$ and let X, W be two sets.

(a) If $\Delta : 2^X \to 2^W$ is any duality, and $f : X \to \overline{A}$ is a function, the conjugate of type Lau of f, with respect to Δ, is the function $f^{L(\Delta)} : W \to \overline{A}$ defined by

$$f^{L(\Delta)}(w) := \left[\inf_{\substack{x\in X\\ w\in W\setminus\Delta(\{x\})}} f(x)\right]^{-1} = \sup_{\substack{x\in X\\ w\in W\setminus\Delta(\{x\})}} f^{-1}(x) \quad (w \in W)\ . \tag{6.9}$$

(b) If $\psi : X \times W \to A$ is a coupling function, the conjugate of type Lau of f with respect to ψ, denoted $f^{L(\psi)}$, is the conjugate of type Lau $f^{L(\Delta_\psi)}$ of f with respect to the duality $\Delta_\psi : 2^X \to 2^W$ defined by (6.1), that is,

$$f^{L(\psi)}(w) := f^{L(\Delta_\psi)}(w) = \left[\inf_{\substack{x\in X\\ e\ll\psi(x,w)}} f(x)\right]^{-1} = \sup_{\substack{x\in X\\ e\ll\psi(x,w)}} f^{-1}(x) \quad (w \in W)\ . \tag{6.10}$$

Remark 6.1. We recall that, for any sets X and W, a mapping $M : \overline{A}^X \to \overline{A}^W$, $f \mapsto f^M$ is called (see [13, 14]) a *conjugation*, if

$$(\inf_{i\in I} f_i)^M = \sup_{i\in I} f_i^M \quad (\{f_i\}_{i\in I} \subseteq \overline{A}^X)\ , \tag{6.11}$$

$$(a \dot{\otimes} f)^M = f^M \underset{\cdot}{\otimes} a^{-1} \quad (f \in \overline{A}^X, a \in \overline{A})\ . \tag{6.12}$$

In particular, it is easy to see that for any duality $\Delta : 2^X \to 2^W$, the conjugation of type Lau $L(\Delta)$ is a conjugation in the sense (6.11), (6.12). Among other results, in [13, 14] it has been shown that a mapping $M : \overline{A}^X \to \overline{A}^W$ is a conjugation (in the sense (6.11), (6.12)) if, and only if, there exists a (necessarily unique) coupling function $\psi : X \times W \to \overline{A}$ such that

$$f^M(w) = \sup_{x\in X} \{\psi(x,w) \underset{\cdot}{\otimes} [f(x)]^{-1}\} \quad (w \in W)\ . \tag{6.13}$$

Such general conjugations, and some particular classes of conjugations, have been studied in [13, 14]. For $X = W = \mathbb{R}^n$ and the coupling function $\psi = \varphi$ of (0.3), the conjugations (6.13) have been characterized in [26].

The *dual* of any mapping $u : \overline{A}^X \to \overline{A}^W$ is the mapping $u' : \overline{A}^W \to \overline{A}^X$ defined by

$$h^{u'} := \inf_{\substack{f\in\overline{A}^X\\ f^u\le h}} f \quad (h \in \overline{A}^W)\ . \tag{6.14}$$

If u is a conjugation in the sense (6.11), (6.12), then u' is a conjugation, too.

Lemma 6.1. *Let A be a c.c.l.o.g., with canonical enlargement $\overline{A}$. If $X = W$ and $\psi : X \times X \to A$ is a symmetric coupling function* (*i.e., satisfying* (0.10)), *then the conjugation of type Lau $L(\psi) : \overline{A}^X \to \overline{A}^X$ of* (6.10) *is self-dual, i.e., for the dual* (6.14) *of $L(\psi)$, we have*

$$L(\psi)' = L(\psi). \tag{6.15}$$

Proof. By (6.10) and (0.10), for any f, $g \in \overline{A}^X$ we have the equivalences

$$f^{L(\psi)} \le g \Leftrightarrow \sup_{\substack{x\in X\\ e\ll\psi(x,w)}} f^{-1}(x) \le g(w) \quad \forall w \in W$$

$$\begin{aligned}&\Leftrightarrow f^{-1}(x) \le g(w) \quad \forall x \in X,\ w \in W,\ e \ll \psi(x,w)\\&\Leftrightarrow g^{-1}(w) \le f(x) \quad \forall x \in X,\ w \in W,\ e \ll \psi(w,x)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Leftrightarrow g^{L(\psi)} \le f\ .\end{aligned}$$

Hence, by (6.14), we get that

$$g^{L(\psi)'} = \inf_{f\in \overline{A}^X,\ f^{L(\psi)}\le g} f = \inf_{f\in \overline{A}^X,\ g^{L(\psi)}\le f} g = g^{L(\psi)}. \ \blacksquare$$

6.2. **Application to the coupling function φ.** Let us now give some properties of the conjugations of type Lau with respect to the coupling function $\psi = \varphi$ of (0.35).

Lemma 6.2. *Let A be a c.c.l.o.g., with canonical enlargement $\overline{A}$ and let $\varphi : A^n \times A^n \to A$ be the coupling function of* (0.35)*. For any function $f : A^n \to \overline{A}$, we have*

$$f^{L(\varphi)}(w) = \sup_{\substack{x\in A^n\\ w^{-1}\ll x}} f^{-1}(x) = \sup_{\substack{x\in A^n\\ w^{-1}\ll x^{-1}}} f^{-1}(x^{-1}) = \sup_{\substack{x\in A^n\\ x\ll w}} f^{-}(x) \quad (w \in A^n)\ . \tag{6.16}$$

Proof. The first equality holds by (6.10) and Corollary 2.3, the second one by the bijectivity of $x \in A^n \mapsto x^{-1} \in A^n$, and the last one by Corollary 2.1 ($w^{-1} \ll x^{-1} \Leftrightarrow x \ll w$). ∎

Lemma 6.3. *Let A be a continuous c.c.l.o.g., with canonical enlargement $\overline{A}$ and let $\varphi : A^n \times A^n \to A$ be the coupling function of* (0.35)*. For all $f : A^n \to \overline{A}$, $f^{L(\varphi)}$ is Scott continuous, that is, continuous with respect to the Scott topologies of A^n and $\overline{A}$.*

Proof. Since A is continuous, so is A^n (see Example 1.4). Let D be an upper bounded directed subset of A^n. Then, by (6.16) we get:

$$\sup f^{L(\varphi)}(D) = \sup_{d\in D}\left(\sup_{\substack{x\in A^n\\ x\ll d}} f^{-}(x)\right) = \sup_{\substack{x\in A^n\\ \exists d\in D,\, x\ll d}} f^{-}(x)\ .$$

On the other hand, by Proposition 1.2, $x \ll \sup D$ implies that there exists $d \in D$ such that $x \ll d$, since A^n is continuous. Moreover, the converse implication is true in any lattice. Hence,

$$\sup f^{L(\varphi)}(D) = \sup_{\substack{x\in A^n\\ x\ll \sup D}} f^{-}(x) = f^{L(\varphi)}(\sup D)\ .$$

This shows that $f^{L(\varphi)}$ preserves upper bounded directed sups, and hence, by Proposition 4.1, $f^{L(\varphi)}$ is Scott continuous. ∎

Theorem 6.1. *Let A be a continuous c.c.l.o.g., with canonical enlargement $\overline{A}$ and let $\varphi : A^n \times A^n \to A$ be the coupling function of* (0.35)*. For a function $f : A^n \to \overline{A}$, the following statements are equivalent:*

1°. *The function f is Scott continuous.*
2°. *The function f is l.s.c. for the Scott topology of A^n.*

3°. *The function f is isotone and l.s.c. for the Lawson topology of A^n.*
4°. *The function f is isotone and l.s.c. for the bi-Scott topology of A^n.*
5°. *The function f is $\Delta'_\varphi\Delta_\varphi$-quasi-convex (i.e., all lower level sets $S_d(f)$, $d \in A$, are $\Delta'_\varphi\Delta_\varphi$-convex).*
6°. *The function f is (A^n, φ)-quasi-convex (i.e., all lower level sets $S_d(f), d \in A$, are (A^n, φ)-convex).*
7°. *We have*

$$f = f^{L(\varphi)L(\varphi)} . \tag{6.17}$$

8°. *We have*

$$f = f^{L(\varphi)L(\varphi)'} . \tag{6.18}$$

Proof. Applying Theorem 5.1 to $G = S_d(f)$, for all $d \in \overline{A}$, and using the definition (6.3) of (A^n, φ)-convex sets and the characterization (6.2) of $\Delta'_\varphi\Delta_\varphi$-convex sets, we obtain the equivalence between 5°, 6°, and the property that all lower level sets $S_d(f)$ of f are Scott closed. Using the equivalences 1° to 5° in Theorem 4.3, we get the equivalence between 1°, 2°, 3°, 4°, and the property that all lower level sets $S_d(f)$ of f are Scott closed. Hence the assertions 1° to 6° are equivalent.

¿From Lemma 6.3, we get the implication 7° ⇒1°.

Let us show the implication 1° ⇒ 7°. Assume that f is Scott continuous. By (6.16) and (1.11), we have

$$\begin{aligned} f^{L(\varphi)L(\varphi)}(x) &= \sup_{\substack{w \in A^n \\ w \ll x}} (f^{L(\varphi)})^{-1}(w^{-1}) = \sup_{\substack{w \in A^n \\ w \ll x}} \left(\sup_{\substack{y \in A^n \\ w \ll y}} f^{-1}(y) \right)^{-1} \\ &= \sup_{\substack{w \in A^n \\ w \ll x}} \left(\inf_{\substack{y \in A^n \\ w \ll y}} f(y) \right) . \end{aligned} \tag{6.19}$$

But, if $w \ll y$, then $w \leq y$ (by Proposition 1.1,1°), whence, since f is isotone, $f(w) \leq \inf_{y \in A^n, w \ll y} f(y)$, and hence, by (6.19),

$$f^{L(\varphi)L(\varphi)}(x) \geq \sup_{\substack{w \in A^n \\ w \ll x}} f(w) . \tag{6.20}$$

Let $D = \{w \in A^n \mid w \ll x\}$. The set D is clearly upper bounded by x, and it is a sup-semilattice (by Proposition 1.1, 3°), hence it is an upper bounded directed set. Since f is Scott continuous, we get, by Proposition 4.1, that $f(\sup D) = \sup f(D)$. Since A is continuous, so is A^n, hence $\sup D = x$. By (6.20), we get $f^{L(\varphi)L(\varphi)}(x) \geq \sup f(D) = f(\sup D) = f(x)$. Since the reverse inequality is true for all functions f, we obtain 7°. Finally, the equivalence 7° ⇔ 8° follows from (6.15) (for $\psi = \varphi$). This concludes the proof of the theorem. ∎

Remark 6.2. (a) The equivalences 1° ⇔ 5°, 2° ⇔ 6° and 3° ⇔ 7° in Theorem 4.3 show that in Theorem 6.1 the lower semicontinuity of f can be defined either by using Definition 4.1 or by the condition that all lower level sets $S_d(f)$ are closed for the respective topologies.

(b) Considering, in particular, the totally ordered group $A = \mathbb{R} = (\mathbb{R}, \leq, +)$, from the equivalence between statements 3° or 4° to 7° of Theorem 6.1 we obtain again Theorem 0.2 of the Introduction. Indeed, as noticed in Remark 5.1, the Lawson and bi-Scott topologies of $\mathbb{R}$ and $\mathbb{R}^n$ are the usual topologies. Similarly, considering, in particular, the totally ordered group $A = \mathbb{R}_{++} = (\mathbb{R}_{++}, \leq, \times)$, from Theorem 6.1 we obtain again Theorem 0.5 of the Introduction.

The following result gives a large class of functions for which the conjugate of type Lau with respect to φ has a particularly simple form.

Theorem 6.2. *Let A be a continuous c.c.l.o.g., with canonical enlargement $\overline{A}$ and let $\varphi : A^n \times A^n \to A$ be the coupling function of* (0.35)*. For a function $f : A^n \to \overline{A}$, the following statements are equivalent:*

1°. *We have $f^{L(\varphi)} = f^-$, that is,*

$$f^{L(\varphi)}(w) = [f(w^{-1})]^{-1} \quad (w \in A^n) . \tag{6.21}$$

2°. *The function f^- is Scott continuous.*
3°. *The function f^- is l.s.c. for the Scott topology of A^n.*
4°. *The function f^- is isotone and l.s.c. for the Lawson topology of A^n.*
5°. *The function f^- is isotone and l.s.c. for the bi-Scott topology of A^n.*
6°. *The function f is dual Scott continuous, that is, continuous with respect to the dual Scott topologies of A^n and $\overline{A}$.*
7°. *The function f is u.s.c. for the dual Scott topology of A^n.*
8°. *The function f is isotone and u.s.c. for the dual Lawson topology of A^n.*
9°. *The function f is isotone and u.s.c. for the bi-Scott topology of A^n.*

Proof. The implication 1° ⇒2° follows from Lemma 6.3.

Let us prove the implication 2° ⇒1°. Assume that f^- is Scott continuous. By (6.16), $f^{L(\varphi)}(w) = \sup f^-(D)$, where $D = \{x \in A^n \mid x \ll w\}$. The set D is clearly upper bounded by w, and it is a sup-semilattice (by Proposition 1.1, 3°), hence it is an upper bounded directed set. Since f^- is Scott continuous, we get, by Proposition 4.1, that $f^-(\sup D) = \sup f^-(D) = f^{L(\varphi)}(w)$. Since A is continuous, so is A^n, hence $\sup D = w$ and $f^{L(\varphi)}(w) = f^-(w)$.

Theorem 4.3 applied to $M = A^n$ (which is continuous, since A is continuous), $L = \overline{A}$ and f^-, shows the equivalence between 2°, 3°, 4° and 5°. Also Theorem 4.3 applied to $M = (A^n)^{\mathrm{op}}$ (which is continuous, by Theorem 2.1, since A^n is continuous), $L = (\overline{A})^{\mathrm{op}}$ and f, shows the equivalence between 6°, 7°, 8° and 9°.

Finally, let us prove the equivalence 6° ⇔ 2°. Let $o : A^n \to A^n$, $x \mapsto x^{-1}$ and $\overline{o} : \overline{A} \to \overline{A}$, $x \mapsto x^{-1}$. Applying Lemma 4.3 to the lattice ordered group A^n (resp. $\overline{A}$), we obtain that o (resp. $\bar{o}$) is bicontinuous from A^n (resp. $\overline{A}$) endowed with the Scott topology, to A^n (resp. $\overline{A}$) endowed with the dual Scott topology. Hence the dual Scott continuity of f is equivalent to the Scott continuity of $f^- = \overline{o}^{-1} \circ f \circ o$. Note that, similarly, 7° ⇔ 3°, 8° ⇔ 4° and 9° ⇔ 5° can be proved using (2.3), (1.11) and Lemma 4.3. ∎

Remark 6.3. (a) By the same arguments as in Remark 6.2 (a), we get that in Theorem 6.2, the lower (resp. upper) semicontinuity of f^- (resp. f) can be defined either by using Definition 4.1 or by the condition that all lower level sets $S_a(f^-) = \{x \in A^n \mid f^-(x) \leq a\}$ (resp. $S_a^{\mathrm{op}}(f) := \{x \in A^n \mid f(x) \geq a\}$) ($a \in \overline{A}$) are closed for the respective topologies.

(b) Also, by the same arguments as in Remarks 6.2 (c) and 5.1, considering the totally ordered group $A = \mathbb{R} = (\mathbb{R}, \leq, +)$, the equivalence $8^\circ \Leftrightarrow 1^\circ$ of Theorem 6.2 shows again Theorem 0.3 of the Introduction. Similarly, considering, in particular, the totally ordered group $A = \mathbb{R}_{++} = (\mathbb{R}_{++}, \leq, \times)$, from Theorem 6.2 we obtain again Theorem 0.6 of the Introduction.

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