EXPLICIT CHARACTERIZATION OF THE
SUPPORT OF NON-LINEAR INCLUSIONS

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Abstract. We study inverse problems for non-linear penetrable media in the
context of scattering theory and impedance tomography. Using a general
description of the range of the non-linear far-field operator we show an explicit
characterization of the support of a weakly non-linear inhomogeneous scatter-
ing object. The application of the same technique to the impedance tomogra-
phy problem for a monotonic non-linear inclusion yields a characterization of
the inclusion’s support from the non-linear Neumann-to-Dirichlet operator.

1. Introduction. Many materials in modern scientific and industrial applications
obey non-linear laws. Thus, there arises the natural question how to deal with
non-linear materials when one seeks to determine their physical parameters, or
when one wants to image them, brief, when one faces any kind of inverse prob-
lem for a non-linear structure. When compared to the vast amount of literature
for inverse problems related to linear models there is little theory related to the
inversion of non-linear models. Several papers [4, 21, 15, 19, 24] deal with inverse
conductivity problems \( \text{div}(k(u)\nabla u) = 0 \) for a temperature-depending conductivity.
In [13, 12], semi-linear elliptic inverse boundary value problems are investigated and
uniqueness results based on the knowledge of the non-linear Dirichlet-to-Neumann
map are shown. Further, there is theory on inverse scattering for the non-linear
Schrödinger operator [25, 22, 23] and the non-linear Helmholtz equation [14], par-
tially based on Born approximation, high-frequency asymptotics (Saito’s formula)
or small-amplitude limits.

The non-linear Helmholtz equation
\[
\Delta u + k^2 (n_L u + n_N(u)) = 0
\]  \( (1) \)
serves, with \( n_N(x,u) = \alpha(x)|u|^2 u \), as a model for propagation of time-harmonic laser
beams in media with a Kerr-type non-linearity [7, 1]. We refer to [7] for a derivation
of this equation in lossless photonic media. Using paraxial approximation of the
envelope function one can derive a non-linear Schrödinger equation from (1), see,
e.g., [20] for a discussion of this approximation. We take (1) as a model problem
and restrict ourselves to non-linearities of linear growth. Existence theory for such
non-linearities is easily available and avoids technicalities. Our goal is then to

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give a characterization of the support of a weakly non-linear scattering object in terms of far-field measurements, without simplifying the inverse problem by further asymptotics. Note that [22] treats inverse problems for a non-linearity of type 
\[ n_N(u) = |u|^2u/(1 + a|u|^2), \]
which fits into our framework. According to [22], this model describes non-linear optics for Kerr-like non-linear dielectric films.

The non-linear conductivity equation

\[ \text{div } a(\nabla u) = 0 \]  \hspace{1cm} (2)

serves as a model for electric current flow inside a material that depends on the current flow \( \nabla u \). Since Ohm’s law does not apply, such materials are sometimes called non-Ohmic. This model describes for instance non-linear current flow in superconductors [8], with \( a(x, p) = \alpha(x)|p|^\beta p \). We note that [19, 21] treat inverse problems for the non-linear conductivity equation \( \text{div } (a(u)\nabla u) \) where the material depends on the potential \( u \) rather than on the current flow \( \nabla u \). As for the scattering problem above, we restrict ourselves in this paper to a model problem and consider (2) for a class of monotonic non-linearities growing linearly in \( p \).

The main results of this paper are formulas for the support of non-linear penetrable inclusions in wave propagation and current flow models in terms of the non-linear measurement operators. First, we give a characterization result for the support of a bounded weakly non-linear penetrable scatterer in terms of the far-field operator. Second, the same technique allows to explicitly characterize a monotonic and penetrable non-linear inclusion inside a homogeneous conducting body from the Neumann-to-Dirichlet operator on the boundary. Our technique is related to the so-called infimum criterion arising in Factorization methods for linear scattering problems [18, 17, 16]. To this end, we show that the non-linear far-field operator and the non-linear Neumann-to-Dirichlet operator both obey a factorization. Further, we determine a couple of properties of the operators in this factorization that allow to prove an infimum criterion: The obstacle is characterized as the union of those points in space where the infimum of a certain functional involving the data is positive. The characterization of the support of the scattering object is quite explicit and it does not rely on any simplification of the inverse problem by asymptotics in the frequency or in material parameters (compare, e.g., [22, 14]).

Our analysis shows that several elements of the analysis of the linear factorization method do not depend on the linear structure of the measurement operator—roughly speaking all those where the adjoint of the data operator is not required. For the so-called infimum criterion, the adjoint is not needed.

The structure of this paper is as follows: In Section 2 we present the infimum criterion for non-linear factorizations. Section 3 treats non-linear scattering problems using an integral equation approach. The obtained results are used to characterize the support of non-linear inclusions in Section 4. Section 5 transfers this characterization to the impedance tomography problem with non-linear current flow. Finally, Appendix A introduces a non-linear transmission eigenvalue problem arising in the analysis of the non-linear scattering problem.

**Notation:** Evaluation of non-linear operators is emphasized using square brackets, e.g., \( T[\cdot] \). The standard \( L^2 \)-based Sobolev space of order \( \ell \) in a domain \( \Omega \) is denoted by \( H^\ell(\Omega) \), and the \( L^p \)-based Sobolev spaces are \( W^{\ell,p}(\Omega) \). By \( H^\ell_{\text{loc}}(\Omega) \) we denote functions that belong to \( H^\ell(B) \) for all open balls \( B \subset \Omega \). The space \( H^\ell_{0}(\Omega) \) is the closure of smooth functions with compact support in \( \Omega \) in the norm of \( H^\ell(\Omega) \).
The domain of definition and the range of an operator $T : X \to Y$ are denoted by $\mathcal{D}(T)$ and $\text{Rg}(T) = \{ T[\phi], \phi \in \mathcal{D}(T) \}$, respectively.

2. **Infimum Criterion for Non-Linear Factorizations.** In this section, we investigate factorizations of a non-linear operator $F$ on a Hilbert space $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ of the form $F = H^*T[H]$

\[
\begin{array}{ccc}
\mathcal{V} & \overset{F}{\longrightarrow} & \mathcal{V} \\
\downarrow H & & \uparrow H^* \\
X & \overset{T}{\longrightarrow} & X^*
\end{array}
\]

where $H : \mathcal{V} \to X$ is linear and compact and $T$ is a non-linear operator from $X$ to $X^*$. The spaces $X \subset X^*$ form a Gelfand triple over some Hilbert space. Actually, we only need $F$ to be defined on $\mathcal{D}(F) := \{ g \in \mathcal{V}, \| g \|_{\mathcal{V}} = 1 \}$, and thus set $\mathcal{D}(T) = \overline{H(\mathcal{D}(F))}$ for simplicity.

Our interest in this kind of non-linear factorizations is motivated by the following observation: Measurement operators in inverse scattering theory often satisfy a factorization that sometimes allows to express the range of $H^*$ in terms of $F$. This in turn allows to characterize the scattering object by special testfunctions. Our aim in this section is to derive conditions on non-linear factorizations that allow similar statements. We start with the simplest setting, assuming that $T$ is coercive.

**Theorem 2.1.** Let $F : \mathcal{D}(F) = \{ g \in \mathcal{V} : \| g \|_{\mathcal{V}} = 1 \} \subset \mathcal{V} \to \mathcal{V}$ satisfy the factorization (3). We assume that $T : \mathcal{D}(T) = \overline{H(\mathcal{D}(F))} \subset X \to X^*$ is bounded and coercive, that is, $\| T[\phi] \|_{X^*} \leq C_T \| \phi \|_X$ and $\text{Re} \langle T[\phi], \phi \rangle_{X^* \times X} \geq \alpha \| \phi \|^2_X$ for $\phi \in \mathcal{D}(T)$. Further, assume that $H : \mathcal{V} \to X$ is linear and compact. Then $f \in \text{Rg}(H^*)$ if and only if

\[
\inf \left\{ \frac{|\langle F[g], g \rangle_{\mathcal{V}}|}{(f,g)^2} : \| g \|_{\mathcal{V}} = 1, \langle f,g \rangle_{\mathcal{V}} \neq 0 \right\} > 0. \tag{4}
\]

**Proof.** Obviously,

\[
|\langle F[g], g \rangle| = |\langle T[Hg], Hg \rangle| \geq \text{Re} \langle T[Hg], Hg \rangle \geq \alpha \| Hg \|^2_X > 0
\]

for all $g \in \mathcal{D}(F) \subset \mathcal{V}$. Assume that $f \in \text{Rg}(H^*)$, say, $f = H^*(\phi)$, and let $g \in \mathcal{D}(F)$ such that $\langle f,g \rangle \neq 0$. The definition of $\mathcal{D}(F)$ implies that such $g$ exists. Then

\[
\frac{|\langle F[g], g \rangle|}{(f,g)^2} \geq \frac{\| Hg \|^2_X}{|\langle H^*(\phi), g \rangle|^2} \geq \frac{\alpha}{\| \phi \|^2_{X^*}}.
\]

Therefore, the infimum in (4) is positive.

Consider now $f \in \mathcal{V}$ that does not belong to $\text{Rg}(H^*)$. Boundedness of $T$ implies

\[
\frac{|\langle F[g], g \rangle|}{(f,g)^2} \leq \frac{C_T \| Hg \|^2_X}{(f,g)^2}, \quad g \in \mathcal{D}(F).
\]

The proof that the infimum in (4) vanishes is now achieved by constructing a sequence $\{ g_j \} \subset \mathcal{V}$ such that $\langle f, g_j \rangle = 1$ and $\| Hg_j \| \to 0$ as $j \to \infty$. Construction of this sequence works exactly as in the proof of the corresponding result on linear factorizations [18, Theorem 1.16]: First one shows that the set $\{ Hg, g \in \mathcal{V}, \langle g,f \rangle = 0 \}$ is dense in the range of $H : \mathcal{V} \to X$ and then exploits the Hahn-Banach theorem to find a $\hat{g} \in \mathcal{V}$ such that $\langle \hat{g}, f \rangle = 1$. Choose a sequence $\hat{g}_j$ in $\{ g \in \mathcal{V}, \langle g,f \rangle = 0 \}$ such
that $H\hat{g}_j \to -H\hat{g}$ as $j \to \infty$. Then $g_j := \hat{g}_j + \hat{g}$ satisfies $\langle f, g_j \rangle = 1$ and $Hg_j \to 0$ in norm as $j \to \infty$. We set $\hat{y}_j = g_j/\|g_j\| \in \mathcal{D}(F)$ and obtain that
\[
\left| \frac{\langle F(\hat{y}_j), g_j \rangle}{\langle f, g_j \rangle} \right| \leq C_T \|H\hat{y}_j\|^2 / \|\langle f, g_j \rangle\|^2 \to 0 \quad \text{as } j \to \infty.
\]

We prove a variant of Theorem 2.1 that is adapted to some weakly non-linear inverse scattering problems we consider later on. Recall that $\mathcal{D}(F) = \{g \in \mathcal{V}, \|g\|_\mathcal{V} = 1\}$.

**Theorem 2.2.** Let $F : \mathcal{D}(F) \subset \mathcal{V} \to \mathcal{V}$ satisfy $F = H^*T[H]$ with $T : \mathcal{D}(T) = H(\mathcal{D}(F)) \subset \mathcal{X} \to \mathcal{X}^*$ and $H : \mathcal{V} \to \mathcal{X}$ linear and compact. We assume that $T$ is bounded, $\|T[\phi]\|_{\mathcal{X}^*} \leq C_T \|\phi\|_{\mathcal{X}}$ for $\phi \in \mathcal{D}(T)$, and has a decomposition $T = T_{1N} + T_{2N}$ such that

(a) $T_{1N} : \mathcal{D}(T) \subset \mathcal{X} \to \mathcal{X}^*$ is bounded and $\|T_{1N}[\phi]\|_{\mathcal{X}^*} \leq \alpha \|\phi\|_{\mathcal{X}}$ for $\phi \in \mathcal{D}(T)$.

(b) If $0 < \|\phi_j\|_\mathcal{X} \to 0$ then it holds that $\|T_{2N}[\phi_j]\|_{\mathcal{X}^*} / \|\phi_j\|_\mathcal{X} \to 0$ as $j \to \infty$.

(c) $T[\phi], \phi \in \mathcal{X}^* : \{z \in \mathbb{C}, 0 \leq \arg(z) < \pi/2\} \setminus \{0\}$ for $0 \neq \phi \in \mathcal{D}(T)$.

(d) If $\mathcal{D}(T) \ni \phi_j \to \phi$ in $\mathcal{X}$, then there is a subsequence $\{\phi_{j_k}\}$ such that $T[\phi_{j_k}] \to T[\phi]$ in $\mathcal{X}^*$ as $k \to \infty$.

Then $f \in \text{Rg}(H^*)$ if and only if
\[
\inf \left\{ \frac{\|\langle F[g], g \rangle\|_\mathcal{V}}{\|\langle f, g \rangle\|_\mathcal{V}} : \|g\|_\mathcal{V} = 1, \langle f, g \rangle_\mathcal{V} \neq 0 \right\} > 0. \tag{5}
\]

**Proof.** Let $g \in \mathcal{D}(F)$. We show that there is $c > 0$ such that $|\langle T[Hg], Hg \rangle| \geq c\|Hg\|^2_{\mathcal{X}}$. If such $c$ does not exist, then there is a sequence $\{g_j\} \subset \mathcal{D}(F)$ such that $\|Hg_j\|_{\mathcal{X}} \neq 0$ and
\[
|\langle T[Hg_j], Hg_j \rangle| / \|Hg_j\|^2_{\mathcal{X}} \to 0 \quad \text{as } j \to \infty.
\]
Especially, $|\langle T[Hg_j], Hg_j \rangle| \to 0$, since $\|Hg_j\|_{\mathcal{X}}$ is bounded by $\|H\|_{\mathcal{V} \to \mathcal{X}} < \infty$. By assumption, $H$ is a compact operator, that is, we can extract a convergent subsequence $\phi_j := Hg_j$ such that $\phi_j \to \phi$ in $\mathcal{X}$. By possibly extracting another subsequence, assumption (d) yields that $|\langle T[\phi_j], \phi_j \rangle| \to |\langle T[\phi], \phi \rangle| = 0$. We conclude that $\langle T[\phi], \phi \rangle = 0$ and by assumption (c) that $\phi = 0$. Now,
\[
\frac{|\langle T[\phi_j], \phi_j \rangle|}{\|\phi_j\|^2} = \frac{|\langle T_{1N}[\phi_j], \phi_j \rangle|}{\|\phi_j\|^2} - \frac{|\langle T_{2N}[\phi_j], \phi_j \rangle|}{\|\phi_j\|^2} \geq \alpha - \frac{|\langle T_{2N}[\phi_j], \phi_j \rangle|}{\|\phi_j\|^2} \to 0 \quad \text{by (d)},
\]
which yields a contradiction. The remainder of the proof follows as in the proof of Theorem 2.1.

\[ \square \]

3. **Weakly Non-Linear Wave Scattering.** In this section we treat weakly non-linear scattering problems using an integral equation approach and Banach’s fixed point theorem. All results are classical, however, we indicate several details that we require later on when treating the inverse problem.

Let $n : \mathbb{R}^m \times \mathbb{C} \to \mathbb{C}$ be a given measurable function such that the support of the contrast function
\[
x \mapsto q(x, z) := n(x, z) - z, \quad x \in \mathbb{R}^m,
\]
is the closure of a bounded Lipschitz domain $D \subset \mathbb{R}^m$, $m = 2, 3$, for any $z \in \mathbb{C}$ different from zero. In the sequel we do not always explicitly denote the dependence of $n$ or $q$ on $x$. As a general assumption, we assume that
\[ |q(x, z)| \leq C |z|, \quad x \in \mathbb{R}^m, \ z \in \mathbb{C}. \tag{6} \]

For wave number $k > 0$ we denote by $u^i$ a smooth incident wave field, that is, an entire solution to the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^m$. Consider the non-linear scattering problem to find a total field $u$ such that $\Delta u + k^2 n(u) = 0$ in $\mathbb{R}^m$ and such that the difference $u^s = u - u^i$ is a radiating function. The scattered field $u^s$ hence solves the scattering problem
\[ \Delta v + k^2 v = -k^2 q(u^i + v) \quad \text{in} \quad \mathbb{R}^m, \quad \lim_{|x| \to \infty} |x|^{(m-1)/2} \left( \frac{\partial v(x)}{\partial |x|} - i k v(x) \right) = 0. \tag{7} \]

We understand (7) in a weak sense and seek for $v \in H^1_{\text{loc}}(\mathbb{R}^m)$. The second condition on $v$ in (7) is the well-known Sommerfeld radiation condition. Solutions to the Helmholtz equation $\Delta v + k^2 v = 0$ outside some ball $\{ |x| < r \}$ that satisfy this condition are called radiating solutions. The formulation of this radiation condition is well-defined for a weak solution to the Helmholtz equation by standard elliptic regularity results. We note that radiating solutions $v$ have an expansion
\[ v(x) = \beta \frac{\exp(i k |x|)}{|x|^{(m-1)/2}} \left( v^\infty(\hat{x}) + O(|x|^{-1}) \right) \quad \text{as} \quad |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}, \]

where $\beta = 1/(4\pi)$ for $m = 3$ and $\beta = \exp(i\pi/4)/\sqrt{8\pi k}$ for $m = 2$, see [5]. The function $v^\infty \in L^2(\mathbb{S})$ is called the far-field pattern of $v$.

**Remark 3.1.** In linear scattering theory one usually writes the Helmholtz equation as $\Delta u + k^2 n^2 u = 0$, where $n$ denotes the refractive index. For notational simplicity we nevertheless prefer to write $\Delta u + k^2 n(u) = 0$ for the non-linear Helmholtz equation.

We first show equivalence of problem (7) to an integral equation and recall the outgoing free-space fundamental solution $\Phi(x, y), x \neq y$, to the Helmholtz equation: $\Phi(x, y) := i H_0^{(1)}(k|x-y|)/4$ and $\Phi(x, y) = \exp(i k |x-y|)/(4\pi |x-y|)$ for $m = 2$ and $m = 3$, respectively. This fundamental solution allows to introduce the volume potential
\[ V(\phi) = \int_D \Phi(\cdot, y) \phi(y) \, dy. \]

**Proposition 3.2.** (a) The potential $V$ defines a bounded operator from $L^2(D)$ into $H^2(B)$ for any open ball $B \subset \mathbb{R}^m$ and $w = \Phi \phi \in H^2_{\text{loc}}(\mathbb{R}^m)$ is the unique radiating solution to $\Delta w + k^2 w = -\phi$ in the distributional sense.

(b) A function $v \in H^1_{\text{loc}}(\mathbb{R}^m)$ solves (7) for an incident field $u^i$ if and only if the restriction $v|_D \in H^1(D)$ solves the integral equation $v - k^2 V(q(u^i + v)) = 0$ in $L^2(D)$. In the latter case, the extension of $v$ by $k^2 V(q(u^i + v))$ to $\mathbb{R}^m$ belongs to $H^2_{\text{loc}}(\mathbb{R}^m)$.

Proof. (a) This is a well-known result, see, e.g., [5].

(b) Due to (6), $V(q(u^i, \cdot))$ is well-defined and continuous as an operator on $L^2(D)$, respectively. If $v|_D \in H^1(D)$ satisfies the integral equation, Proposition 3.2 yields that $v$ is a solution to (7), and also that $v \in H^2_{\text{loc}}(\mathbb{R}^m)$. If $v \in H^1_{\text{loc}}(D)$ solves (7), then the right-hand side of (7) belongs to $L^2(D)$ and Proposition 3.2 yields $v = -k^2 V(q(u^i + u^s))$. \hfill \rlap{\hbox to 1cm{\hfill □}}}
We investigate the non-linear integral equation from the last proposition for more general excitation terms,

\[ v = k^2 V(q(f + v)) \quad \text{for } f \in L^2(D). \tag{8} \]

**Proposition 3.3.** Assume that \( |q(x, z_1) - q(x, z_2)| \leq L_q |z_1 - z_2| \) for \( x \in D \) and \( z_{1,2} \in \mathbb{C} \), and that

\[ k^2 L_q \|V\|_{L^2 \rightarrow L^2} < 1. \tag{9} \]

Then there is a unique solution \( v \in H^2(D) \) to (8). The extension of \( u^* \) by the right hand side of (8) is a solution of (7) in \( H^2_{\text{loc}}(\mathbb{R}^m) \): For any open ball \( B \subset \mathbb{R}^m \) there is a constant \( C \) independent of \( f \) and \( v \) such that

\[ \|v\|_{H^2(B)} \leq \frac{C\|V(q(f))\|_{L^2(D)}}{1 - k^2 L_q \|V\|_{L^2 \rightarrow L^2}}. \]

**Proof.** First we note that Lipschitz continuity of \( q \) implies that the integral equation (8) is well-defined in \( L^2(D) \). Define \( A[v] = k^2 V(q(f + v)) \). Exploiting (9) yields

\[ \|A[v_1] + A[v_2]\|_{L^2(D)} \leq k^2 \|V\|_{L^2 \rightarrow L^2} \|q(f + v_1) - q(f + v_2)\|_{L^2(D)} \]

\[ \leq k^2 \|V\|_{L^2 \rightarrow L^2} L_q \|v_1 - v_2\|_{L^2(D)}. \]

Under the assumption that \( k^2 L_q \|V\|_{L^2 \rightarrow L^2} < 1 \), \( A \) is hence a strict contraction and possesses a unique fixed point due to Banach’s fixed point theorem, that is, (8) has a unique solution \( v \in L^2(D) \). By Proposition 3.2, \( v \in H^2(D) \). The standard error estimate for the fixed point iteration yields

\[ \|v\|_{L^2(D)} \leq k^2 \|V(q(f))\|_{L^2(D)}/(1 - k^2 L_q \|V\|_{L^2 \rightarrow L^2}). \tag{10} \]

Combination of this error estimate with the latter regularity result gives the stated bound on \( v \).

The last proposition states, roughly speaking, that (8) is solvable if the contrast \( q \) is small. The same technique can be applied to show a similar result for a contrast where the non-linear part is small compared to the linear part. We assume that

\[ q(x, z) = q_L(x)z + q_N(x, z) \]

with \( q_L \in L^\infty(D) \) such that \( \text{Im}(q_L) \geq 0 \).

**Proposition 3.4.** Assume that \( |q_N(x, z_1) - q_N(x, z_2)| \leq L_N |z_1 - z_2| \) for \( x \in D \) and \( z_{1,2} \in \mathbb{C} \). Denote by \( v_L \) the solution to the linear integral equation \( v_L = k^2 V(q_L(f + v_L)) \) in \( L^2(D) \) and assume that

\[ k^2 L_N \|V\|_{L^2 \rightarrow L^2} \left\| \left[ I - k^2 V(q_L) \right]^{-1} \right\|_{L^2 \rightarrow L^2} < 1. \tag{11} \]

Then there is a unique solution \( v \in H^2(D) \) of (8). The extension of \( v \) by the right hand side of (8) is a solution of (7) in \( H^2_{\text{loc}}(\mathbb{R}^m) \).

**Proof.** The inverse of \( I - k^2 V(q_L) \) is a bounded operator on \( L^2(D) \); this follows, e.g., from unique solvability of the corresponding linear scattering problem with coefficient \( q_L \), see, e.g., [18]. Let \( v_L \in H^1(D) \) be a solution to

\[ v_L = k^2 V(q_L(f + v_L)). \tag{12} \]

Set \( v = v_N + v_L \). Rewriting (8) as

\[ v_L - k^2 V(q_L(f + v_N + v_L)) - k^2 V(q_N(f + v_N + v_L)) = 0 \]
we find
\[ v_N - k^2 \left[ I + k^2 V(q_N) \right]^{-1} V(q_N(f + v_N + v_L)) = 0 \]  \hspace{1cm} (13)
and the claim follows as in the proof of Proposition 3.3.

We discuss an example for a non-linearity where the last proposition applies. Consider
\[ q(x, z) = q_L(x)z + \alpha q_N(x, z) \]
where \( \alpha > 0 \) is a parameter and \( \mathcal{N} \) is a bounded weak derivative. Under this assumption,
\[ |q_N(x, z_1) - q_N(x, z_2)| \leq \int_0^1 \left| \frac{\partial q_N}{\partial z}(x, z_2 + t(z_2 - z_1)) \right| dt |z_1 - z_2| \leq C |z_1 - z_2|. \]
If \( \alpha \) is small enough this non-linearity is hence admissible for Proposition 3.4.

4. Inverse Scattering for Non-Linear Media. In this section we define a non-linear far-field operator and we show that this operator obeys a factorization. Under suitable assumptions on the non-linear contrast, this factorization meets the requirements of the range identities from Section 2, yielding a uniqueness result for the support of the non-linear medium.

Definition of the non-linear far-field operator requires the notion of Herglotz waves. For a density \( g \in L^2(\mathcal{S}) \) the Herglotz wave function
\[ v_g(x) = \int_{\mathcal{S}} \exp(i k \theta \cdot x) g(\theta) \, ds, \quad g \in L^2(\mathcal{S}), \]
is a superposition of plane waves. The corresponding linear operator is the Herglotz operator
\[ H : L^2(\mathcal{S}) \to L^2(D), \quad Hg = v_g|_{\mathcal{D}}. \]
One can show that \( H \) is a compact and injective operator, see [5].

4.1. Characterization of Weakly Scattering Objects. In this entire section, we assume that either the assumptions of Proposition 3.3 or of Proposition 3.4 hold, to guarantee existence and uniqueness of solution for the direct scattering problems. We start with the simpler assumptions of Proposition 3.3, requiring that
\[ |q(x, z_1) - q(x, z_2)| \leq L_q |z_1 - z_2| \quad \text{and} \quad k^2 L_q \|V\|_{L^2 \to L^2} \leq \alpha < 1 \]  \hspace{1cm} (14)
for \( x \in D, z_1, z_2 \in \mathbb{C} \). Moreover, we require an additional assumption on the contrast,
\[ \text{Im} \int_D k^2 q(f)^T d\mathcal{E} \geq 0 \quad \text{for} \quad f \in L^2(D). \]  \hspace{1cm} (15)
Further, we assume that the complement of \( D \) is connected, since our method is not able to determine “holes” inside an inclusion, compare Figure 1.

![Figure 1](image-url)

**Figure 1.** A penetrable (shaded) scattering object \( D \) is illuminated by a Herglotz wave function that generates a scattered field \( u^s \). The exterior domain \( \mathbb{R}^m \setminus \overline{D} \) is assumed to be connected.
Due to (14) we can solve the direct non-linear scattering problem (8) for all incident fields \( f = Hg \) with \( g \in D(F) = \{ g \in L^2(\mathbb{S}) : \|g\|_{L^2(\mathbb{S})} = 1 \} \). Denote by \( v \in H^1_{\text{loc}}(\mathbb{R}^m) \) the extension of this solution of (8). The radiating function \( v \) possesses a far-field pattern \( v^\infty \). We define the non-linear far-field operator \( F : D(F) \subset L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S}) \) by \( F[g] = v^\infty \), where \( v^\infty \) is the far-field pattern of \( v \). Further, by construction, \( v = k^2V(q(u^i + v)) \) and it is not difficult to see that \( v^\infty = H^*(k^2q(Hg + v)) \). Indeed, since \( \exp(-ik\theta \cdot y) \) is the far-field pattern of \( \Phi(\cdot, y) \),

\[
(H^*\phi)(\theta) = \int_D \exp(-ik\theta \cdot y)\phi(y) \, dy = \left( \int_D \Phi(\cdot, y)\phi(y) \, dy \right)^\infty(\theta) = (V\phi)^\infty(\theta).
\]

Recall that \( D(F) = \{ g \in L^2(\mathbb{S}) : \|g\|_{L^2(\mathbb{S})} = 1 \} \) and define \( T : D(T) \subset L^2(D) \rightarrow L^2(D) \) by \( D(T) = H(D(F)) \) (roughly speaking, the closure of the set of Herglotz wave functions in \( L^2(D) \)) and

\[
T(f) = k^2q(f + v)
\]

where \( v = v(f) \) solves (8). Then

\[
F[g] = H^*T[Hg].
\]

The following result checks some of the assumptions of Theorem 2.2 for the factorization (17). In analogy to the inverse scattering problem for linear models, the non-linear interior transmission eigenvalue problem plays a role. This problem is introduced in Appendix A.

**Proposition 4.1.** Consider the non-linear operator \( T : D(T) \subset L^2(D) \rightarrow L^2(D) \) defined in (16) and \( f \in D(T) \).

(a) \( T \) satisfies \( ||T[f]||_{L^2(D)} \leq C_T ||f||_{L^2(D)} \).

(b) \( \text{Im} \langle T[f], f \rangle_{L^2(D)} \geq 0 \).

(c) If \( k^2 \) is no non-linear interior transmission eigenvalue, then \( \text{Im} \langle T[f], f \rangle_{L^2(D)} > 0 \) for \( f \neq 0 \), and consequently \( \langle T[f], f \rangle_{L^2(D)} \in \{ z \in \mathbb{C}, 0 \leq \arg(z) < \pi/2 \} \setminus \{ 0 \} \).

**Proof.** (a) This follows directly from the definition of \( T \) and Propositions 3.3 or 3.4.

(b) We show that \( \text{Im} \langle T[f], f \rangle_{L^2(D)} \geq 0 \) for any \( f \in L^2(D) \). Recall that

\[
\langle T[f], f \rangle = \int_D k^2q(f + v)(f) \, dx = \int_D k^2q(f + v)(f + v) \, dx - \int_D k^2q(f + v)v \, dx
\]

where \( v = v(f) \) is given by the solution to the integral equation (8). Choose a cut off function \( \chi \in C_0^\infty(\mathbb{R}^m) \) such that \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) in \( B_R = \{ |x| < R \} \), and \( \chi = 0 \) in \( B_{R+1} \). Using Green’s first identity twice we find that

\[
\int_D k^2q(f + v)v \, dx = \int_{B_{R+1}} k^2q(f + v)\chi v \, dx = -\int_{B_{R+1}} (\Delta v + k^2v)\chi v \, dx
\]

\[
= \int_{B_R} (|\nabla v|^2 - k^2|v|^2) \, dx + \int_{\{R+1<|x|<R\}} (\nabla v \cdot \nabla (\chi v) - k^2\chi |v|^2) \, dx
\]

\[
= \int_{B_R} (|\nabla v|^2 - k^2|v|^2) \, dx - \int_{\Gamma_R} \frac{\partial v}{\partial r} \eta \, ds.
\]
Now we exploit the structural assumption (15) and the Sommerfeld radiation condition which implies that
\[ \int_{\Gamma_R} \frac{\partial v}{\partial r} v \, ds \to \frac{1}{4\pi} \int_{\Sigma} |v^\infty|^2 \, ds \quad \text{as } R \to \infty \] (18)
to obtain that
\[ \text{Im} \langle T[f], f \rangle = \text{Im} \int_D k^2 q(f + v)(f + v) \, dx + \frac{1}{4\pi} \int_{\Sigma} |v^\infty|^2 \, ds \geq 0. \]

(c) Assume that Im \( \langle T[f], f \rangle = 0 \) for \( f \in \mathcal{D}(T) \subset L^2(D) \). From (15) we infer that the far-field \( v^\infty \) vanishes, and by unique continuation this implies \( v \equiv 0 \) in \( \mathbb{R}^m \setminus D \).

Since \( f \in \mathcal{D}(T) = \mathcal{H}(\mathcal{D}(F)) \subset L^2(D) \) is a distributional solution of the Helmholtz equation \( \Delta u + k^2 u = 0 \) in \( D \), \( k^2 \) is hence a non-linear transmission eigenvalue according to the definition in Appendix A.

\[ \text{Lemma 4.2.} \] Assume that \( q(x, z) = q_L(x)z + q_N(x, z) \) is small enough to satisfy (14). Define \( c^-_L = \text{ess inf}_D(q_L), \ C^+_N = \text{ess sup}_D(q_L) \) and introduce a constant \( C_N \) such that \( |q_N(x, z)| \leq C_N|z| \) for \( x \in D \) and \( z \in \mathbb{C} \). Assume further that
\[ c^-_L - C_N - k^2(C^+_L + C_N)^2 \frac{\|V\|_{L^2(D)\to L^2(D)}}{1 - k^2 L_q\|V\|_{L^2(D)\to L^2(D)}} > 0. \] (19)

Then \( T : L^2(D) \to L^2(D) \) is coercive on \( L^2(D) \).

\[ \text{Proof.} \] Since we only work in \( L^2(D) \) in this proof there is no danger of confusion and we just write \( \| \cdot \| \) for vector and operator norms. Recall from (10) that
\[ \|v\| \leq k^2 \frac{\|V(q(f))\|}{1 - k^2 L_q\|V\|} \leq k^2 \frac{(C^+_L + C_N)\|V\|}{1 - \alpha} \|f\| \]
for \( \alpha = k^2 L_q\|V\| \). We compute
\[ \text{Re} \langle T[f], f \rangle = k^2 \int_D q_L|f|^2 \, dx + k^2 \int_D q_N|f|^2 \, dx + k^2 \int_D q_N(f + v)^2 \, dx \]
\[ \geq k^2 c^-_L \|f\|^2 - k^2 C^+_L \|v\|\|f\| - k^2 \|q_N(f + v)\|\|f\| \]
\[ \geq k^2 c^-_L \|f\|^2 - k^2 C^+_L \|V\|\|f\|^2 - k^2 C_N \|f + v\| \]
\[ \geq k^2 \left( c^-_L \|f\|^2 - k^2(C^+_L + C_N)^2 \frac{\|V\|}{1 - \alpha} \|f\|^2 - C_N \|f\|^2 \right) \]
\[ \geq k^2 \left( c^-_L - C_N - k^2(C^+_L + C_N)^2 \frac{\|V\|}{1 - \alpha} \|f\|^2. \] 

The following corollary is now an easy consequence of the last propositions together with Theorem 2.1. We test the range of \( F \) with testfunctions
\[ \phi_y(\theta) := \exp(-ik\cdot y), \quad y \in \mathbb{R}^m, \quad \theta \in \mathbb{S}. \] (20)

\[ \text{Corollary 4.3.} \] Assume that inequalities (14) and (19) hold. Then \( y \in D \) if and only if
\[ \inf \left\{ \frac{\langle F[y], g \rangle_{L^2(\mathbb{S})}}{\langle \phi_y, g \rangle_{L^2(\mathbb{S})}^2} \mid \|g\|_{L^2(\mathbb{S})} = 1, \langle \phi_y, g \rangle_{L^2(\mathbb{S})} \neq 0 \right\} > 0. \] (21)
Proof. It is well-known that $\phi_y \in \text{Rg}(H^*)$ if and only if $y \in D$, see, e.g., [18]. Therefore the claimed criterion is just a reformulation of (4) in Theorem 2.1. Propositions 4.1 and 4.2 verify the assumptions of the latter theorem. \hfill $\square$

4.2. Characterization of Scattering Objects with Small Non-Linearity. The result in Corollary 4.3 is limited to weak scattering since the contrast or the wave number need to be small. Next we use Theorem 2.2 to extend the last corollary to contrasts where the non-linear part is sufficiently small compared to the linear part of the contrast. We consider contrasts of the form

$$q(x, z) = q_L(x)z + q_N(x, z)$$

with $q_N(x, z) = b(x)h(|z|)z$,

where again we assume that $|q_N(x, z)| \leq \|b\|_{L^{\infty}(D)}C_N|z|$ for $x \in D$ and $z \in \mathbb{C}$. However, solution theory in $L^2(D)$ will not be sufficient to check assumption (b) from Theorem 2.2 and therefore we require some $L^p$ theory for $p \in (1, \infty)$.

From [9, Theorem 9.9] we know that the volume potential with the fundamental solution of the Laplace equation as kernel is bounded from $L^p(D)$ into $W^{2,p}(B)$ for all open balls $B \subset \mathbb{R}^m$ and $p \in (1, \infty)$. Since the difference of the fundamental solution for the Laplace and the Helmholtz equation is smooth, the above-introduced volume potential $V$ is also bounded on $L^p(D)$. Moreover, since the Helmholtz equation differs from the Laplace equation only in the lowest-order terms, it is not difficult to see that $V$ is bounded from $L^p(D)$ into $W^{2,p}(B)$ for all balls $B \subset \mathbb{R}^m$. Especially, $V$ is a compact operator on $L^p(D)$. The same holds for $v \mapsto V(q_Lv)$ and the Fredholm alternative and uniqueness of solution imply that $I - k^2 V(q_L \cdot)$ is boundedly invertible on $L^p(D)$. If we further suppose that

$$|q_N(x, z_1) - q_N(x, z_2)| \leq L_N |z_1 - z_2|$$

for $x \in D$, $z_{1,2} \in \mathbb{C}$, and

$$k^2 L_N \|V\|_{L^p(D) \rightarrow L^p(D)} \left\| \left[I - k^2 V(q_L \cdot)\right]^{-1}\right\|_{L^p(D) \rightarrow L^p(D)} < 1,$$  \hfill (22)

then essentially the same technique as used above shows that the Lippmann-Schwinger equation $v - k^2 V(q(u^t + v)) = 0$ is well-posed in $L^p(D)$. Note that we still assume that (15) holds.

Following Proposition 3.4 we split the unique solution $v$ of $v = k^2 V(q(Hg + v))$ into $v = v_L + v_N$ where $v_L \in W^{2,p}(D)$ solves (7) for $q$ replaced by $q_L$, and $v_N = k^2 \left[I - k^2 V(q_L \cdot)\right]^{-1}_{L^p(D) \rightarrow L^p(D)}V(q_N(Hg + v_L + v_N))$. Then

$$\|v_N\|_{L^p} \leq k^2 \left\| \left[I - k^2 V(q_L \cdot)\right]^{-1}\right\|_{L^p \rightarrow L^p} \|V\|_{L^p \rightarrow L^p} \|b\|_{L^{\infty}(D)}C_N \|Hg + v_L\|_{L^p} \leq k^2 \left\| \left[I - k^2 V(q_L \cdot)\right]^{-1}\right\|_{L^p \rightarrow L^p} \|V\|_{L^p \rightarrow L^p} \|b\|_{L^{\infty}(D)}C_N (1 + C) \|Hg\|_{L^p}. \hfill (23)$$

In the following, we only consider $p \geq 2$ and turn now to the far-field operator and its factorization. First, under the above assumptions, $F$ is well-defined as an operator on $D(F) \subset L^2(S)$, because the direct scattering problem for incident Herglotz wave functions is uniquely solvable. Second, note that the Herglotz operator $H$ is compact from $L^2(S)$ into $L^p(D)$. Third, the middle operator

$$T(f) = k^2 q(f + v) = k^2 \left( q_L(f + v_L) + q_N(v_N) \right)$$

is bounded from

$$D(T) = \{Hg : \|g\|_{L^2(S)} = 1\} \subset L^p(D)$$
into $L^p(D)$. Due to Hölder’s inequality, $T$ is hence also a bounded operator from $L^p(D)$ into $L^{p'}(D)$ where

$$p' \text{ is defined by } 1/p + 1/p' = 1.$$ 

For the last argument, we used that $p \geq 2$. The factorization $F = H^*T[H]$ now follows as in Section 4.1, but note that $T$ maps now $L^p(D)$ into $L^{p'}(D)$. For the next proposition we recall that $q_N(x, z) = b(x)h(|z|)z$.

**Proposition 4.4.** (a) If $D(T) \ni \phi_j \to \phi$ with convergence in $L^p(D)$ for $p > 2$, then there is a subsequence $\{\phi_{j_k}\}$ of $\{\phi_j\}$ such that $T[\phi_{j_k}] \to T[\phi]$ in $L^{p'}(D)$.

(b) Assume that $|h(|z|)| \leq C|z|^\gamma$ for $z \in \mathbb{R}$ and $0 < \gamma < (2 - p')/p'$ for $p > 2$. If $f_j \in \tilde{T}(D) \subset L^p(D)$ and $0 < \|f_j\|_{L^p(D)} \to 0$, then $\|T[\tilde{f}_j]\|_{L^{p'}(D)} \to 0$ as $j \to \infty$.

**Proof.** (a) Let $D(T) \ni f_j \to f$ in $L^p(D)$, $T(f_j) = q(f_j + v_j)$ and $T(f) = q(f + v)$. Then $v_{L,j} \to v_{L,j}$ in $L^p(D)$ by linearity. Moreover,

$$v_{N,j} = k^2 \left[1 - k^2V(q_{L'})^{-1}\right]_{L^p(D) \to L^{p'}(D)} V(q_{N,j}(f + v_{L,j} + v_{N,j})),$$

which shows, by (23) and boundedness of the volume potential from $L^p(D)$ into $W^{2,p}_0(\mathbb{R}^m)$, that we can extract a subsequence such that $v_{N,j}$ converges in $L^p(D)$ to $\tilde{v}_N$. Taking the limit of left and right-hand side of the last equation shows that

$$\tilde{v}_N = k^2 \left[1 - k^2V(q_{L'})^{-1}\right]_{L^p(D) \to L^{p'}(D)} V(q_{N,j}(f + v_{L,j} + \tilde{v}_N)),$$

and unique solvability of this integral equation implies $\tilde{v}_N = v_N$. Hence, $v_j = v_{L,j} + v_{N,j} \to v_L + v_N = v$.

(b) For $\|f_j\|_{L^p(D)} = \delta_j \to 0$ we have $\|v_{L,j}\|_{L^p(D)} \leq C\delta_j \to 0$ and (see (23)) $\|v_{N,j}\|_{L^p(D)} \leq C\delta_j \to 0$ in $L^p(D)$ as $j \to \infty$. Hence, setting $\tilde{f}_j := f_j + v_{L,j} + v_{N,j}$ we have $\|\tilde{f}_j\|_{L^p(D)} \leq C\delta_j \to 0$. Hölder’s inequality implies that $\|\tilde{f}_j\|_{L^{p'}(D)} \leq |D|^{1/p' - 1/p} \|\tilde{f}_j\|_{L^p(D)} \to 0$. The assumption of part (b) of the theorem yields that

$$\|bh([\tilde{f}_j])\|_{L^{p'}(D)} \leq \|b\|_{L^\infty(D)} \|\tilde{f}_j\|_{L^{p'(1+\gamma)}(D)}^{1+\gamma} \leq \|b\|_{L^\infty(D)} |D|^{1/(p'(1+\gamma)) - 1/p} \|\tilde{f}_j\|^{1+\gamma}_{L^p(D)} \leq C\delta_j^{1+\gamma}.$$

Therefore

$$\frac{\|T[\tilde{f}_j]\|_{L^{p'}(D)}}{\|\tilde{f}_j\|_{L^p(D)}} = \frac{\|bh([\tilde{f}_j])\|_{L^{p'}(D)}}{\|\tilde{f}_j\|_{L^p(D)}} \leq C\delta_j \to 0 \text{ as } j \to \infty.$$

**Remark 4.5.** The assumption of Proposition 4.4(b) that $|h(|z|)| \leq C|z|^\gamma$ is only an assumption on the behavior of $h$ close to zero, since, anyway, $h$ needs to be bounded in order that $|q_N(x, z)| = |b(x)h(|z|)z| \leq \|b\|_{L^\infty(D)} C_N |z|$.

For the following characterization result we recall the definition of the testfunctions $\phi_p$ from (20).

**Theorem 4.6.** Suppose that $k^2$ is not a linear or a non-linear interior transmission eigenvalue in the sense of Appendix A. Further, assume that $q(x, z) = q_L(x)z + b(x)h(|z|)z$ satisfies (15) and (22) and that the assumption of Proposition 4.4(b)
on $h$ holds. Finally, let $q_L \geq c > 0$ in $D$. Then there exists $\beta_0 > 0$ such that for $\|b\|_{L^\infty(D)} < \beta_0$ it holds that $y \in D$ if and only if

$$\inf \left\{ \left( \frac{\langle F[g], g \rangle_{L^2(S)}}{\langle \phi_y, g \rangle_{L^2(S)}} \right), \|g\|_{L^2(S)} = 1, \langle \phi_y, g \rangle_{L^2(S)} \neq 0 \right\} > 0. \tag{24}$$

Proof. We apply Theorem 2.2 with $V = L^2(S)$, $X = L^p(D)$ with $p > 2$ and $X^* = L^{p'}(D)$. The factorization $F = H^*T[H]$ holds with $H : D(F) = \{ g \in L^2(S) : \|g\|_{L^2(S)} = 1 \} \subset L^2(S) \to L^p(D)$ and $T : D(T) = \{ Hg : \|g\|_{L^2(S)} = 1 \} \subset L^p(D) \to L^{p'}(D)$ defined by

$$Tg = k^2q_Lf + k^2q_Lv_L + k^2q_Lv_N + k^2q_N(f + v_L + v_N).$$

We need to check the assumptions (a)–(d) on these operators made in Theorem 2.2.

Assumption (a) partly follows from [10, Lemma 4.2] (see also [18]): Since $T_0 : L^p(D) \to L^{p'}(D)$ is linear and coercive, since $T_{1L} : L^p(D) \to L^{p'}(D)$ is linear and compact, and since $k^2$ is not a linear eigenvalue for $q_L$, it holds that

$$\|\langle (T_0 + T_{1L})f, f \rangle_{L^{p'}(D) \times L^p(D)}\| \geq \alpha \|f\|^2_{L^p(D)}.$$

Thus, assumption (a) of Theorem 2.2 holds if $\|b\|_{L^\infty(D)}$ is small enough: Due to (23),

$$\|T_{1N}[f]\|_{L^{p'}(D)} \leq k^2\|q_L\|_{L^\infty(D)}\|D^{1/p'-1/p}\|v_N\|_{L^p(D)} \leq k^2\|D\|^{1/p'-1/p}\|q_L\|_{L^\infty(D)}\|b\|_{L^\infty(D)}C\|f\|_{L^p(D)}.$$

Hence, we conclude that $T_0 + T_{1L} + T_{1N}$ is coercive under the condition that

$$\|b\|_{L^\infty(D)} < \frac{\alpha}{k^2\|D\|^{1/p'-1/p}C\|q_L\|_{L^\infty(D)}} =: \beta_0.$$

The constant $C$ can be given explicitly using (23), but since the coercivity constant $\alpha$ from above is anyway not explicitly known, we skip details.

Finally, assumptions (b) and (d) of Theorem 2.2 are shown in Proposition 4.4, and assumption (c) has been checked in Proposition 4.1(b), under the assumption that $k^2$ is not a non-linear transmission eigenvalue for $q$. Since $L^p(D) \subset L^2(D)$ for $p > 2$, Proposition 4.1(b) also applies to $f \in \text{Rg}(T) \subset L^p(D)$.

The assumptions on $q_L$ are satisfied for $n_N(u) = b(x)|u|^2u/(1 + |u|^2)$ as long as $\|b\|_{L^\infty(D)}$ is small enough. See [22] for the analysis of a different inverse scattering problem for this non-linearity using high-frequency asymptotics.

5. Impedance Tomography for Non-Linear Current Flow. The characterization technique we used to treat inverse scattering problems for weakly non-linear models also applies to impedance tomography with a non-linear conductivity term. This non-linearity models for example materials where the conductivity depends on the temperature and thus on the heating of the material by the current flow through the body. The governing equation for current flow through such a material is

$$\text{div} a(\nabla u) = 0 \quad \text{in} \quad \Omega$$

where $a : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is a vector field and $\Omega \subset \mathbb{R}^m$, $m = 2$ or 3, is assumed to be a Lipschitz domain. Note that this problem is potentially an isotropic (but real valued, for simplicity). To avoid technicalities we simplify the setting by assuming that $a(x, p) = p$ outside some bounded Lipschitz domain $D \subset \Omega$, such that $\Omega \setminus \overline{D}$ is connected, compare Figure 2. The task in impedance tomography is then to
characterize the inclusion $D$ from measured potentials when applying currents on the boundary of $\Omega$. If $a(x,p)$ is linear in $p$, this problem can be solved, e.g., using the factorization method, see, e.g., [2, 11].

For simplicity, we often abbreviate $a(x,p) = a(p)$ in the sequel. Further, we formally set $v|_{\partial D}^+$ and $v|_{\partial D}^-$ to be the traces of $v$ on $\partial D$ from the exterior and interior of $D$, respectively, and denote by $[v]_{\partial D}^+ - [v]_{\partial D}^-$ the corresponding jump through $\partial D$. Duality products between fractional Sobolev spaces on $\partial \Omega$ and $\partial D$ are simply denoted by $\langle \cdot, \cdot \rangle_{\partial \Omega}$ and $\langle \cdot, \cdot \rangle_{\partial D}$, respectively.

The crucial assumptions on the measurable function $a$ are (strict) monotonicity and a linear growth bound. More precisely, for constants $\alpha > 1$ and $\beta > 0$ we assume that

\[
(a(x,p) - a(x,q)) \cdot (p - q) \geq \alpha|p - q|^2 \quad \text{for } x \in D, p \in \mathbb{R}^m, \quad |a(x,p)| \leq C(1 + |p|) \quad \text{and} \quad a(x,p) \cdot p \geq \beta|p|^2 \quad \text{for } x \in D, p \in \mathbb{R}^m, \quad (25)
\]

For $f \in H^{1/2, -1}_0(\partial \Omega) = \{g \in H^{1/2}(\partial \Omega), \langle g, 1 \rangle_{\partial \Omega} = 0\}$ we seek for a solution $u \in H^1_{\Omega}(\Omega) = \{v \in H^1(\Omega), \langle v|_{\partial \Omega}, 1 \rangle_{\partial \Omega} = 0\}$ to

$$
\text{div } a(\nabla u) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial \Omega.
$$

Here, $\nu$ is the exterior normal to $\Omega$, and we remark that we also denote the exterior normal to $D$ by $\nu$ subsequently. The variational formulation of this problem is

$$
\int_\Omega a(\nabla u) \cdot \nabla v \, dx = \langle f, v \rangle_{\partial \Omega} \quad \text{for all } v \in H^1_{\partial \Omega}(\Omega). \quad (26)
$$

**Theorem 5.1.** For $f \in H^{-1/2, 1}_0(\partial \Omega)$ there is a unique solution $u \in H^1_{\Omega}(\Omega)$ to (26) that satisfies $\|u\|_{H^1_{\Omega}(\Omega)} \leq C\|f\|_{H^{-1/2, -1}_0(\partial \Omega)}$.

**Proof.** The proof relies on the method of Browder and Minty. For completeness, we briefly recall the main steps, following [6, Chapter 9.1].

For any orthonormal basis $\{w_j\}_{j \in \mathbb{N}}$ of $H^1_{\partial \Omega}(\Omega)$ we first find a solution $u_n = \sum_{j=1}^n c_j w_j$, $n \in \mathbb{N}$, of the finite dimensional problem

$$
\int_\Omega a(\nabla u_n) \cdot \nabla v_j \, dx = \langle f, w_j \rangle_{\partial \Omega}, \quad j = 1, \ldots, n. \quad (27)
$$
The trick is to use the assumption \( a(x,p) \cdot p \geq \beta |p|^2 \) to estimate that the vector field \( F : \mathbb{R}^n \to \mathbb{R}^n \),
\[
F_j(c) = \int_\Omega a \left( \sum_{i=1}^n c_i \nabla w_i \right) \cdot \nabla v_j \, dx, \quad j = 1, \ldots, n,
\]
satisfies \( F(c) \cdot c \geq C_1 |c|^2 - C_2 \). In this situation, a technical lemma [6, Ch. 9.1, p. 493] shows that there is \( c = (c_1, \ldots, c_n)^T \) such that \( F(c) = 0 \), which gives the finite-dimensional approximate solution \( u_n \).

The energy estimate \( \|u_n\|_{H^1(\Omega)} \leq C \|f\|_{H^{-1/2}_0(\partial \Omega)} \) follows directly from the assumption \( a(x,p) \cdot p \geq \beta |p|^2 \) and a Poincaré-type lemma. Thus, we can extract a weakly convergent subsequence \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \), \( u_n \to u \) in \( L^2(\Omega) \), and \( a(\nabla u_n) \to \xi \) in \( L^2(\Omega)^n \). Now, the monotonicity assumption serves to pass to the limit inside the non-linearity: First, we note that \( \int_\Omega \xi \cdot \nabla v \, dx = \int_{\partial \Omega} fv \, ds \) for all \( v \in H^1_0(\Omega) \). Then the discrete problem (27) and the monotonicity \( \int_\Omega (a(\nabla u_n) - a(\nabla w)) \cdot (\nabla u_n - \nabla w) \, dx \geq 0 \) imply that
\[
\int_\Omega (\xi - a(\nabla w)) \cdot \nabla (u - w) \, dx \geq 0 \quad \text{for all } w \in H^1_0(\Omega).
\]
Plugging in \( w = u + \lambda v \) and letting \( \lambda \to 0 \) yields that
\[
\int_\Omega a(\nabla u) \cdot \nabla v \, dx = \int_\Omega \xi \cdot \nabla v \, dx = \langle f, v \rangle_{\partial \Omega}
\]
for all \( v \in H^1_0(\Omega) \). Uniqueness of the solution \( u \) follows from monotonicity. \( \Box \)

In impedance tomography one tries to recover information on the interior conductivity distribution by applying currents to the object and measuring the potential \( u \), solution to (26), on the boundary. Mathematically, the available information in impedance tomography is encoded in the Neumann-to-Dirichlet operator \( \Lambda : f \to u|_{\partial D} \). The last theorem shows that \( \Lambda \) is well-defined and bounded from \( H^{-1/2}_0(\partial \Omega) \) into \( H^{1/2}_0(\partial \Omega) = \{ g \in H^{1/2}(\partial \Omega), \langle g, 1 \rangle_{\partial \Omega} = 0 \} \), \( \|\Lambda f\|_{H^{1/2}_0(\partial \Omega)} \leq C \|f\|_{H^{-1/2}_0(\partial \Omega)} \). By \( \Lambda_0 \) we denote the corresponding operator where the non-linearity \( a(x,p) \) is replaced by \( p \), such that the governing equation becomes a Laplace equation. Our characterization result indeed requires not only the data \( \Lambda \) by also the background measurements \( \Lambda_0 \). Since the domain \( \Omega \) is assumed to be known, these additional data can, in principle, be computed numerically. We also note that the assumption that \( \alpha > 1 \) is crucial for the method.

We follow the technique that we used to tackle the inverse scattering problem for non-linear materials: First we show a factorization of the measurement operator \( \Lambda - \Lambda_0 \) and then prove the necessary properties to obtain an explicit characterization of \( D \) in terms of \( \Lambda - \Lambda_0 \). To formulate the factorization, we need a couple of auxiliary operators. Define
\[
G : H^{-1/2}_0(\partial D) := \{ g \in H^{-1/2}(\partial D), \langle g, 1 \rangle_{\partial \Omega} = 0 \} \to H^{1/2}_0(\partial \Omega)
\]
by \( G \phi = u|_{\partial \Omega} \) where \( u \in H^1_0(\Omega \setminus \overline{\mathcal{T}}) = \{ v \in H^1(\Omega \setminus \overline{\mathcal{T}}), \langle v, 1 \rangle_{\partial \Omega} = 0 \} \) is a (variational) solution to
\[
\Delta u = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{T}}, \quad \frac{\partial u}{\partial \nu} = \phi \quad \text{on } \partial D, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\]
The adjoint operator $G^*$ is bounded from $H^{-1/2}_0(\partial \Omega)$ to

$$H^{-1/2}_0(\partial D) = \{ \phi \in H^{1/2}(\partial D), \langle \phi, 1 \rangle_{\partial D} = 0 \}.$$ 

For $g \in H^{-1/2}_0(\partial \Omega)$, $G^* g = u|_{\partial D}$ where $u \in H^1_0(\Omega \setminus \overline{D})$ solves

$$\Delta u = 0 \text{ in } \Omega \setminus \overline{D}, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D, \quad \frac{\partial u}{\partial \nu} = -g \text{ on } \partial \Omega. \quad (29)$$

We also define $M : H^{1/2}_0(\partial D) \to H^{-1/2}_0(\partial D)$ by setting $M[\phi] = \partial u/\partial \nu|_{\partial D}$ where $u \in H^1_0(\Omega \setminus D) = \{ v \in H^1(\Omega \setminus D), \langle v, 1 \rangle_{\partial \Omega} = 0 \}$ solves the following transmission problem,

$$\text{div } a(\nabla u) = 0 \text{ in } \Omega \setminus \partial D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \quad [u]|_{\partial D} = \phi, \quad [\nu \cdot a(\nabla u)]|_{\partial D} = 0. \quad (30)$$

The variational form corresponding to this problem is to find $u \in H^1_0(\Omega \setminus \partial D)$ such that $[u]|_{\partial D} = \phi$ and

$$\int_{\Omega} a(\nabla u) \cdot \nabla v \, dx = 0 \quad \text{for all } v \in H^1_0(\Omega). \quad (31)$$

Using Browder’s and Minty’s technique it can be shown that this problem is uniquely solvable for all $\phi \in H^{1/2}_0(\partial D)$ and that the solution operator is bounded. To this end, we note that by linearity of the problem in $\Omega \setminus \overline{D}$ choosing any function $w \in H^1_0(\Omega \setminus \overline{D})$ with $w = \phi$ on $\partial D$ and $w = 0$ on $\partial \Omega$ allows to rewrite (31) for the new unknown $\tilde{u} \in H^1_0(\Omega)$, $\tilde{u} = u$ in $D$ and $\tilde{u} = u - w \in H^1(\Omega \setminus \overline{D})$, as

$$\int_{\Omega} a(\nabla \tilde{u}) \cdot \nabla v \, dx = -\int_{\Omega \setminus \overline{D}} \nabla w \cdot \nabla v \, dx \quad \text{for all } v \in H^1_0(\Omega). \quad (32)$$

By $M_0$ we denote the corresponding operator where $a(\nabla u)$ is replaced by $\nabla u$ in (30).

**Theorem 5.2.** The factorization $\Lambda - \Lambda_0 = G(M - M_0)|G^*$ holds. The operators $G : H^{-1/2}_0(\partial D) \to H^{1/2}_0(\partial \Omega)$ and $G^* : H^{-1/2}_0(\partial \Omega) \to H^{1/2}_0(\partial D)$ are compact and injective. Further, $M$ is bounded, continuous, and $M_0 - M : H^{1/2}_0(\partial D) \to H^{-1/2}_0(\partial D)$ is coercive on any bounded subset of $H^{1/2}_0(\partial D)$.

**Proof.** Define the auxiliary operator $L : H^{-1/2}_0(\partial \Omega) \to H^{-1/2}_0(\partial D)$ by $L(g) = \partial u/\partial \nu|_{\partial D}$ where $u \in H^1_0(\Omega)$ solves

$$\text{div } a(\nabla u) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial \Omega. \quad (33)$$

The operator $L_0$ is again defined via the corresponding problem where $a(\nabla u)$ is replaced by $\nabla u$. We note that $\Lambda - \Lambda_0 = G(L - L_0)$. Indeed, let $\Lambda[g] = u|_{\partial \Omega}$ and $\Lambda_0 g = u_0|_{\partial \Omega}$ where $u$ solves (31) and $u_0$ solves the corresponding Laplace problem.

Then $w = u - u_0$ solves the following problem in $\Omega \setminus \overline{D}$,

$$\Delta w = 0 \quad \text{in } D, \quad \frac{\partial w}{\partial \nu} = \frac{\partial}{\partial \nu}(u - u_0) \quad \text{on } \partial D, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega.$$ 

Thus, (28) shows that

$$G \left( \frac{\partial(u - u_0)}{\partial \nu} \right) = u - u_0|_{\partial \Omega} = (\Lambda - \Lambda_0)[g].$$
Consequently, \( A - A_0 = G(L - L_0) \). Now we show that \( L = MG^* \) (the proof that \( L_0 = M_0[G^*] \) being similar and the factorization \( A - A_0 = G(M - M_0)[G^*] \) follows. Let \( Lg = \partial w/\partial \nu|_{\partial D} \) where \( w \in H^1_0(\Omega) \) solves (33), that is,

\[
\text{div} a(\nabla w) = 0 \quad \text{in} \ \Omega, \quad \frac{\partial w}{\partial \nu} = g \quad \text{on} \ \partial \Omega.
\]  

(34)

Further, let \( MG^*g = \partial v/\partial \nu|_{\partial D} \) where \( v \in H^1_0(\Omega \setminus \partial D) \) satisfies (30), that is,

\[
\text{div} a(\nabla v) = 0 \quad \text{in} \ \Omega \setminus \partial D, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \ \partial D, \quad [v]|_{\partial D} = G^*g, \quad \left[ \frac{\partial v}{\partial \nu} \right]_{\partial D} = 0.
\]

Finally, let \( G^*g = u|_{\partial D} \) where \( u \in H^1_0(\Omega \setminus \partial D) \) satisfies

\[
\Delta u = 0 \quad \text{in} \ \Omega \setminus \partial D, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial D, \quad \frac{\partial u}{\partial \nu} = -g \quad \text{on} \ \partial \Omega.
\]

Set

\[
z = \begin{cases} 
v - u & \text{in} \ \Omega \setminus \partial D, \\
v & \text{in} \ \partial D.
\end{cases}
\]

Then \( v \in H^1_0(\Omega) \) and the equations satisfied by \( u \) and \( v \) imply that

\[
\text{div} a(\nabla z) = 0 \quad \text{in} \ \Omega, \quad z|_{\partial D}^+ - z|_{\partial D}^- = 0,
\]

\[
\frac{\partial z}{\partial \nu}|_{\partial D}^+ - \nu \cdot a(\nabla z)|_{\partial D}^- = 0, \quad \frac{\partial z}{\partial \nu} = g \quad \text{on} \ \partial \Omega.
\]

Hence, \( z \) solves the same boundary value problem as \( w \) from (34). Theorem 5.1 ensures unique solvability of this problem, thus \( z = w \) and hence \( Lg = MG^*g \).

For a proof of compactness and injectivity of \( G \) and its adjoint \( G^* \) we refer to [18, Theorem 6.3]. Boundedness and continuity of \( M \) follow from monotonicity of \( a \) and, e.g., (32) and we will not detail the proof here, but we show the coercivity of \( M_0 - M \). Assume that \( \phi \in H^{1/2}(\partial D) \) and denote

\[
M[\phi] = \frac{\partial u}{\partial \nu}|_{\partial D}, \quad M_0[\phi] = \frac{\partial u_0}{\partial \nu}|_{\partial D}
\]

for \( u \in H^1_0(\Omega \setminus \partial D) \) solving (30) for data \( \phi \) and \( u_0 \) solving the same problem with \( a(\nabla u) \) replaced by \( \nabla u_0 \). Green’s formula implies

\[
\langle M_0[\phi] - M[\phi], \phi \rangle_{\partial D} = -\left( \frac{\partial u_0}{\partial \nu} - \frac{\partial u}{\partial \nu}, \ u_0^+ - u^- \right)_{\partial D}
\]

\[
= \int_{\Omega} a(\nabla u) \cdot \nabla u \ dx - \int_{\Omega} |\nabla u_0|^2 \ dx
\]

\[
\geq (\alpha - 1) \| \nabla u \|^2_{L^2(\Omega)} + \| \nabla u_0 \|^2_{L^2(\Omega)} - \| \nabla u_0 \|^2_{L^2(\Omega)}
\]

\[
= (\alpha - 1) \| \nabla u \|^2_{L^2(\Omega)} + \int_{\Omega} (|\nabla u|^2 - |\nabla u_0|^2) \ dx
\]

\[
= (\alpha - 1) \| \nabla u \|^2_{L^2(\Omega)} + \| \nabla(u - u_0) \|^2_{L^2(\Omega)}
\]

\[
- 2 \int_{\Omega} \nabla u_0 \cdot \nabla (u - u_0) \ dx.
\]

\[=0\]
Now, assume that $\langle M_0\phi - M[\phi], \phi \rangle_{\partial D} = 0$ for $\phi \in H_0^{1/2}(\partial D)$. Then $\nabla u = 0$ in $D$ and hence $\nabla u_0 = 0$ in $D$, too. Note that $u_0 \in H^2_0(\Omega \setminus \overline{D})$ solves a linear problem

$$\Delta u_0 = 0 \quad \text{in} \quad \Omega \setminus \partial D, \quad \frac{\partial u_0}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \quad [u_0]|_{\partial D} = \phi, \quad \left[\frac{\partial u_0}{\partial \nu}\right]|_{\partial D} = 0.$$

Since $u_0$ is constant inside $D$, $\partial u_0/\partial \nu$ vanishes on $\partial D$ and unique solvability of the Neumann problem in $H^2_0(\Omega \setminus \overline{D})$ implies that $u$ vanishes in $\Omega \setminus \overline{D}$. Hence, $\phi$ equals a constant and the constraint $\langle \phi, 1 \rangle_{\partial \Omega} = 0$ yields $\phi = 0$, contradiction. Consequently, $\langle M_0\phi - M[\phi], \phi \rangle_{\partial D} > 0$ for $\phi \in H_0^{1/2}(\partial D)$. We claim that for any $C > 0$ there is $c > 0$ such that

$$\langle M_0\phi - M[\phi], \phi \rangle_{\partial D} \geq c\|\phi\|_{H_0^{1/2}(\partial D)}^2 \quad \text{for} \quad \phi \in H_0^{1/2}(\partial D), \|\phi\|_{H_0^{1/2}(\partial D)} \leq C.$$

Indeed, otherwise, there is a sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset H_0^{1/2}(\partial D)$ such that $\langle M_0\phi - M[\phi], \phi \rangle_{\partial D}$ tends to $0$ but $\|\phi_n\|_{H_0^{1/2}(\partial D)}$ does not tend to $0$. Then $\nabla u_n$ and $\nabla u_{0,n}$ tend to zero in $L^2(D)$. In consequence, $u_{0,n}$ converges to a constant in $D$ and $\partial u_{0,n}/\partial \nu|_{\partial D}$ converges to $0$ as $n \to \infty$. Since $u_{0,n} \in H^2_0(\Omega \setminus \overline{D})$ solves a Neumann problem for the Laplace equation in $\Omega \setminus \overline{D}$ with homogeneous Neumann data on $\partial \Omega$ and Neumann data $\partial u_{0,n}/\partial \nu|_{\partial D}$ on $\partial D$ we obtain that $u_{0,n}$ tends to zero in $H^2_0(\Omega \setminus \overline{D})$ as $n \to \infty$. This implies that $\phi_n = [u_{0,n}]_{\partial D}$ converges to a constant value in $H^{1/2}_0(\partial D)$. Again, the constraint $\langle \phi_n, 1 \rangle_{\partial D} = 0$ implies that this constant vanishes, and yields a contradiction.

Again, we characterize the non-linear inclusion by special test functions that are obtained from a Green's function for the background (Laplace) problem. We define $N(x,y), x \neq y \in \Omega$ to be the Neumann function for the Laplacian in $\Omega$. In the distributional sense,

$$\Delta N(\cdot, y) = -\delta_y \quad \text{in} \quad \Omega, \quad \frac{\partial N(\cdot, y)}{\partial \nu} = -\frac{1}{|\partial \Omega|}, \quad \int_{\partial \Omega} N(\cdot, y) \, ds = 0 \quad \text{for} \quad y \in \Omega.$$

For a unit vector $d \in \mathbb{R}^m$ we set

$$\phi_y(x) = d \cdot \nabla_y N(x,y), \quad x \in \partial \Omega, \quad y \in \Omega.$$

**Theorem 5.3.** Assume that Assumption (25) holds. Then a point $y \in \Omega$ belongs to $D$ if and only if

$$\inf \left\{ \frac{\langle (\Lambda - \Lambda_0)[g], g \rangle_{L_2^0(\partial \Omega)}}{\langle \phi_y, g \rangle_{L_2^0(\partial \Omega)}} \mid \|g\|_{L_2^0(\partial \Omega)} = 1, \langle \phi_y, g \rangle_{L_2^0(\partial \Omega)} \neq 0 \right\} > 0.$$

**Proof.** We apply Theorem 2.1 to the factorization $\Lambda_0 - \Lambda = G(M_0 - M)[G^*]$, with $F = \Lambda_0 - \Lambda, H = G^*$ and $T = M_0 - M$. However, to fit into the theoretical framework of Theorem 2.1 we need to restrict the domain of definition of $\Lambda - \Lambda_0$ to the Hilbert space $L_2^0(\partial \Omega)$. Consider $\Lambda - \Lambda_0 : L_2^0(\partial \Omega) \to L_2^0(\partial \Omega), \quad G^* : L_2^0(\partial \Omega) \to H_0^{1/2}(\partial D)$ and $G : H_0^{-1/2}(\partial D) \to L_2^0(\partial \Omega)$. Following the notations of Theorem 2.1 we also set $T = M_0 - M$. The assumptions concerning $G^*$ and $M_0 - M$ have been checked in Theorem 5.2. Thus, we obtain that $f \in L_2^0(\partial \Omega)$ belongs to the range of $G$ if and only if

$$\inf \left\{ \frac{\langle (\Lambda - \Lambda_0)[g], g \rangle_{L_2^0(\partial \Omega)}}{\langle f, g \rangle_{L_2^0(\partial \Omega)}} \mid \|g\|_{L_2^0(\partial \Omega)} = 1, \langle f, g \rangle_{L_2^0(\partial \Omega)} \neq 0 \right\} > 0.$$

Finally, Theorem 6.6 in [18] shows that \( \phi_y \) belongs to the range of \( G \) if and only if \( y \in D \). Combining the last two facts shows the claim. \( \square \)

**Appendix A. Non-Linear Transmission Eigenvalues.** We call \( k^2 > 0 \) a non-linear interior transmission eigenvalue if there are functions \( u \in L^2(D) \) and \( w \in L^2(D) \) such that \( u - w \in H^1_0(D) \) and
\[
\Delta u + k^2 n(u) = 0 \quad \text{and} \quad \Delta w + k^2 w = 0 \quad \text{in } D.
\]
Both equations are interpreted in the distributional sense, that is,
\[
\int_D (u \Delta \psi + k^2 n(u) \psi) \, dx = 0 \quad \text{and} \quad \int_D u (\Delta \overline{\psi} + k^2 \overline{\psi}) \, dx = 0
\]
for all \( \psi \in C_0^\infty(D) \). Note that linear transmission eigenvalues are defined analogously, replacing \( n(u) \) by \( n_L u \), see, e.g., [3]. The difference \( v = u - w \) can be extended by zero outside \( D \) and the extension solves
\[
\Delta v + k^2 v = -k^2 q(v + w) \quad \text{in } \mathbb{R}^m,
\]
and, trivially, also the Sommerfeld radiation condition. By Proposition 3.2, this problem is equivalent to find a solution \( v \in H^1_0(D) \) to
\[
v - k^2 V(q(v + w)) = 0
\]
in \( D \). We show in this section that, similar to the linear situation, absorption of the medium implies that there are no real interior transmission eigenvalues. Further, we show that there are no transmission eigenvalues if the material contrast is small enough.

Assume that \( k^2 \) is a transmission eigenvalue with eigenfunction \( (u, w) \neq (0,0) \), set \( v = u - w \) and extend \( v \) by zero to \( \mathbb{R}^m \). In the proof of Proposition 4.1 we showed that
\[
\text{Im} \langle [w], w \rangle_{L^2(D)} = \text{Im} \int_D k^2 q(v + w)(\nabla \overline{w}) \, dx + \frac{1}{4\pi^2} \int_S |v^\infty|^2 \, ds \quad \text{(37)}
\]
The integral \( \int_S |v^\infty|^2 \, ds \) vanishes since \( v \) vanishes outside \( D \). Due to the assumption (15) on \( q \), also \( \text{Im} \int_D k^2 q(v + w)(\nabla \overline{w}) \, dx \) needs to vanish. Assume now additionally that (15) holds with strict inequality,
\[
\text{Im} \int_D q(f) \overline{f} \, dx > 0 \quad \text{for all } 0 \neq f \in L^2(D). \quad \text{(38)}
\]
Then (37) yields that \( v + w \) vanishes inside \( D \). But then \( \Delta v + k^2 v = 0 \) and the Cauchy data of \( v \) vanish on \( \partial D \), which implies that \( v \) vanishes in \( D \). Necessarily, \( w \) vanishes, too. This contradicts our initial assumption that \( (u, w) \neq (0,0) \).

**Corollary A.1.** If the non-linear medium is absorbing, that is, (38) holds, then there are no non-linear interior transmission eigenvalues.

Let us now replace assumption (38) by the assumption that \( q \) is small enough to satisfy condition (19). Then Lemma 4.2 implies that \( T \) is coercive on \( L^2(D) \). However, if \( k^2 \) is a transmission eigenvalue with eigenfunction \( (u, w) \neq (0,0) \),
\[
\langle [w], w \rangle_{L^2(D)} = k^2 \int_D q(v + w) \overline{\omega} \, dx = - \int_D (\Delta v + k^2 v) \overline{\omega} \, dx = 0
\]
due to the second equation in (35) and density of \( C_0^\infty(D) \) in \( H^1_0(D) \). Since \( T \) is coercive, this implies that \( w \) vanishes on \( D \) and, furthermore, that \( v \) vanishes on \( D \), contradiction.
Corollary A.2. If the non-linear contrast $q$ and the wave number $k$ satisfy (19), then $k^2$ is not a non-linear interior transmission eigenvalue.

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