

The Operator Equations of Lippmann-Schwinger Type for Acoustic and Electromagnetic Scattering Problems in L^2

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Abstract

This paper is concerned with the scattering of acoustic and electromagnetic time harmonic plane waves by an inhomogeneous medium. These problems can be translated into volume integral equations of the second kind – the most prominent example is the Lippmann-Schwinger integral equation. In this work, we study a particular class of scattering problems where the integral operator in the corresponding operator equation of Lippmann-Schwinger type fails to be compact. Such integral equations typically arise if the modeling of the inhomogeneous medium necessitates space dependent coefficients in the highest order terms of the underlying partial differential equation. The two examples treated here are acoustic scattering from a medium with a space dependent material density and electromagnetic medium scattering where both the electric permittivity and the magnetic permeability vary. In these cases, Riesz theory is not applicable for the solution of the arising integral equations of Lippmann-Schwinger type. Therefore we show that positivity assumptions on the relative material parameters allow to prove positivity of the arising volume potentials in tailor-made weighted spaces of square integrable functions. This result merely holds for imaginary wavenumber and we exploit a compactness argument to conclude that the arising integral equations are of Fredholm type, even if the integral operators themselves are not compact. Finally, we explain how the solution of the integral equations in L^2 affects the notion of a solution of the scattering problem and illustrate why the order of convergence of a Galerkin scheme set up in L^2 does not suffer from our L^2 setting, compared to schemes in higher order Sobolev spaces.

1 Introduction

Coercivity and compactness are two important properties for the analysis of integral equations. For equations of the second kind such as the Lippmann-Schwinger integral equation, exploiting compactness of the integral operator by using Riesz theory is of

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course favorable whenever it is possible. However, there are important scattering problems where the corresponding integral operator fails to be compact. Take for instance electromagnetic medium scattering where both the electric permittivity ϵ and the magnetic permeability μ in Maxwell's equations

$$\operatorname{curl} E - i\omega\mu H = 0, \quad \operatorname{curl} H + i\omega\epsilon E = \sigma E \quad \text{in } \mathbb{R}^3, \quad (1)$$

are space dependent [11]. Acoustic scattering problems for media with constant speed of sound but varying material density a in the governing equation

$$\operatorname{div} (a\nabla u) + k^2 u = 0 \quad \text{in } \mathbb{R}^3, \quad (2)$$

also lead to a second-kind integral equation of Lippmann-Schwinger type where the integral operator lacks compactness on any reasonable function space. The same integral equation appears in a two dimensional setting when one studies electromagnetic scattering from an orthotropic medium [2].

In this paper, we analyze such volume integral equations by replacing the lacking compactness of the integral operators by positivity, which is implicitly introduced via positivity constraints on the coefficients of the partial differential equations. In our opinion, a natural setting for this analysis are weighted spaces of square integrable functions. The correct choice of a weight function is shown to be crucial to obtain positivity of the integral operator in a weighted space for wavenumber $k = i$. Hence, the sum of the identity and the integral operator is coercive for complex wavenumber. Afterwards, we use a compactness argument to show that the integral equation of the second kind is of Fredholm type for arbitrary positive k . This procedure applies for both the acoustic and electromagnetic medium scattering problems described above. However, note that we have to accept smoothness conditions on the coefficients: Roughly speaking, in the acoustic case, the relative material density $a > 0$ needs to be such that the non-negative function $a - 1$ possesses a piecewise weakly differentiable square root. In the electromagnetic case, the electric permittivity ϵ needs to satisfy a similar differentiability and sign condition. The magnetic permeability μ in the above formulation of Maxwell's equations merely needs to meet a sign but no differentiability condition. These low assumptions on regularity of the coefficients allow to solve the scattering problems in spaces with low regularity. We explain, for the acoustic and electromagnetic case, that the solutions we obtain can be interpreted as ultra weak solutions of the scattering problem – especially, they are in general weaker than weak solutions.

Finally, we briefly discuss Galerkin methods in L^2 for the numerical approximation of the investigated integral equations of Lippmann-Schwinger type. Our main objective in this topic is to explain that the analysis and numerical approximation of the operator equation in L^2 does not destroy the order of convergence of the approximate solution, even in higher order Sobolev spaces if the right-hand side of the problem is smooth enough. We would like to point out here that one of the main motivations of our study is indeed the numerical analysis of medium scattering problems. To be more precise, the results in this paper might serve as a gateway to interpret Vainikko's fast solver for the Lippmann-Schwinger equation [16, 15, 5] as a Galerkin method in L^2 for all the above mentioned electromagnetic and acoustic scattering problems. This is, to the best of our knowledge,

not yet possible today, and for such an interpretation, the L^2 theory established in this paper seems to be crucial.

Most similar to our study of scattering problems leading to non-compact volume integral operators seems to be the paper of Potthast [14]. He studies electromagnetic scattering from an orthotropic medium which leads to a two-dimensional anisotropic scattering problem with governing equation analogous to (2). In his paper, he reformulates the Lippmann-Schwinger equation as a system of strongly singular integral equations. Similar to our study, the paper [14] also uses spaces of square integrable functions and similar to our detour to wavenumber $k = i$, Potthast uses special results for the wavenumber $k = 0$ in his arguments. Apart from integral equation methods, there are a couple of papers which solve scattering problems corresponding to equations (1) or (2) by variational methods using exterior Dirichlet-to-Neumann maps, see, e.g., [13, 4, 2]. We also want to put emphasize on the recent work [6], where, roughly speaking, similar electromagnetic scattering problems as in this present paper are investigated using different methods in more regular spaces. To put it in a nutshell, the analysis in [6] uses variational methods to show existence of weak solutions of the volume integral equations in $H(\text{curl})$. In the present work we go the other way round: we prove solvability of the integral equation in L^2 to show existence of ultra weak solutions which satisfy again a variational principle.

Let us briefly explain the structure of this paper. We start in the next section with a derivation of the Lippmann-Schwinger-like integral equation describing acoustic scattering from a medium with space dependent material density. This equation is first briefly analyzed in a weighted H^1 space to present our concepts in a more well-known setting. Afterwards, in Section 3 we investigate the same integral equations in a weighted L^2 space and discuss the notion of solution which we obtain for the scattering problem. Section 4 generalized this theory to Maxwell's equations and the last Section 5 contains some remarks concerning the numerical analysis of the operator equations of Lippmann-Schwinger type in the acoustic case.

2 An Operator Equation of Lippmann-Schwinger Type

This section deals with the derivation of a volume integral equation of the second kind to describe scattering of acoustic waves from a medium with constant speed of sound but varying material density. The material density is characterized by a real-valued function $a \in L^\infty(\mathbb{R}^3)$ such that $a = 1$ outside some region $D \subset \mathbb{R}^3$. We assume that the exterior of D is connected, the boundary of D is Lipschitz and that there exists $\alpha_0 > 0$ such that $a(x) \geq \alpha_0$ for almost all $x \in D$. In this paper we restrict ourselves to the three dimensional case. We note, however, that we do not use any particular feature of \mathbb{R}^3 .

A time harmonic plane wave $u^i(x) = \exp(ikx \cdot \hat{\theta})$ is described by the wave number $k = \omega/c$ with frequency ω , speed of sound c , and the direction of incidence $\hat{\theta} \in S^2$. In the case of a being smooth, i.e. $a|_D \in C^1(\bar{D})$, the scattering of u^i by the medium is modeled by the classical interpretation of the equation

$$\operatorname{div}(a\nabla u) + k^2u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial D \tag{3}$$

for the total field $u = u^i + u^s$. The scattered field u^s has to satisfy the Sommerfeld

radiation condition

$$\frac{\partial u^s}{\partial r} - ik u^s = \mathcal{O}\left(\frac{1}{r^2}\right), \quad r = |x| \rightarrow \infty, \quad (4)$$

uniformly with respect to $\hat{x} = x/|x|$ in the unit sphere S^2 . Furthermore, on the interface ∂D the following transmission conditions have to be satisfied: $u_+ = u_-$ and $\partial u_+/\partial \nu = (\nu^\top a \nabla u)_-$ where $\nu = \nu(x)$ denotes the exterior unit normal vector at $x \in \partial D$ and the subscripts $+$ and $-$ indicate that the trace on ∂D is taken from the exterior and interior of D , respectively.

For general $a \in L^\infty(\mathbb{R}^3)$ the solution $u \in H_{\text{loc}}^1(\mathbb{R}^3)$ has to be understood in the variational sense,

$$\iint_{\mathbb{R}^3} [a \nabla \psi^* \nabla u - k^2 \bar{\psi} u] dx = 0 \quad (5)$$

for all $\psi \in H^1(\mathbb{R}^3)$ with compact support. Here, $H^1(\Omega)$ denotes the usual Sobolev space of order 1 for any region Ω , and

$$H_{\text{loc}}^1(\mathbb{R}^3) = \{u : \mathbb{R}^3 \rightarrow \mathbb{C}, u|_B \in H^1(B) \text{ for every ball } B\}.$$

There are several results on uniqueness and existence of solutions of this scattering problem. Potthast (in [14]) studied the two-dimensional case and proved uniqueness for $a \in C^1(\mathbb{R}^2)$ by a unique continuation argument. He showed existence by transforming the problem to a singular integral equation. Piana (in [13]) allowed inhomogeneous transmission conditions, and Hähner (in [4]) coupled the variational equation with the exterior Dirichlet-Neumann map.

We will derive the same integral equation as in [14] but treat it in a different and, in our opinion, more natural way. The following lines repeat the arguments of [7].

We formulate the variational equation (5) for the scattered field u^s as

$$\iint_{\mathbb{R}^3} [a \nabla \psi^* \nabla u^s - k^2 \bar{\psi} u^s] dx = - \iint_D \nabla \psi^* \nabla u^i q dx \quad (6)$$

for all $\psi \in H^1(\mathbb{R}^3)$ with compact support. Here, we have set $q = a - 1$ and used the fact that the incident field satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in \mathbb{R}^3 and thus, by Greens formula,

$$\iint_{\mathbb{R}^3} [\nabla \psi^* \nabla u^i - k^2 \bar{\psi} u^i] dx = 0$$

for all $\psi \in H^1(\mathbb{R}^3)$ with compact support. Equation (6) is a special case of

$$\iint_{\mathbb{R}^3} [a \nabla \psi^* \nabla v - k^2 \bar{\psi} v] dx = - \iint_D \nabla \psi^* f dx \quad (7)$$

for all $\psi \in H^1(\mathbb{R}^3)$ where $f \in L^2(D)^3$ is any given function. (For (6) set $f = q \nabla u^i$.) Equation (7) is the variational form of

$$\text{div}(a \nabla v) + k^2 v = -\text{div } f \quad \text{in } \mathbb{R}^3 \setminus \partial D, \quad (8)$$

and $v_+ = v_-$ and $(a\nu^\top \nabla v)_- - \partial v_+ / \partial \nu = -\nu^\top f$ on ∂D .

Let Φ_k be the fundamental solution of the scalar Helmholtz equation,

$$\Phi_k(x, y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, \quad x \neq y.$$

As shown in [7] the following operator $L_k : L^2(D)^3 \rightarrow H^1(D)$ is well defined and bounded:

$$(L_k g)(x) = \operatorname{div} \int_D g(y) \Phi_k(x, y) dy, \quad x \in D.$$

Furthermore, the extension $w \in H_{\text{loc}}^1(\mathbb{R}^3)$ of $L_k g$ to \mathbb{R}^3 ,

$$w(x) = \operatorname{div} \int_D g(y) \Phi_k(x, y) dy, \quad x \in \mathbb{R}^3,$$

solves

$$\iint_{\mathbb{R}^3} [\nabla \psi^* \nabla w - k^2 \bar{\psi} w] dx = - \iint_D \nabla \psi^* g dx$$

for all $\psi \in H^1(\mathbb{R}^3)$ with compact support.

Now we turn back to the variational equation (7) and write it as

$$\iint_{\mathbb{R}^3} [\nabla \psi^* \nabla v - k^2 \bar{\psi} v] dx = - \iint_D \nabla \psi^* (f + q \nabla v) dx,$$

which we can write as

$$v = L_k(f + q \nabla v) \tag{9}$$

in $H^1(D)$. This operator equation (or integro-differential equation) corresponds to the Lippmann-Schwinger equation, see [3]. To study this equation we rewrite it as

$$v - L_i(q \nabla v) + (L_i - L_k)(q \nabla v) = L_k f.$$

It is easily seen that the operator $v \mapsto v - L_i(q \nabla v)$ is an isomorphism from $H^1(D)$ onto itself. We even show that – in the case that $a > 1$, that is, $q > 0$, on D – it is coercive with respect to the inner product

$$\langle f, g \rangle_{H_q^1(D)} = \iint_D [q \nabla g^* \nabla f + f \bar{g}] dx.$$

By $H_q^1(D)$ we denote the completion of $H^1(D)$ with respect to the norm $\|f\|_{H_q^1(D)} = \langle f, f \rangle_{H_q^1(D)}^{1/2}$. We note that $\|f\|_{H_q^1(D)}$ is an equivalent norm in $H^1(D)$ provided that q is bounded below by some positive constant $q_0 > 0$.

Lemma 1. *Let $q > 0$ on D .*

- (a) *The operator $v \mapsto L_k(q\nabla v)$ is bounded also as an operator from $H_q^1(D)$ into itself.*
(b) *The operator $v \mapsto v - L_i(q\nabla v)$ is coercive in $H_q^1(D)$,*

$$\operatorname{Re} \langle v - L_i(q\nabla v), v \rangle_{H_q^1(D)} \geq \frac{1}{2} \|v\|_{H_q^1(D)}^2 \quad \text{for all } v \in H_q^1(D). \quad (10)$$

- (c) *The operator $Kv = (L_k - L_i)(q\nabla v)$ is compact in $H^1(D)$ as well as in $H_q^1(D)$.*

Proof. (a) We have seen above that L_k is bounded from $L^2(D)^3$ into $H^1(D)$. Furthermore, $v \mapsto q\nabla v$ is bounded from $H_q^1(D)$ into $L^2(D)^3$. Finally, the imbedding $H^1(D) \subset H_q^1(D)$ is also bounded.

- (b) Let $v \in H_q^1(D)$ and define w by

$$w = \operatorname{div} \int_D q(y) \nabla v(y) \Phi_i(\cdot, y) dy \quad \text{in } \mathbb{R}^3.$$

Then (see [8]) $w \in H^1(\mathbb{R}^3)$ decays exponentially to zero as $|x|$ tends to infinity, and $\Delta w - w = -\operatorname{div}(q\nabla v)$ in the weak sense,

$$-\iint_{\mathbb{R}^3} [\nabla \psi^* \nabla w + \bar{\psi} w] dx = \iint_D q \nabla \psi^* \nabla v dx \quad \text{for all } \psi \in H^1(\mathbb{R}^3). \quad (11)$$

Therefore, substituting $\psi = w$ yields

$$\begin{aligned} \operatorname{Re} \langle v - L_i(q\nabla v), v \rangle_{H_q^1(D)} &= \operatorname{Re} \iint_D [q |\nabla v|^2 + |v|^2 - q \nabla v^* \nabla w - w \bar{v}] dx \\ &= \iint_D [q |\nabla v|^2 + |v|^2 - \operatorname{Re}(w \bar{v})] dx + \iint_{\mathbb{R}^3} [|\nabla w|^2 + |w|^2] dx \\ &\geq \iint_D [q |\nabla v|^2 + |v|^2 + |w|^2 - \operatorname{Re}(w \bar{v})] dx \\ &\geq \frac{1}{2} \|v\|_{H_q^1(D)}^2 + \frac{1}{2} \iint_D \underbrace{[|v|^2 + |w|^2 - 2 \operatorname{Re}(w \bar{v})]}_{=|v-w|^2} dx \\ &\geq \frac{1}{2} \|v\|_{H_q^1(D)}^2. \end{aligned}$$

- (c) This follows from the compactness of $L_k - L_i$ as an operator from $L^2(D)^3$ into $H^1(D)$ and the same arguments than in (a). \square

We note that this Lemma provides an estimate of Gårding's type,

$$\operatorname{Re} \langle v - L_k(q\nabla v), v \rangle_{H_q^1(D)} \geq \frac{1}{2} \|v\|_{H_q^1(D)}^2 - \operatorname{Re} \langle Kv, v \rangle_{H_q^1(D)} \quad \text{for all } v \in H_q^1(D), \quad (12)$$

where K is compact in $H_q^1(D)$. This follows from the decomposition

$$\langle v - L_k(q\nabla v), v \rangle_{H_q^1(D)} = \langle v - L_i(q\nabla v), v \rangle_{H_q^1(D)} - \langle Kv, v \rangle_{H_q^1(D)}.$$

The Riesz-Fredholm theory yields the following main result.

Theorem 2. *Let $a \in L^\infty(\mathbb{R}^3)$ satisfy $a \geq 1$ on D and $a = 1$ on $\mathbb{R}^3 \setminus D$. Furthermore, assume that the problem (7) has at most one radiating solution in $H_{\text{loc}}^1(\mathbb{R}^3)$. Then for each $f \in L^2(D)^3$ there exists a unique solution $v \in H_{\text{loc}}^1(\mathbb{R}^3)$ of (7). The restriction $v|_D$ satisfies (9).*

We note that this result yields, in particular, Fredholmness of $v \mapsto v - L_k(q\nabla v)$ which can be proven without the assumption $a \geq 1$. Indeed, it is easy to see by using the equivalence with the variational form that $v \mapsto v - L_i(q\nabla v)$ is an isomorphism. The assumption $a \geq 1$ is merely necessary for proving that $v \mapsto v - L_i(q\nabla v)$ is coercive.

The question of uniqueness is closely related to the validity of the unique continuation property of solutions of (3). It is known (cf. [10]) that this holds for $q|_D \in W^{1,\infty}(D)$.

3 Investigation of the Operator Equation in L^2

Gårding's inequality (12) provides optimal error estimates for projection methods in $H_q^1(D)$. For numerical purposes it might be more convenient to set up a numerical method in the space of square integrable functions. Moreover, numerical experiments [12] indicate that a Galerkin method in L^2 for a corresponding one dimensional problem is better conditioned than the corresponding Galerkin method in H^1 .

Therefore it is the aim of this section to study equation (9) in a weighted L^2 -space instead of $H^1(D)$ or $H_q^1(D)$. In this section we require more smoothness of q . We make the general assumption that $q > 0$ in D and \sqrt{q} is piecewise weakly differentiable in D . More precisely, we assume that $\overline{D} = \bigcup_{j=1}^N \overline{D}_j$ where D_j are domains with Lipschitz boundaries and $D_j \cap D_k = \emptyset$ for $j \neq k$. Furthermore, we suppose that the restriction $\sqrt{q}|_{D_j}$ has a bounded weak derivative for each D_j which we write as $\sqrt{q}|_{D_j} \in W^{1,\infty}(D_j)$, $j = 1, \dots, N$. Globally, \sqrt{q} belongs to

$$\tilde{W}^{1,\infty}(D) = \left\{ \rho \in L^\infty(D), \rho|_{D_j} \in W^{1,\infty}(D_j) \text{ for all } j = 1, \dots, N \right\}$$

with norm $\|\rho\|_{\tilde{W}^{1,\infty}(D)} = \max_{j=1,\dots,N} \|\rho\|_{W^{1,\infty}(D_j)}$. Hence, we allow q to have jumps or to vanish on the boundaries of D_j .

We introduce the weighted inner product $\langle u, v \rangle_{L_q^2(D)}$ as

$$\langle u, v \rangle_{L_q^2(D)} = \iint_D u(x) \overline{v(x)} q(x) dx \quad (13)$$

which yields the norm

$$\|u\|_{L_q^2(D)} = \sqrt{\iint_D |u(x)|^2 q(x) dx}.$$

By $L_q^2(D)$ we denote the completion of $C^\infty(\overline{D})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L_q^2(D)}$ and norm $\|\cdot\|_{L_q^2(D)}$ and note that $L_q^2(D)$ coincides with $L^2(D)$ (with equivalent norms) provided q is bounded away from zero on \overline{D} . We note that the space

$$\tilde{C}_0^\infty(D) := \left\{ \psi \in C_0^\infty(D), \psi|_{D_j} \in C_0^\infty(D_j) \text{ for all } j = 1, \dots, N \right\} \quad (14)$$

is dense in $L^2_q(D)$. Indeed, let $v \in L^2_q(D)$, i.e. $v\sqrt{q} \in L^2(D)$. Then $(v\sqrt{q})|_{D_j} \in L^2(D_j)$ and can be approximated by some $\psi_j \in C_0^\infty(D_j)$. Then the function ψ which coincides with ψ_j on D_j approximates $v\sqrt{q}$ in D .

Lemma 3. *The mapping*

$$v \mapsto \iint_D \nabla v(y) \Phi_k(\cdot, y) q(y) dy \quad (15)$$

has a bounded extension to an operator from $L^2_q(D)$ into $H^1(D)^3$ with bound $C\|\sqrt{q}\|_{\tilde{W}^{1,\infty}(D)}$ for some C independent of q .

Proof. First, we consider again the case $k = i$. Let $v \in \tilde{C}_0^\infty(D)$ and define for some $\ell \in \{1, 2, 3\}$

$$w = \iint_D \frac{\partial v(y)}{\partial y_\ell} \Phi_i(\cdot, y) q(y) dy \quad \text{on } \mathbb{R}^3.$$

Then $w \in C^1(\mathbb{R}^3)$ and

$$\Delta w - w = -q \frac{\partial v}{\partial x_\ell} \quad \text{on } \mathbb{R}^3 \setminus \partial D$$

where we have extended q by zero outside of \bar{D} . Furthermore, $w(x)$ decays exponentially to zero as $|x|$ tends to infinity, uniformly with respect to $x/|x|$. We compute

$$\begin{aligned} \|w\|_{H^1(D)}^2 &\leq \iint_{\mathbb{R}^3} [|\nabla w|^2 + |w|^2] dx = \iint_{\mathbb{R}^3} \bar{w} (w - \Delta w) dx \\ &= \iint_D \bar{w} q \frac{\partial v}{\partial x_\ell} dx = \sum_{j=1}^N \iint_{D_j} \bar{w} q \frac{\partial v}{\partial x_\ell} dx \\ &= - \sum_{j=1}^N \iint_{D_j} v \frac{\partial}{\partial x_\ell} (q \bar{w}) dx \quad (16) \\ &= - \sum_{j=1}^N \iint_{D_j} v \sqrt{q} \left[2\bar{w} \frac{\partial}{\partial x_\ell} \sqrt{q} + \sqrt{q} \frac{\partial \bar{w}}{\partial x_\ell} \right] dx \end{aligned}$$

since $v = 0$ on ∂D_j . Therefore, by the inequality of Cauchy-Schwarz,

$$\|w\|_{H^1(D)}^2 \leq 2 \|\sqrt{q}\|_{\tilde{W}^{1,\infty}(D)} \|v\|_{L^2_q(D)} \|w\|_{H^1(D)} \quad (17)$$

which proves the boundedness for $k = i$.

We note that the difference $\Phi_k - \Phi_i$ has the form

$$\Phi_k(x, y) - \Phi_i(x, y) = \alpha(|x - y|^2) + |x - y| \beta(|x - y|^2) \quad (18)$$

with some analytic functions $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{C}$. From this we observe that

$$\begin{aligned} v &\mapsto \iint_D \nabla v(y) (\Phi_k(\cdot, y) - \Phi_i(\cdot, y)) q(y) dy \\ &= - \sum_{j=1}^N \iint_{D_j} v(y) \nabla_y [(\Phi_k(\cdot, y) - \Phi_i(\cdot, y)) q(y)] dy \end{aligned}$$

has, by the same arguments as above, a bounded extension from $L^2_q(D)$ into $H^1(D)^3$ with operator bound $C \|\sqrt{q}\|_{\tilde{W}^{1,\infty}(D)}$ for some C independent of q . \square

Therefore, as a corollary we have shown:

Corollary 4. *The operator $v \mapsto L_k(q\nabla v)$ has a bounded extension from $L^2_q(D)$ into $L^2(D)$ with bound $C \|\sqrt{q}\|_{\tilde{W}^{1,\infty}(D)}$ for some C independent of q .*

Before we prove Gårding's estimate for the operator $v \mapsto v - L_k(q\nabla v)$ in $L^2_q(D)$ we recall the following positivity result for the volume potential:

Lemma 5. *The operator of the volume potential for $k = i$,*

$$V_i \phi = \iint_D \phi(y) \Phi_i(\cdot, y) dy$$

is non-negative in $L^2(D)$,

$$\langle V_i \phi, \phi \rangle_{L^2(D)} \geq 0 \quad \text{for all } \phi \in L^2(D).$$

Proof. Let $\phi \in C_0^\infty(D)$ and define w by $w = \iint_D \phi(y) \Phi_i(\cdot, y) dy$ in \mathbb{R}^3 . Then $\Delta w - w = -\phi$ in \mathbb{R}^3 and thus by Green's first formula (note that $w(x)$ decays exponentially to zero as $|x|$ tends to infinity)

$$\begin{aligned} \langle V_i \phi, \phi \rangle_{L^2(D)} &= \iint_{\mathbb{R}^3} w \bar{\phi} dx = \iint_{\mathbb{R}^3} w [\bar{w} - \Delta \bar{w}] dx \\ &= \iint_{\mathbb{R}^3} [|w|^2 + |\nabla w|^2] dx = \|w\|_{H^1(\mathbb{R}^3)}^2 \geq 0. \end{aligned}$$

For $\phi \in L^2(D)$ the assertion follows from the density of $C_0^\infty(D)$ in $L^2(D)$. \square

Theorem 6. *There exists a compact operator K from $L^2_q(D)$ into itself such that*

$$\operatorname{Re} \langle v - L_k(q\nabla v), v \rangle_{L^2_q(D)} \geq \|v\|_{L^2_q(D)}^2 + \operatorname{Re} \langle K v, v \rangle_{L^2_q(D)} \quad (19)$$

for all $v \in L^2_q(D)$.

Proof. Again, first we study the case $k = i$. Let $v \in \tilde{C}_0^\infty(D)$ and define the vector valued function $w \in C^1(\mathbb{R}^3)^3$ by

$$w = \iint_D \nabla v(y) \Phi_i(\cdot, y) q(y) dy \quad \text{in } \mathbb{R}^3. \quad (20)$$

Then $w = V_i(q\nabla v)$ in D and, by the previous lemma,

$$\begin{aligned} \operatorname{Re} \langle v - L_i(q\nabla v), v \rangle_{L_q^2(D)} &= \|v\|_{L_q^2(D)}^2 - \operatorname{Re} \sum_{j=1}^N \iint_{D_j} (\operatorname{div} w) \bar{v} q dx \\ &= \|v\|_{L_q^2(D)}^2 + \operatorname{Re} \sum_{j=1}^N \iint_{D_j} \nabla(qv)^* w dx + \\ &\quad + \operatorname{Re} \underbrace{\sum_{j=1}^N \int_{\partial D_j} q \bar{v} w^\top \nu ds}_{=0} \\ &= \|v\|_{L_q^2(D)}^2 + \langle V_i(q\nabla v), (q\nabla v) \rangle_{L^2(D)} \\ &\quad + \operatorname{Re} \sum_{j=1}^N \iint_{D_j} \bar{v} w^\top \nabla q dx \\ &\geq \|v\|_{L_q^2(D)}^2 + \operatorname{Re} \iint_D \bar{v} w^\top \nabla q dx \end{aligned}$$

where $\nabla q \in L^\infty(D)^3$. Therefore,

$$\begin{aligned} \operatorname{Re} \langle v - L_k(q\nabla v), v \rangle_{L_q^2(D)} &\geq \operatorname{Re} \langle (L_i - L_k)(q\nabla v), v \rangle_{L_q^2(D)} + \|v\|_{L_q^2(D)}^2 \\ &\quad + \operatorname{Re} \iint_D \bar{v} w^\top \nabla q dx \\ &= \|v\|_{L_q^2(D)}^2 + \operatorname{Re} \langle (L_i - L_k)(q\nabla v) + w \cdot \nabla q/q, v \rangle_{L_q^2(D)}. \end{aligned}$$

It remains to show compactness of the operator

$$K : v \mapsto (L_i - L_k)(q\nabla v) + \frac{\nabla q}{q} \cdot V_i(q\nabla v) \quad (21)$$

in $L_q^2(D)$. First, we consider the second part of the operator, $v \mapsto (\nabla q/q) \cdot V_i(q\nabla v)$. Let (v_n) converge to zero weakly in $L_q^2(D)$. Then $V_i(q\nabla v_n)$ converges weakly to zero in $H^1(D)^3$ by Lemma 3 and thus $V_i(q\nabla v_n) \rightarrow 0$ in $L^2(D)^3$. Since $\nabla q/\sqrt{q} = 2\nabla\sqrt{q} \in L^\infty(D)$ we conclude that also $(\nabla q/\sqrt{q}) \cdot V_i(q\nabla v_n) \rightarrow 0$ in $L^2(D)$, i.e. $(\nabla q/q) \cdot V_i(q\nabla v_n) \rightarrow 0$ in $L_q^2(D)$.

Finally, we show compactness of $(L_i - L_k)(q\nabla v)$ in $L_q^2(D)$ by a variational approach. Set $w_1 = L_k(q\nabla v)$ and $w_2 = L_i(q\nabla v)$ for $v \in \tilde{C}_0^\infty(D)$ and choose $R > 0$ such that

$\bar{D} \subset B(0, R)$. The difference $w_1 - w_2$ solves

$$-\Delta(w_1 - w_2) + (w_1 - w_2) = (1 + k^2)w_1 \quad \text{in } B(0, R). \quad (22)$$

First, we estimate $w_1 - w_2$ on the boundary $\Gamma := \partial B(0, R)$ of $B(0, R)$ and use integration by parts in the explicit representation:

$$\begin{aligned} w_1 - w_2 &= \operatorname{div} \iint_D q(y) \nabla v(y) (\Phi_k - \Phi_i)(\cdot, y) dy \\ &= -\operatorname{div} \iint_D v(y) \nabla_y [q(y) (\Phi_k - \Phi_i)(\cdot, y)] dy \\ &= -\operatorname{div} \iint_D v(y) \nabla_y (\Phi_k - \Phi_i)(\cdot, y) q(y) dy \\ &\quad - 2 \iint_D \nabla_y (\Phi_k - \Phi_i)(\cdot, y)^\top \nabla \sqrt{q(y)} v(y) \sqrt{q(y)} dy. \end{aligned}$$

Since Γ has positive distance from D the kernel of the integral operator is smooth and we can estimate

$$\|w_1 - w_2\|_{H^{1/2}(\Gamma)} \leq C \|v\|_{L^2_q(D)}$$

for some constant $C > 0$ independent of v . Now we use the fact that L_k is bounded from $L^2_q(D)$ into $L^2(D)$. Classical regularity results for (22) yield

$$\|w_1 - w_2\|_{H^1(D)} \leq C \|w_1\|_{L^2(D)} + C \|w_1 - w_2\|_{H^{1/2}(\Gamma)} \leq C \|v\|_{L^2_q(D)}.$$

Consequently, Rellich's lemma implies compactness of $v \mapsto (L_k - L_i)(q\nabla v)$ from $L^2_q(D)$ into $L^2(D)$ and thus also into $L^2_q(D)$. \square

Under the assumptions of Lemma 6, the operator $I - L_k(q\nabla \cdot)$ is hence Fredholm of index 0 on $L^2_q(D)$, as sum of a coercive and a compact operator. Hence, uniqueness in $L^2_q(D)$ implies existence in $L^2_q(D)$. Before we summarize the result we make a remark in which sense an L^2 -solution v of the integral equation

$$v - L_k(q\nabla v) = L_k f \quad (23)$$

can be interpreted as a solution of the partial differential equation (8).

Surely, v cannot be a weak solution since the variational formulation

$$\iint_{\mathbb{R}^3} [a \nabla \psi^* \nabla v - k^2 \bar{\psi} v] dx = - \iint_D \nabla \psi^* f dx \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^3) \quad (24)$$

requires $v \in H^1(D)$ to be well-defined. However, taking $\psi \in \tilde{C}_0^\infty(D)$ and integrating another time by parts in the latter equation we obtain (note that $a \in \tilde{W}^{1,\infty}(D)$)

$$\iint_{\mathbb{R}^3} v [\operatorname{div} (a \nabla \bar{\psi}) + k^2 \bar{\psi}] dx = \iint_D \nabla \psi^* f dx. \quad (25)$$

Now we define $v \in L^2(D)$ to be an ultra weak solution of (8) in D if it satisfies the latter equation for all $\psi \in \tilde{C}_0^\infty(\mathbb{R}^3)$. Of course, the integral on the right-hand side of the ultra weak formulation needs to be well defined. Since the testfunction ψ in this formulation requires two derivatives, it seems natural to choose $f \in H_0^{-1}(D)^3$, such that the right hand side in the previous equation is well defined for $\psi \in H^2(D)$. Recall that $H_0^{-1}(D)^3$ is the dual space of $H^1(D)^3$ for the inner product of $L^2(D)^3$, see [9]. Since we have chosen smooth testfunctions, the choice of the space of f is not that important for the variational formulation of the ultra weak solution. For equation (23), however, this choice is crucial. Indeed, we are going to show next that L_k is bounded from $H_0^{-1}(D)^3$ into $L_{\text{loc}}^2(\mathbb{R}^3)$. Afterwards we show for such f the existence of a unique ultra weak solution of the scattering problem (3) which satisfies the ultra weak formulation (25).

Lemma 7. *The operator L_k is bounded from $H_0^{-1}(D)^3$ into $L^2(B)$ for any ball $B \subset \mathbb{R}^3$.*

Proof. We first recall that the volume potential V_k is bounded from $L^2(\Omega)$ into $H_{\text{loc}}^2(\mathbb{R}^3)$ for any Lipschitz domain Ω . Especially, $V_k : L^2(\Omega) \rightarrow H^2(\Omega)$ is bounded. The $L^2(\Omega)$ -adjoint V_k^* is in consequence bounded from $H_0^{-2}(\Omega)$ into $L^2(\Omega)$. We recall that the dual space $H_0^{-2}(\Omega)$ of $H^2(\Omega)$ for the inner product of $L^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $H^{-2}(\Omega)$ [9]. Computing the adjoint explicitly, we obtain

$$V_k^* \phi = \iint_{\Omega} \overline{\Phi(y, \cdot)} \psi(y) dy$$

and observe that the structure of V_k and V_k^* is the same. In particular, the mapping properties of V_k^* and V_k are the same. Therefore, V_k is bounded from $L^2(\Omega)$ into $H^2(\Omega)$ and from $H_0^{-2}(\Omega)$ into $L^2(\Omega)$. The real interpolation result (see [9, Appendix B]) yields boundedness of V_k from $H_0^{-1}(\Omega)$ into $H^1(\Omega)$.

Let now $\phi \in C_0^\infty(D)$. Then the choice $\Omega = B(0, R)$ allows to estimate

$$\begin{aligned} \|V_k \phi\|_{H^1(B(0,R))} &\leq C \|\phi\|_{H_0^{-1}(B(0,R))} = C \sup_{\psi \neq 0} \frac{|\langle \phi, \psi \rangle_{L^2(B(0,R))}|}{\|\psi\|_{H^1(B(0,R))}} \\ &\leq C \sup_{\psi \neq 0} \frac{|\langle \phi, \psi \rangle_{L^2(D)}|}{\|\psi\|_{H^1(D)}} = C \|\phi\|_{H_0^{-1}(D)}. \end{aligned}$$

In consequence, V_k is bounded from $H_0^{-1}(D)$ into $H^1(B)$ and $L_k = \text{div } V_k$ is bounded from $H_0^{-1}(D)^3$ into $L^2(B)$ for any ball $B \subset \mathbb{R}^3$. \square

The previous lemma shows that the right hand side of (23) is well defined in $L_{\text{loc}}^2(\mathbb{R}^3)$ for $f \in H_0^{-1}(D)$. Under the assumption, that the corresponding homogeneous equation has only the trivial solution, we show that any solution of (23) is the unique ultra weak solution in the sense of (25).

Theorem 8. *Assume that the problem (7) has at most one radiating solution in $H_{\text{loc}}^1(\mathbb{R}^3)$. Then for any $f \in H_0^{-1}(D)$ there exists a solution $v \in L^2(D)$ of $v - L_k(q \nabla v) = L_k f$. Furthermore, v can be extended to an ultra weak solution of (3) in the sense of (25).*

Proof. From Lemma 1 and Theorem 6 we know that $v \mapsto v - L_k(q\nabla v)$ is a Fredholm operator in $H_q^1(D)$ as well as in $L_q^2(D)$. Furthermore, by the assumption and the equivalence of the variational equation (7) with equation (9) by Theorem 2 we conclude that $v \mapsto v - L_k(q\nabla v)$ is even an isomorphism from $H_q^1(D)$ onto itself. A quite general result from functional analysis (see Appendix) yields that this operator is also an isomorphism from $L_q^2(D)$ onto itself. If the right hand side g of $v - L_k(q\nabla v) = g$ belongs to $L^2(D)$ then also $v \in L^2(D)$ because $v \mapsto L_k(q\nabla v)$ maps $L_q^2(D)$ into $L^2(D)$ (see Corollary 4).

Let now $f \in H_0^{-1}(D)$. Then the right hand side $L_k f$ of (23) belongs to $L^2(D)$ and, by the above remarks, a unique solution $v \in L^2(D)$ of (23) exists. We choose an approximating sequence $f_n \in C_0^\infty(D)$ such that $f_n \rightarrow f$ in $H_0^{-1}(D)$ and denote the solutions to (23) for right hand side $L_k f_n$ by v_n . From Theorem 2 and [7] we know that v_n can be extended to $v_n \in H_{\text{loc}}^1(\mathbb{R}^3)$ and satisfies (24). Integration by parts shows that the weak solution v_n also satisfies (25),

$$\iint_{\mathbb{R}^3} v_n [\operatorname{div} (a \nabla \bar{\psi}) + k^2 \bar{\psi}] dx = \iint_D \nabla \psi^* f_n dx$$

for all $\psi \in \tilde{C}_0^\infty(D)$. As $n \rightarrow \infty$, the left hand side tends to $\iint_{\mathbb{R}^3} v [\operatorname{div} (a \nabla \bar{\psi}) + k^2 \bar{\psi}] dx$ and the right hand side to $\iint_D \nabla \psi^* f dx$, which shows that v is indeed an ultra weak solution. Uniqueness of the ultra weak solution follows from unique solvability of the integral equation. \square

4 L^2 -Theory for Maxwell's Equations

In this section, we extend the L^2 -theory for the Lippmann-Schwinger integral equation to Maxwell's equations. For frequency $\omega > 0$, electric permittivity ϵ , magnetic permeability μ and conductivity σ , these equations for the electric field E and the magnetic field H read

$$\operatorname{curl} E - i\omega\mu H = 0, \quad \operatorname{curl} H + i\omega\epsilon E = \sigma E \quad \text{in } \mathbb{R}^3. \quad (26)$$

Additionally, the tangential components of the fields across material jump interfaces need to be continuous. In the electromagnetic scattering problem we consider incident electromagnetic fields (E^i, H^i) which satisfy the homogeneous Maxwell system

$$\operatorname{curl} E^i - i\omega\mu_0 H^i = 0, \quad \operatorname{curl} H^i + i\omega\epsilon_0 E^i = 0 \quad \text{in } \mathbb{R}^3$$

with constant background parameters ϵ_0, μ_0 and seek for a total field (E, H) which satisfies (26). Moreover, the scattered fields $(E^s, H^s) = (E, H) - (E^i, H^i)$ need to satisfy the Silver-Müller radiation condition

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \rightarrow \infty \quad (27)$$

uniformly with respect to all directions $\hat{x} = x/|x| \in S^2$.

We eliminate the electric field E from (26) and find the following equation for the total magnetic field,

$$\operatorname{curl} \left(\frac{1}{\epsilon_r} \operatorname{curl} H \right) - k^2 \mu_r H = 0,$$

where we introduced the relative electric permeability $\epsilon_r = (\epsilon + i\sigma)/\epsilon_0$ and the relative magnetic permeability $\mu_r = \mu/\mu_0$ as well as the wavenumber $k = \omega\sqrt{\epsilon_0\mu_0}$. As a general assumption, we suppose that the support of ϵ_r and ϵ_μ is included in the bounded Lipschitz domain $D \subset \mathbb{R}^3$. The scattered magnetic field H^s in turn solves

$$\operatorname{curl} \left(\frac{1}{\epsilon_r} \operatorname{curl} H^s \right) - k^2 \mu_r H^s = k^2 (\mu_r - 1) H^i + \operatorname{curl} \left((1 - \epsilon_r^{-1}) \operatorname{curl} H^i \right).$$

This motivates to consider the following source problem with data $g, h : \mathbb{R}^3 \rightarrow \mathbb{C}^3$

$$\operatorname{curl} \left(\frac{1}{\epsilon_r} \operatorname{curl} v \right) - k^2 \mu_r v = k^2 h + \operatorname{curl} g \quad (28)$$

for the unknown $v : \mathbb{R}^3 \rightarrow \mathbb{C}^3$ which satisfies in addition a reformulation of the Silver-Müller condition (27),

$$\operatorname{curl} v(x) \times \frac{x}{|x|} - ikv(x) = \mathcal{O} \left(\frac{1}{|x|^2} \right), \quad |x| \rightarrow \infty.$$

Next we define the contrasts

$$p(x) = \mu_r(x) - 1 \quad \text{and} \quad q(x) = 1 - \frac{1}{\epsilon_r},$$

which vanish outside of D . As in the first sections, we assume that D is the finite union of non-intersecting Lipschitz domains $D_j \subset \mathbb{R}^3$, $\overline{D} = \bigcup_{j=1}^N \overline{D}_j$. As in (14), we define a special set of testfunctions for this situation,

$$\tilde{C}_0^\infty(D)^3 := \left\{ \psi \in C_0^\infty(D)^3 : \psi|_{D_j} \in C_0^\infty(D_j)^3 \text{ for all } j = 1, \dots, N \right\}$$

and note that $\tilde{C}_0^\infty(D)^3$ is dense in $L^2(D)^3$.

Moreover, both material parameters are greater than one in the interior of the bounded inhomogeneity D . Therefore, our assumption on $p \in L^\infty(D)$ is that $p > 0$ almost everywhere in D . In contrast, we suppose that $q < 0$ almost everywhere in D to be able to show solvability of Maxwell's equations for magnetic materials in a weighted space. Moreover, we merely consider contrasts q such that $\sqrt{|q|}$ belongs to

$$\tilde{W}^{1,\infty}(D) := \left\{ \rho \in L^\infty(D) : \rho|_{D_j} \in W^{1,\infty}(D_j) \text{ for all } j = 1, \dots, N \right\}$$

with norm $\|\rho\|_{\tilde{W}^{1,\infty}(D)} = \max_{j=1,\dots,N} \|\rho\|_{W^{1,\infty}(D_j)}$. The electric permeability is hence assumed to be piecewise weakly differentiable. To obtain a solution in the unweighted space $L^2(D)^3$ we additionally require $q \leq c_q < 0$ and $p \geq c_p > 0$ in D .

Now, consider the following integro-differential equation in Lippmann-Schwinger form with unknown v and data g, h ,

$$\begin{aligned} v(x) = & (k^2 + \nabla \operatorname{div}) \iint_D [p(y)v(y) + h(y)] \Phi_k(x, y) dy \\ & + \operatorname{curl} \iint_D [q(y) \operatorname{curl} v(y) + g(y)] \Phi_k(x, y) dy, \quad x \in D. \quad (29) \end{aligned}$$

As in the previous sections where we investigated the integro-differential equation for the acoustic scattering problem, the integral operators in the operator equation lack compactness due to their composition with differential operators. Again, our aim is to solve the latter equation in $L^2(D)^3$, where variational formulations fail to help, and therefore our technique to compensate for the lack of compactness is imposing positivity implicitly through assumptions on p and q .

To show the relation between equation (29) and the source problem (28), we cite the following theorem from [6], which states that a solution of the integro-differential equation gives rise to a solution of the differential equation and vice versa. This result is formulated in the space $H(\text{curl}, D) := \{u \in L^2(D)^3 : \text{curl } u \in L^2(D)^3\}$.

Theorem 9. *Let $k \in \mathbb{C} \setminus \{0\}$ with $\text{Re } k \geq 0$ and $\text{Im } k \geq 0$ and $g, h \in L^2(D)^3$.*

(a) *Let $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ be a radiating solution of (28). Then $v|_D \in H(\text{curl}, D)$ solves (29).*

(b) *Let $v \in H(\text{curl}, D)$ solve (29). Then v can be extended by the right-hand side of (29) to a radiating solution of (28) in $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$.*

Moreover, reference [6] contains several conditions on p and q such that the Lippmann-Schwinger equation (29) is uniquely solvable in $H(\text{curl}, D)$. Our aim in this section is to extend these solvability results from $H(\text{curl}, D)$ to $L^2(D)^3$. Let us therefore introduce operator notation for the integral operators. Following [6], we introduce the operators

$$\begin{aligned} (A_k v)(x) &= (k^2 + \nabla \text{div}) \iint_D p(y) v(y) \Phi_k(x, y) dy \quad \text{and} \\ (B_k v)(x) &= \text{curl} \iint_D q(y) \text{curl } v(y) \Phi_k(x, y) dy. \end{aligned}$$

As a first result of this section, we show that both of these operators are bounded on $L^2(D)^3$ and on the weighted spaces $L_p^2(D)^3$ and $L_{|q|}^2(D)^3$, which are defined analogously to (13). (Recall that $q < 0$ in D , hence we apply the absolute value to q in the definition of the weighted space $L_{|q|}^2(D)^3$.) In case that merely $p > 0$ and $q < 0$ without strict boundedness away from zero, the norms on the latter weighted spaces are weaker than the norm on $L^2(D)^3$.

Lemma 10. (a) *Assume that $p \in L^\infty(D)$ without sign restriction. Then the operator A_k is bounded on $L^2(D)^3$. If, moreover, $p > 0$ in D , then A_k is also bounded on $L_p^2(D)^3$.*

(b) *Assume that $q \in \tilde{W}^{1,\infty}(D)$. Then B_k is bounded on $L^2(D)^3$. If, moreover, $q < 0$ almost everywhere in D and $\sqrt{|q|} \in \tilde{W}^{1,\infty}(D)$, then B_k is also bounded on $L_{|q|}^2(D)^3$.*

Proof. (a) Define the volume potential V_k by

$$(V_k u)(x) := \iint_D u(y) \Phi(x, y) dy, \quad x \in D. \quad (30)$$

Then for any two cutoff functions $\chi_1 \in C_0^\infty(D)$ and $\chi_2 \in C_0^\infty(\mathbb{R}^3)$, the operator $\chi_2 V_k \chi_1$ is bounded from $H^s(\mathbb{R}^3)$ into $H^{s+2}(\mathbb{R}^3)$ [9, Chapter 6]. Therefore we obtain that V_k maps

$L^2(D)^3$ boundedly into $H^2(D)^3$. We already observe here that A_k is bounded: Since $V_k(pv) \in H^2(D)^3$, $\nabla \operatorname{div} V_k(pv) \in L^2(D)^3$ and moreover

$$\|A_k v\|_{L^2(D)^3} \leq C \|V_k(pv)\|_{H^2(D)^3} \leq C \|pv\|_{L^2(D)^3} \leq C_A \|p\|_\infty^{1/2} \|v\|_{L_p^2(D)^3} \quad (31)$$

where C_A does not depend on p . To obtain boundedness on $L_p^2(D)^3$ we note that $\|A_k v\|_{L_p^2(D)^3} \leq \|p\|_\infty^{1/2} \|A_k v\|_{L^2(D)^3}$.

(b) For $q < 0$ in D we already showed in the proof of Lemma 3 that the $H^1(D)^3$ norm of

$$w = \iint_D q(y) \frac{\partial v(y)}{\partial y_\ell} \Phi_k(\cdot, y) dy, \quad \ell = 1, 2, 3,$$

is bounded in terms of v i.e. $\|w\|_{H^1(D)^3} \leq \|\sqrt{q}\|_{\tilde{W}^{1,\infty}(D)} \|v\|_{L_{|q|}^2(D)^3}$. From this bound it follows directly by linear combination that the H^1 norm of

$$V_k(q \operatorname{curl} v) = \iint_D q(y) \operatorname{curl} v(y) \Phi_k(\cdot, y) dy$$

is bounded by $\|v\|_{L_{|q|}^2(D)^3}$. Clearly, this implies an L^2 bound for the curl of $V_k(q \operatorname{curl} v)$, yielding

$$\|B_k v\|_{L^2(D)^3} = \|\operatorname{curl} V_k(q \operatorname{curl} v)\|_{L^2(D)^3} \leq C_B \|\sqrt{q}\|_{\tilde{W}^{1,\infty}(D)} \|v\|_{L_{|q|}^2(D)^3} \quad (32)$$

with a constant C_B independent of q . Now we note that the weighted norm on $L_{|q|}^2(D)^3$ is weaker than the norm on $L^2(D)^3$ to obtain boundedness of B_k on $L_{|q|}^2(D)^3$ and also on $L^2(D)^3$.

Boundedness of B_k on $L^2(D)^3$ for arbitrary $q \in \tilde{W}^{1,\infty}(D)$ (without sign restriction) follows from (16) because of $\|w\|_{H^1(D)^3} \leq C \|q\|_{\tilde{W}^{1,\infty}(D)} \|v\|_{L^2(D)^3}$. This in turn implies, with the same arguments as above, that $\|B_k v\|_{L^2(D)^3} \leq C \|v\|_{L^2(D)^3}$. \square

By the previous lemma it makes sense to consider the equation

$$v - A_k v - B_k v = f \quad (33)$$

in a space of square-integrable functions, that is, either in $L^2(D)^3$ or in the weighted spaces $L_p^2(D)^3$ and $L_{|q|}^2(D)^3$. Solvability of the operator equation (33) of the second kind is established in the following by Riesz-Fredholm theory, similar as in Section 3 for the acoustic case.

Lemma 11. (a) For $p > 0$ almost everywhere in D , there is a compact operator K_A on $L_p^2(D)^3$ such that for all $v \in L_p^2(D)^3$ it holds

$$-\operatorname{Re} \langle (A_k - K_A)v, v \rangle_{L_p^2(D)^3} \geq 0 \quad \text{for all } v \in L_p^2(D)^3.$$

(b) For $q < 0$ almost everywhere in D , there is a compact operator K_B on $L_{|q|}^2(D)^3$ such that

$$-\operatorname{Re} \langle (B_k - K_B)v, v \rangle_{L_{|q|}^2(D)^3} \geq 0 \quad \text{for all } v \in L_{|q|}^2(D)^3.$$

Proof. By the density of $\tilde{C}_0^\infty(D)^3$ both in $L_p^2(D)^3$ and $L_{|q|}^2(D)^3$ it is sufficient to show parts (a) and (b) merely for such functions.

(a) For $v \in \tilde{C}_0^\infty(D)^3$ we know from [6, Lemma 2.2] that $w = A_k v$ solves $\operatorname{curl}^2 w - k^2 w = k^2 p v$. Using this result for $k = i$ we can rewrite the inner product $\langle A_i v, v \rangle_{L_p^2(D)^3}$ as

$$\begin{aligned} -\langle A_i v, v \rangle_{L_p^2(D)^3} &= - \iint_{\mathbb{R}^3} p v^* A_i v \, dx = \iint_{\mathbb{R}^3} (\operatorname{curl}^2 w + w)^* w \, dx \\ &= \iint_{\mathbb{R}^3} (|\operatorname{curl} w|^2 + |w|^2) \, dx \geq 0. \end{aligned}$$

Since $k = i$, the function w is exponentially decaying, which validates the integration by parts in the last computation.

The difference $A_k - A_i$ is moreover compact on $L_p^2(D)^3$. To see this we proceed exactly as in the proof of Lemma 6 for the volume potential V_k from (30). Choose some $v \in C_0^\infty(D)^3$. The difference of $w_1 = V_k v$ and $w_2 = V_i v$ solves

$$-\Delta(w_1 - w_2) + (w_1 - w_2) = (1 + k^2) w_1 \quad \text{in } B(0, R)$$

for some ball $B(0, R)$ with radius R large enough. We denote by Γ the boundary $\partial B(0, R)$. We see directly from the definitions of $V_k v$ and $V_i v$ that $\|w_1 - w_2\|_{H^{5/2}(\Gamma)^3} \leq C \|p\|_\infty^{1/2} \|v\|_{L_p^2(D)^3}$. Then standard regularity results again yield $w_1 - w_2 \in H^3(B(0, R))^3$ and

$$\begin{aligned} \|w_1 - w_2\|_{H^3(B(0,R))^3} &\leq C \|w_1\|_{H^1(B(0,R))^3} + C \|w_1 - w_2\|_{H^{5/2}(\Gamma)} \\ &\leq C \|w_1\|_{H^1(B(0,R))^3} + C \|v\|_{L_p^2(D)^3}. \end{aligned}$$

Finally, we use the boundedness of $v \mapsto V_k(pv)$ from $L_p^2(D)^3$ into $H^2(D)^3$ which yields that $V_k - V_i$ is bounded from $L_p^2(D)^3$ into $H^3(D)^3$. Therefore, $A_k - A_i$ is bounded from $L_p^2(D)^3$ into $H^1(D)^3$ and thus compact into $L^2(D)^3$. With $K_A := A_k - A_i$ this finishes part (a).

(b) Now we consider the operator B_k and note that for $v \in \tilde{C}_0^\infty(D)^3$ the potential $w = V_k(q \operatorname{curl} v)$ solves $\Delta w + k^2 w = -q \operatorname{curl} v$. Hence, for $k = i$, an integration by parts shows as above that

$$\begin{aligned} \langle B_i v, v \rangle_{L_{|q|}^2(D)^3} &= - \iint_D q v^* B_i v \, dx \\ &= - \iint_D (q \operatorname{curl} v)^* V_i(q \operatorname{curl} v) \, dx - \iint_D (\nabla q \times v)^* V_i(q \operatorname{curl} v) \, dx. \end{aligned} \tag{34}$$

Note that no boundary terms occur in the latter integration by parts since $v \in \tilde{C}_0^\infty(D)^3$, see [11, Theorem 3.24]. Using the vector identity $a^\top(b \times c) = (a \times b)^\top c$ and, again, that $\nabla q = -\nabla|q| = -2\sqrt{|q|} \nabla\sqrt{|q|}$ we can write for the last integral

$$\iint_D (\nabla q \times v)^* V_i(q \operatorname{curl} v) \, dx = -2 \iint_D v^* [V_i(q \operatorname{curl} v) \times \nabla\sqrt{|q|}] \sqrt{|q|} \, dx.$$

Therefore we have the splitting

$$-\langle B_k v, v \rangle_{L^2_{|q|}(D)^3} = \langle (B_i - B_k)v, v \rangle_{L^2_{|q|}(D)^3} + \iint_D (q \operatorname{curl} v)^* V_i(q \operatorname{curl} v) dx - \langle K v, v \rangle_{L^2_{|q|}(D)^3}$$

where

$$K v = 2 V_i(q \operatorname{curl} v) \times \nabla \sqrt{|q|}.$$

The integral $\iint_D (q \operatorname{curl} v)^* V_i(q \operatorname{curl} v) dx$ is non-negative by Lemma 5. It remains to show compactness of the operators K and $B_k - B_i$ from $L^2_{|q|}(D)^3$ into $L^2(D)^3$ (and thus also into $L^2_{|q|}(D)^3$).

First, we comment on K . Since $v \mapsto V_i(q \operatorname{curl} v)$ is bounded from $L^2_{|q|}(D)^3$ into $H^1(D)^3$ by Lemma 3 it is compact as an operator into $L^2(D)^3$. Furthermore, the cross product with $\nabla \sqrt{|q|}$ is continuous on $L^2(D)^3$. Therefore, we conclude that K is compact from $L^2_{|q|}(D)^3$ into $L^2(D)^3$.

To show compactness of $B_k - B_i$ we set again $w_1 = B_k v$ and $w_2 = B_i v$ for $v \in \tilde{C}_0^\infty(D)^3$. The difference $w_1 - w_2$ solves

$$\operatorname{curl}^2(w_1 - w_2) + (w_1 - w_2) = (1 + k^2)w_1 \quad \text{in } \mathbb{R}^3,$$

and thus, since $\operatorname{div}(w_1 - w_2) = 0$,

$$-\Delta(w_1 - w_2) + (w_1 - w_2) = (1 + k^2)w_1 \quad \text{in } \mathbb{R}^3.$$

Now we proceed exactly as in the proof of Lemma 6. First, by the explicit form of the potentials $B_k v$ and B_i we show by partial integration an estimate of the form

$$\|w_1 - w_2\|_{H^{3/2}(\Gamma)^3} \leq C \|v\|_{L^2_{|q|}(D)^3},$$

where again $\Gamma = \partial B(0, R)$ and the ball is chosen such that it contains \bar{D} in its interior. Then regularity results and the boundedness of B_k yields

$$\|w_1 - w_2\|_{H^2(D)^3} \leq C \|w_1 - w_2\|_{H^{3/2}(\Gamma)^3} + C \|w_1\|_{L^2(D)^3} \leq C \|v\|_{L^2_{|q|}(D)^3}.$$

The compact imbedding of $H^2(D)^3$ into $L^2(D)^3$ yields compactness of $B_k - B_i$. This ends the proof of part (b). \square

From this result we conclude that $I - A_k$ and $I - B_k$ are Fredholm of index zero in $L^2_p(D)^3$ and $L^2_{|q|}(D)^3$, respectively. Unfortunately, due to our approach in two *different* weighted spaces $L^2_p(D)^3$ and $L^2_{|q|}(D)^3$, we need to impose rather restrictive assumptions on the electric permittivity and the magnetic permeability if both of these material parameters are different from zero. Roughly speaking, we are able to treat the cases where p and $|q|$ are close, where one of both parameters is small, or where p and $|q|$ are “nearly” linearly dependent. This follows from the following proposition.

Theorem 12. Assume that $p \in L^\infty(D)$ and $q \in \tilde{W}^{1,\infty}(D)$ such that $p > 0$ and $q < 0$ and $\sqrt{|q|} \in \tilde{W}^{1,\infty}(D)$. Let $\rho \in L^\infty(D)$ be any positive function. Then

$$\begin{aligned} \langle (I - A_k - B_k)v, v \rangle_{L^2_p(D)^3} &\geq \|v\|_{L^2_p(D)^3}^2 - \langle A_k v, v \rangle_{L^2_p(D)^3} - \langle B_k v, v \rangle_{L^2_{|q|}(D)^3} \\ &- \|v\|_{L^2_p(D)^3}^2 \left[C_A \left\| \sqrt{\rho} - \frac{p}{\sqrt{\rho}} \right\|_\infty \left\| \frac{p}{\rho} \right\|_\infty^{1/2} \|p\|_\infty^{1/2} + C_B \left\| \sqrt{\rho} - \frac{|q|}{\sqrt{\rho}} \right\|_\infty \left\| \frac{|q|}{\rho} \right\|_\infty^{1/2} \|\sqrt{|q|}\|_{\tilde{W}^{1,\infty}(D)} \right] \end{aligned} \quad (35)$$

for all $v \in \tilde{C}_0^\infty(D)^3$.

The constants C_A and C_B have been introduced in (31) and (32), respectively.

Proof. We estimate

$$\begin{aligned} &\langle (I - A_k - B_k)v, v \rangle_{L^2_p(D)^3} \\ &= \|v\|_{L^2_p(D)^3}^2 - \langle A_k v, v \rangle_{L^2_p(D)^3} - \langle (\rho - p)A_k v, v \rangle_{L^2(D)^3} \\ &\quad - \langle B_k v, v \rangle_{L^2_{|q|}(D)^3} - \langle (\rho - |q|)B_k v, v \rangle_{L^2(D)^3} \\ &= \|v\|_{L^2_p(D)^3}^2 - \langle A_k v, v \rangle_{L^2_p(D)^3} - \langle (\sqrt{\rho} - p/\sqrt{\rho})A_k v, \sqrt{\rho}v \rangle_{L^2(D)^3} \\ &\quad - \langle B_k v, v \rangle_{L^2_{|q|}(D)^3} - \langle (\sqrt{\rho} - |q|/\sqrt{\rho})B_k v, \sqrt{q}v \rangle_{L^2(D)^3} \\ &\geq \|v\|_{L^2_p(D)^3}^2 - \langle A_k v, v \rangle_{L^2_p(D)^3} - \left\| \sqrt{\rho} - p/\sqrt{\rho} \right\|_\infty C_A \|p\|_\infty^{1/2} \|v\|_{L^2_p(D)^3} \|v\|_{L^2_p(D)^3} \\ &\quad - \langle B_k v, v \rangle_{L^2_{|q|}(D)^3} - \left\| \sqrt{\rho} - |q|/\sqrt{\rho} \right\|_\infty C_B \|\sqrt{|q|}\|_{\tilde{W}^{1,\infty}(D)} \|v\|_{L^2_{|q|}(D)^3} \|v\|_{L^2_p(D)^3} \\ &\geq \|v\|_{L^2_p(D)^3}^2 - \langle A_k v, v \rangle_{L^2_p(D)^3} - C_A \left\| \sqrt{\rho} - p/\sqrt{\rho} \right\|_\infty \|p\|_\infty^{1/2} \|p/\rho\|_\infty^{1/2} \|v\|_{L^2_p(D)^3}^2 \\ &\quad - \langle B_k v, v \rangle_{L^2_{|q|}(D)^3} - C_B \left\| \sqrt{\rho} - |q|/\sqrt{\rho} \right\|_\infty \|\sqrt{|q|}\|_{\tilde{W}^{1,\infty}(D)} \left\| |q|/\rho \right\|_\infty^{1/2} \|v\|_{L^2_p(D)^3}^2 \end{aligned}$$

where we used the bounds in (31) and (32). \square

We summarize the different cases in the following corollary.

Corollary 13. Assume that $p \in L^\infty(D)$ and $q \in W^{1,\infty}(D)$ satisfy the assumption of the previous theorem.

(a) Then $I - A_k$ and $I - B_k$ are Fredholm operators of index 0 in $L^2_p(D)^3$ and $L^2_{|q|}(D)^3$, respectively.

(b) Assume additionally that $p \geq c_p > 0$ and $q \leq -c_q < 0$ for some positive constants c_p and c_q . Then both operators $I - A_k$ and $I - B_k$ are Fredholm operators of index 0 on $L^2(D)^3$.

(c) Assume that

$$C_A \left\| \sqrt{|q|} - \frac{p}{\sqrt{|q|}} \right\|_\infty \left\| \frac{p}{|q|} \right\|_\infty^{1/2} \|p\|_\infty^{1/2} < 1 \quad \text{and} \quad \left\| \frac{|q|}{p} \right\|_\infty < \infty$$

or

$$C_B \left\| \sqrt{p} - \frac{|q|}{\sqrt{p}} \right\|_\infty \left\| \frac{|q|}{p} \right\|_\infty^{1/2} \|\sqrt{|q|}\|_{\tilde{W}^{1,\infty}(D)} < 1 \quad \text{and} \quad \left\| \frac{p}{|q|} \right\|_\infty < \infty.$$

Then $I - A_k - B_k$ is a Fredholm operator of index 0 in $L^2_{|q|}(D)^3$ or in $L^2_p(D)^3$, respectively.

(d) If, in addition to one of the assumptions of part (c), p and q are bounded away from zero, $I - A_k - B_k$ is a Fredholm operator of index 0 on $L^2(D)^3$.

In all cases the Fredholm operators are sums of a coercive and a compact operator.

Proof. Part (a) of the corollary follows from Lemma 11 yielding a Gårding inequality for $I - A_k$ and $I - B_k$. Part (b) then follows from the norm isomorphy of $L^2(D)^3$ and $L^2_\rho(D)^3$ for $\rho \in L^\infty(D)$ bounded away from zero. Part (c) follows by choosing ρ in (35) to be $|q|$ or p , respectively. Note that the conditions $\|p/|q|\|_\infty < \infty$ and $\||q|/p\|_\infty < \infty$ imply that A_k and B_k are bounded on $L^2_{|q|}(D)^3$ and $L^2_p(D)^3$, respectively. Part (d) holds, as part (b), due to the norm isomorphy of $L^2(D)^3$ and $L^2_\rho(D)^3$ under the announced assumptions. \square

We note that in parts (c) and (d) of Corollary 13 it is even sufficient to replace $\|\sqrt{|q|} - p/\sqrt{|q|}\|_\infty$ or $\|\sqrt{p} - |q|/\sqrt{p}\|_\infty$ by $\|\sqrt{|q|} - tp/\sqrt{|q|}\|_\infty$ or $\|\sqrt{p} - s|q|/\sqrt{p}\|_\infty$, respectively, for some constants $t, s > 0$. Indeed, in the proof of Theorem 12 we just subtract $t\langle A_k v, v \rangle_{L^2_p(D)^3}$ or $s\langle B_k v, v \rangle_{L^2_{|q|}(D)^3}$, respectively.

Due to the previous corollary, existence of solutions for (29) follows from uniqueness. In [6], several assumptions for which uniqueness holds have been investigated. One of those serves now to prove uniqueness of solution in $L^2(D)^3$ for smooth coefficients.

Corollary 14. *Let $\mathbb{R}^3 \setminus \bar{D}$ be connected, $q \in C^{2,\alpha}(D)$, and $p \in C^{1,\alpha}(D)$ for some $\alpha \in (0, 1)$. Under the additional assumptions of parts (a)–(d) of Corollary 13, all Fredholm operators announced in this theorem are isomorphisms on their respective spaces.*

Proof. Consider, for instance, the operator $I - A_k - B_k$ under condition (d). Under our assumptions, Theorem 2.5 in [6] implies that this operator is an isomorphism in $H(\text{curl}, D)$. Again, by a general result from functional analysis (see Lemma 17 in the Appendix), this implies that $I - A_k - B_k$, which is Fredholm by Corollary 13, is also an isomorphism on the respective weighted L^2 -spaces announced in this Corollary. \square

It remains the question in which sense a solution $v \in L^2(D)$ of the integro-differential equation (33) is related to Maxwell's equations (28). In the remainder of this section we show that v can be interpreted as an ultra weak solution of (28). In the same manner as variational formulations for weak solutions to second-order equations are obtained by integrating one derivative on the test function using an integration by parts, the ultra weak solution has to satisfy a variational formulation where both derivatives have been integrated on the test function. The formulation becomes thus necessarily unsymmetric. For the source problem (28) we recall that the weak formulation in $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ for right-hand sides $h, g \in L^2(D)^3$ is

$$\iint_{\mathbb{R}^3} \left[\frac{1}{\epsilon_r} \text{curl } v \cdot \text{curl } \bar{\psi} - k^2 \mu_r v \cdot \bar{\psi} \right] dx = k^2 \iint_D [h \cdot \bar{\psi} + g \cdot \text{curl } \bar{\psi}] dx \quad (36)$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)^3$. Now we introduce the dual of $H(\text{curl}, D)$ with respect to the pivot space $L^2(D)^3$, which we denote by $H_0^{-1}(\text{curl}, D)$.

The problem in its ultra weak formulation in turn is to find $v \in L^2_{\text{loc}}(\mathbb{R}^3)^3$ for $g, h \in H_0^{-1}(\text{curl}, D)$ such that

$$\iint_{\mathbb{R}^3} v \cdot \left[\text{curl} \left(\frac{1}{\epsilon_r} \text{curl} \bar{\psi} \right) - k^2 \mu_r \bar{\psi} \right] dx = k^2 \iint_D [h \cdot \bar{\psi} + g \cdot \text{curl} \bar{\psi}] dx \quad (37)$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)^3$. As for the scalar case, we first show that the source term of equation (33), namely

$$(k^2 + \nabla \text{div}) \iint_D h(y) \Phi_k(x, y) dy + \text{curl} \iint_D g(y) \Phi_k(x, y) dy, \quad (38)$$

is well defined for $g, h \in H_0^{-1}(\text{curl}, D)$.

Lemma 15. *The operators*

$$C_k : h \mapsto (k^2 + \nabla \text{div}) \iint_D h(y) \Phi_k(\cdot, y) dy$$

and

$$D_k : g \mapsto \text{curl} \iint_D g(y) \Phi_k(\cdot, y) dy$$

are bounded from $H_0^{-1}(\text{curl}, D)$ into $L^2(B)^3$ for any ball $B \subset \mathbb{R}^3$.

Proof. Both C_k and D_k (with Ω replacing the region D of integration) are bounded from $L^2(\Omega)^3$ into $H(\text{curl}, \Omega)$ for any Lipschitz domain $\Omega \subset \mathbb{R}^3$, see [6, Lemma 2.2]. We show that the operators are self adjoint with respect to the bilinear form $\langle \psi, \varphi \rangle = \iint_\Omega \psi \varphi dx$. Let $g_1, g_2 \in C_0^\infty(\Omega)$ and define $v_j = C_k g_j$ for $j = 1, 2$. Then v_j satisfies $\text{curl}^2 v_j - k^2 v_j = k^2 g_j$ (see [6, Lemma 2.2]) and thus

$$\langle C_k g_1, g_2 \rangle = \iint_\Omega v_1 g_2 dx = \frac{1}{k^2} \iint_\Omega v_1 (\text{curl}^2 v_2 - k^2 v_2) dx \quad (39)$$

$$= \frac{1}{k^2} \iint_\Omega v_2 (\text{curl}^2 v_1 - k^2 v_1) dx = \iint_\Omega v_2 g_1 dx \quad (40)$$

$$= \langle g_1, C_k g_2 \rangle. \quad (41)$$

The arguments for D_k are the same. Therefore, we conclude that C_k and D_k are also bounded from $H_0^{-1}(\text{curl}, \Omega)$ into $L^2(\Omega)^3$. Using literally the same arguments as in the end of the proof of Lemma 7, we obtain finally that both operators are bounded from $H_0^{-1}(\text{curl}, \Omega)$ into $L^2(B)^3$ for any ball $B \subset \mathbb{R}^3$. \square

Finally, we show that solutions of equation (38) solve the ultra weak formulation (37).

Theorem 16. *Assume that the operator $I - A_k - B_k$ is invertible in $H(\text{curl}, D)$ and that one of the assumptions of part (d) of Corollary 13 holds. Then for any $g, h \in H_0^{-1}(\text{curl}, D)$ there exists a unique solution $v \in L^2(D)^3$ of $v - A_k v - B_k v = C_k h + D_k g$, and v can be extended into \mathbb{R}^3 to an ultra weak solution of (28) in the sense of (37).*

Proof. First, we note that for $h, g \in H_0^{-1}(\text{curl}, D)$, the right-hand side $C_k h + D_k g$ is well defined in $L^2(D)^3$. Second, by the assumption that $I - A_k - B_k$ is invertible in $H(\text{curl}, D)$ we note that $I - A_k - B_k$, which is Fredholm of index 0 on $L^2(D)^3$ by Corollary 13, is an isomorphism from $L^2(D)^3$ onto itself (Corollary 14). For g and $h \in C_0^\infty(D)^3$ the claim of the proposition is clear, since even for g and $h \in L^2(D)^3$ we know from [6] that a solution of the integro-differential equation belongs to $H(\text{curl}, D)$ and satisfies the weak formulation (36), hence especially the ultra weak one. Let now $g, h \in H_0^{-1}(\text{curl}, D)$ be arbitrary and denote by $(h_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ two sequences in $C_0^\infty(D)^3$ converging to h and g , respectively, in $H_0^{-1}(\text{curl}, D)$. For data h_n and g_n we denote by v_n the solution of equation (29), which exists by our assumption on $I - A_k - B_k$. Hence, for v_n we have

$$\iint_{\mathbb{R}^3} v_n \cdot \left[\text{curl} \left(\frac{1}{\epsilon_r} \text{curl} \bar{\psi} \right) - k^2 \mu_r \bar{\psi} \right] dx = k^2 \iint_D [h_n \cdot \bar{\psi} + g_n \cdot \text{curl} \bar{\psi}] dx$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)^3$. Obviously, the right side of this variational equation converges as $n \rightarrow \infty$ since the dual pairing is continuous in its arguments. Moreover, continuous dependence of the L^2 -solution v_n of the integral equation on the data g and h is clear from Corollary 13. Therefore we have that $v_n \rightarrow v$ in $L^2(D)^3$ and we conclude that

$$\iint_D v \cdot \left[\text{curl} \left(\frac{1}{\epsilon_r} \text{curl} \bar{\psi} \right) - k^2 \mu_r \bar{\psi} \right] dx = k^2 \iint_D [h \cdot \bar{\psi} + g \cdot \text{curl} \bar{\psi}] dx$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)^3$, which was the claim of the theorem. \square

5 Convergence of Projection Methods

The coercivity results of the previous sections allow to show convergence properties of projection methods applied to the integro-differential equations. We restrict ourselves to the scalar one, i.e. to (9), but remark that analogous arguments hold also for the case of Maxwell's equation. The corresponding analysis including the implementation and test of numerical examples is under investigation.

For this section we make the general assumption that $q \in W^{1,\infty}(D)$ satisfies $q(x) \geq q_0$ on D for some constant $q_0 > 0$. Then the norms in $H_q^1(D)$ and $L_q^2(D)$ are equivalent to the canonical norms in $H^1(D)$ and $L^2(D)$, respectively.

We first sketch the treatment of the operator equation (9) with right-hand side $f \in H^1(D)$ in the (canonical) Sobolev space $H^1(D)$. Let X_h be the finite dimensional subspace of $H^1(D)$ consisting of linear functions on a regular grid with meshsize h . Then we have the following approximating properties (cf. [1]): There exists $c > 0$ such that

$$\inf_{\psi_h \in X_h} \|u - \psi_h\|_{H^p(D)} \leq c h^{\ell-p} \|u\|_{H^\ell(D)} \quad \text{for all } u \in H^\ell(D) \quad (42)$$

for $p = 0, 1$ and $\ell = 1, 2$ such that $p \leq \ell$.

The projection method is to replace (9) by the equation

$$\langle v_h - L_k(q \nabla v_h), \psi_h \rangle_{H_q^1(D)} = \langle L_k f, \psi_h \rangle_{H_q^1(D)} \quad \text{for all } \psi_h \in X_h \quad (43)$$

for $v_h \in X_h$. Well known convergence results (see [1]) yield that Gårding's inequality (12) implies Céa's estimate

$$\|u - u_h\|_{H^1(D)} \leq \inf_{\psi_h \in X_h} \|u - \psi_h\|_{H^1(D)}, \quad (44)$$

thus, by combining it with (42),

$$\|u - u_h\|_{H^1(D)} \leq ch \|u\|_{H^2(D)} \quad (45)$$

where we note that $u \in H^2(D)$ by well known regularity results.

We note that the numerical implementation of equation (43) is rather complicated because of the H_q^1 -inner product. Therefore, we prefer to treat equation (9) in $L_q^2(D)$, that is, we study the projection method to determine $v_h \in X_h$ such that

$$\langle v_h - L_k(q\nabla v_h), \psi_h \rangle_{L_q^2(D)} = \langle L_k f, \psi_h \rangle_{L_q^2(D)} \quad \text{for all } \psi_h \in X_h, \quad (46)$$

i.e., by using the volume potential $V_k \psi = \iint_D \psi(y) \Phi_k(\cdot, y) dy$,

$$\iint_D [v_h - \operatorname{div} V_k(q\nabla v_h)] \overline{\psi_h} q dx = \iint_D \operatorname{div} (V_k f) \overline{\psi_h} q dx \quad \text{for all } \psi_h \in V_h.$$

Gårding's estimate (19) and Céa's estimate yield

$$\|v - v_h\|_{L^2(D)} \leq ch^2 \|v\|_{H^2(D)}. \quad (47)$$

Again we note that $v \in H^2(D)$. Moreover, using an inverse inequality [1, Theorem 4.5.11], valid due to our $H^1(D)$ -conforming elements, and the Lagrangian interpolation operator I_h on $H^2(D)$ [1, Theorem 4.4.20], we obtain

$$\begin{aligned} \|v - v_h\|_{H^1(D)} &\leq \|v - I_h v\|_{H^1(D)} + \|I_h v - v_h\|_{H^1(D)} \\ &\leq Ch \|v\|_{H^2(D)} + \frac{C}{h} \|I_h v - v_h\|_{L^2(D)} \\ &\leq Ch \|v\|_{H^2(D)} + \frac{C}{h} \|I_h v - v\|_{L^2(D)} + \frac{C}{h} \|v - v_h\|_{L^2(D)} \leq Ch \|v\|_{H^2} \end{aligned}$$

Thus, surprisingly, approximating (9) by a Galerkin method in $L^2(D)$ with $H^1(D)$ -conforming elements yields the same quasi-optimal convergence rates in $H^1(D)$ as the corresponding Galerkin method in $H^1(D)$.

For smooth q , i.e. $q \in C_0^1(\mathbb{R}^3)$, it is more convenient to rewrite the integrals by using partial integration. This yields

$$\iint_D v_h \overline{\psi_h} q dx + \iint_D V_k(q\nabla v_h) \cdot \nabla(\overline{\psi_h} q) dx = - \iint_D V_k f \cdot \nabla(\overline{\psi_h} q) dx \quad \text{for all } \psi_h \in V_h.$$

The boundary terms vanishes since $q = 0$ on ∂D_j .

6 Appendix

We used the following result from functional analysis.

Lemma 17. *Let X, Y be Banach spaces such that X is densely and continuously imbedded in Y . Let $T : Y \rightarrow Y$ and $S : X \rightarrow X$ be linear and bounded operators such that T is an extension of S , i.e. $j \circ S = T \circ j$ where $j : X \rightarrow Y$ denotes the imbedding operator. Furthermore, assume that S is an isomorphism from X onto itself and T a Fredholm operator of index zero. Then also T is an isomorphism from Y onto itself.*

Proof. From $j \circ S = T \circ j$ we conclude that $S' \circ j' = j' \circ T'$ where $S' : X' \rightarrow X'$ and $T' : Y' \rightarrow Y'$ and $j' : Y' \rightarrow X'$ are the duals of S, T and j , respectively. From this injectivity of T' follows. Indeed, let $T'\psi = 0$. Then $S'(j'\psi) = 0$, i.e. $j'\psi = 0$. The denseness of the imbedding $X \subset Y$ yields injectivity of j' , thus $\psi = 0$. Therefore, since also T' is a Fredholm operator of index zero it is an isomorphism in Y' . This, finally, implies that also T is an isomorphism. \square

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