Homogenization of the Navier-Stokes Equations with a Slip Boundary Condition

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Abstract

This paper deals with the homogenization of the Stokes or Navier-Stokes equations in a domain containing periodically distributed obstacles, with a slip boundary condition (i.e., the normal component of the velocity is equal to zero, while the tangential velocity is proportional to the tangential component of the normal stress). We generalize our previous results (see [11]) established in the case of a Dirichlet boundary condition; in particular, for a so-called critical size of the obstacles (equal to $\epsilon^3$ in the three-dimensional case, $\epsilon$ being the inter-hole distance), we prove the convergence of the homogenization process to a Brinkman-type law.

1. Introduction

In a recent article (see [1]) we addressed the problem of the homogenization of the Stokes or Navier-Stokes equations, with a Dirichlet boundary condition, in open sets perforated with tiny holes. The present paper is devoted to the generalization of that previous study to the case of a slip boundary condition (the normal component of the velocity is equal to zero, but the tangential component is proportional to the tangential component of the normal stress). Roughly speaking, all the results of [11] still hold true, including the construction of an extension of the pressure, and the three different limit regimes (Darcy, Brinkman, Stokes). In both cases the critical size of the holes (leading to the homogenized Brinkman-type law) is the same, but the matrix $M$ which appears in the limit is different. From a mathematical viewpoint this paper shows that the same type of results are obtained either if all the $N$ components of the velocity are equal to zero on the boundary of the obstacles (Dirichlet boundary condition in [1]), or if the single normal component is equal to zero (slip boundary condition in the present case). From a physical viewpoint it shows that the slowing effect of the obstacles is mainly due to the fact that the fluid does not penetrate them (zero normal component of the velocity), rather than to the fact that it sticks to the obstacles because of the viscosity (no-slip or Dirichlet boundary condition). Though the results of the present paper are similar to those of [1], the generalization of their proof is not trivial. Some new technical lemmas are required in order to carry out the machinery of [1]. For example, the extension of the velocity is no longer obvious, and we construct it following an idea of C. Conca (see [11]), D. Cioranescu and J. Saint Jean Paulin (see [10]), and L. Tartar (see [27]). Also, due to the presence of the symmetric stress tensor in the boundary condition, the variational formulation of the Stokes equations is different from that in [1], and its coercivity is proved through a Korn-type inequality. Besides recovering the results of [1] for the more restrictive case of a slip boundary condition, we present here a new result concerning the local problem.
in the two-dimensional case. As already pointed out in [1], the 2-D case is completely
different from the other ones. We complete our previous study [1] by introducing
the precise form of the local problem in the plane. This yields a clearer proof of
the paradoxical two-dimensional result which is linked to the well-known Stokes
paradox and to the Finn-Smith paradox; see [13]. Finally, the exposition is here
more simple (and less general) than in [1]; several technical lemmas are merely
quoted, and we refer to [1] for their proof.

We turn now to a brief survey of our main result. We represent solid obsta-
cles in a fluid flow by holes in the fluid domain: \( \Omega \), is obtained by removing
from a given set \( \Omega \), included in \( \mathbb{R}^N \) a collection of periodically distributed holes
\( (T^i)_{i \leq N(e)} \) (their number \( N(e) \) is of order of \( e^{-N} \)). Each hole \( T^i \) is homothetic
with ratio \( a_i \) to the same model hole \( T \). The hole size \( a_i \) is assumed to be much
smaller than the inter-hole distance \( e \) (i.e., \( a_i/e \) goes to zero as \( e \) does). For a given
force \( f \in [L^2(\Omega)]^N \) and a constant positive viscosity \( \mu \), denoting by \( u \), the velocity,
and by \( p \), the pressure, the Stokes equations in \( \Omega \), are

\[
\begin{align*}
\nabla p_e - \mu \Delta u_e &= f & \text{in } \Omega_e \\
\n\nabla \cdot u_e &= 0 & \text{in } \Omega_e 
\end{align*}
\]

We provide the Stokes equations with the so-called slip boundary condition

\[
(BC) \quad \begin{cases}
\quad u_e \cdot n = 0 & \text{on } \partial T^i \\
\quad \frac{\alpha}{a_i} u_e = 2 \left( \frac{\partial u_e}{\partial n} \right) n - (\nabla u_e + \nabla u_e) n & \text{on } \partial T^i, \\
\quad u_e = 0 & \text{on } \partial \Omega 
\end{cases}
\]

where the slip coefficient \( \alpha \) is a positive constant. The first equation in (BC) expresses
that the fluid does not flow through the obstacle \( T^i \). The second one is a balance
relation between the tangential components of the velocity and the infinitesimal
force exerted by the fluid on the obstacle. We define a ratio \( \sigma_e \) between the size of
the holes and the inter-hole distance

\[
\begin{align*}
\sigma_e &= \left( \frac{e^N}{a_i^{N-2}} \right)^{1/2} & \text{for } N \geq 3 \\
\sigma_e &= e \log \left( \frac{a_i}{e} \right)^{1/2} & \text{for } N = 2 
\end{align*}
\]

If the limit of \( \sigma_e \) is strictly positive and finite, then the size of the holes is said to
be critical. If the limit is zero (or infinite), then the size is larger (or smaller) than
the critical one. These three different limits of \( \sigma \), lead to three different homogenized
limits for \( (S) \): a Darcy's law, a Brinkman-type law, and the Stokes equations. More
precisely, using the so-called energy method introduced by L. Tartar in [27] and
adapted by D. Cioranescu and F. Murat in [9], we prove the following:
THEOREM 1.1. There exists an extension \((E, u, v, p)\) of the unique solution \((u, v, p)\) of \((S) - (BC)\), with the following properties.

(i) If \(\lim_{\varepsilon \to 0} \sigma_\varepsilon = +\infty\), then \((E, u, v, p)\) converges strongly to \((u, v, p)\) in 
\([H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]\), where \((u, v, p)\) is the unique solution of the Stokes equations

\[
\begin{align*}
\nabla p - \mu \Delta u &= f & \text{in } \Omega \\
\nabla \cdot u &= 0 & \text{in } \Omega \\
\n\text{on } & & \text{on } \partial \Omega
\end{align*}
\]

(In this case the holes are too small, and nothing happens when passing to the limit.)

(ii) If \(\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 > 0\), then \((E, u, v, p)\) converges weakly to \((u, v, p)\) in 
\([H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]\), where \((u, v, p)\) is the unique solution of the Brinkman-type law

\[
\begin{align*}
\nabla p - \mu \Delta u + \frac{\mu}{\sigma_\varepsilon^2} Mu &= f & \text{in } \Omega \\
\n\nabla \cdot u &= 0 & \text{in } \Omega \\
\n\text{on } & & \text{on } \partial \Omega
\end{align*}
\]

(For this critical size of the holes, an additional term appears when passing to the limit.)

(iii) If \(\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0\), then \((E, u, v/\sigma_\varepsilon^2, p)\) converges strongly to \((u, v, p)\) in 
\([L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]\), where \((u, v, p)\) is the unique solution of Darcy’s law

\[
\begin{align*}
\n\frac{\mu}{M^{-1}} \left( f - \nabla p \right) &= 0 & \text{in } \Omega \\
\n\nabla \cdot u &= 0 & \text{in } \Omega \\
\n\text{on } & & \text{on } \partial \Omega
\end{align*}
\]

(Thus, if the holes are too large, the Stokes flow degenerates to a Darcy flow when passing to the limit.)

The matrix \(M\) appearing in the Brinkman-type law and in the Darcy’s law is the same, and it depends only on the model hole \(T\) and on the slip coefficient \(\alpha\). Moreover, we can compute \(M\) thanks to the following

THEOREM 1.2. In the homogenization of the Stokes equations \((S)\), for any limit value of \(\sigma_\varepsilon\), the so-called local problem is
where $e_k$ is the $k$-th unit basis vector in $\mathbb{R}^N$. For any dimension $N$, the matrix $M$ is defined in terms of the drag force of the above Stokes flow, i.e.,

$$M e_k = \frac{1}{2^N} \int_{\partial T} (q_k n - e(w_k)n),$$

where $n$ is the normal interior vector of $\partial T$. Furthermore it turns out that

for $N \geq 3$, $\alpha e_k M e_i = \frac{1}{2^N} \left[ \frac{1}{2} \int_{\mathbb{R}^N-T} e(w_k) : e(w_i) + \alpha \int_{\partial T} w_k \cdot w_i \right]$;

for $N = 2$, whatever the size and shape of the model hole $T$ and the value of the slip coefficient $\alpha$ are, $M$ is always equal to $\pi Id$.

All other possible scalings of the slip coefficient are examined in Section 3; they yield the same homogenized equations as before. We emphasize that the permeability tensor $M^{-1}$ in the above Darcy’s law is completely different from that obtained by the two-scale expansions method when the holes size $a$, is exactly of order $\epsilon$ (see [2], [15], [19], [25], [26]). However, in a forthcoming paper (see [4]), we shall prove that they coincide in the so-called low volume fraction limit.

We conclude this introduction by referring to C. Conca in [11] and R. Lipton and M. Avellaneda in [20] for the homogenization of the Stokes equations with other types of mixed boundary conditions (different from our slip boundary condition). We also refer to [1], [6], [18], [21], [23], and [24] for the derivation of Brinkman’s law through homogenization of the Stokes equations with a Dirichlet boundary condition (see the introduction to [1] for a brief survey of these works). Finally we mention that in [7] A. Brillard has addressed a similar problem for the Laplacian with a mixed boundary condition. For a general introduction to the homogenization theory, one could see [5], [25], and the references therein. The results of the present paper have been announced in [3].

**Notation**

Throughout this paper, $C$ denotes various real positive constants which never depend on $\epsilon$. The duality products between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and between
[\mathcal{H}^1_0(\Omega)]^N and [\mathcal{H}^{-1}(\Omega)]^N, are both denoted by \langle, \rangle_{\mathcal{H}^1_0(\Omega)'. The canonical basis of \mathbb{R}^N is denoted by \{e_k\}_{k \leq N}. If u(x) is a vector-valued function from \mathbb{R}^N to \mathbb{R}^N, then

- The gradient of u is an \(N \times N\) tensor: \(\nabla u = (\partial u_i / \partial x_j)_{1 \leq i,j \leq N}\).
- We denote by e(u) the \(N \times N\) symmetric tensor defined by: \(e(u) = (\nabla u + \nabla u^T)_{1 \leq i,j \leq N}\).
- The tensorial product of u by v (two vectors in \(\mathbb{R}^N\)) is denoted by: \(u \otimes v = (u_i v_j)_{1 \leq i,j \leq N}\).
- The inner product of two \(N \times N\) tensors \(A = (a_{ij})\) and \(B = (b_{ij})\) is denoted by:

\[
A : B = \text{tr}(AB) = \sum_{1 \leq i,j \leq N} a_{ij} b_{ij}.
\]

2. Formulation of the Problem

Let \(\Omega\) be a bounded, connected, open set in \(\mathbb{R}^N (N \geq 2)\), with Lipschitz boundary \(\partial \Omega\), \(\Omega\) being locally located on one side of its boundary. Let \(\varepsilon\) be a sequence of strictly positive reals which tends to zero. The set \(\Omega\) is covered with a regular mesh of size \(2\varepsilon\), each cell being a cube \(P_i\), identical to \((-\varepsilon, +\varepsilon)^N\). At the center of each cube \(P_i\) entirely included in \(\Omega\) we make a hole \(T_i\) of size \(\alpha_i\). Every hole \(T_i\) is similar to the same model obstacle \(T\) rescaled to size \(\alpha_i\). We assume that \(T\) is a smooth closed set, which contains a small open ball \(B_{r_0}\) (with strictly positive radius \(r_0\)), is strictly included in the unit open ball \(B_1\), and is such that \((B_1 - T)\) is connected. Throughout the present paper, the main assumption is that the size of the holes \(\alpha_i\) is smaller than the inter-hole distance \(\varepsilon\), i.e.,

\[
\lim_{\varepsilon \to 0} \frac{\alpha_i}{\varepsilon} = 0.
\]

In this section, we do not specify the relation between \(\alpha_i\) and \(\varepsilon\), but we define a ratio \(\sigma_\varepsilon\) between \(\alpha_i\) and what will turn out to be the critical size in Section 3.

\[
\sigma_\varepsilon = \begin{cases} \left( \frac{\varepsilon^N}{\alpha_i^{N-3}} \right)^{1/2} & \text{for } N \geq 3 \\ \varepsilon \log \left( \frac{\alpha_i}{\varepsilon} \right)^{1/2} & \text{for } N = 2 \end{cases}
\]

To be precise, if the limit of \(\sigma_\varepsilon\), as \(\varepsilon\) tends to zero, is strictly positive and finite, then the hole size is called critical. An elementary geometrical consideration gives the number of holes

\[
N(\varepsilon) = \frac{\|\Omega\|}{(2\varepsilon)^N} [1 + o(1)].
\]
The open set $\Omega_\varepsilon$ is obtained by removing from $\Omega$ all the holes $(T^\varepsilon_i)^{N_i}_{i=1}$ : $\Omega_\varepsilon = \Omega - \cup_{i=1}^{N_i} T^\varepsilon_i$. Because only the cells entirely included in $\Omega$ are perforated, it follows that no hole meets the boundary $\partial \Omega$. Thus $\Omega_\varepsilon$ is also a bounded connected open set, with a smooth boundary $\partial \Omega_\varepsilon$. This boundary is made of two disconnected parts $\partial \Omega$ and $\Gamma_\varepsilon$

\begin{equation}
\partial \Omega_\varepsilon = \partial \Omega \cup \Gamma_\varepsilon \quad \text{with} \quad \Gamma_\varepsilon = \bigcup_{i=1}^{N_i} \partial T^\varepsilon_i.
\end{equation}

The flow of an incompressible viscous fluid in the domain $\Omega_\varepsilon$ under the action of an exterior force $f$ is described by the following Stokes equations (see Remark 3.4 for the case of the Navier-Stokes equations)

\begin{equation}
\begin{cases}
\Sigma_\varepsilon = -p_\varepsilon \text{Id} + \mu \varepsilon(u_\varepsilon) \\
-\nabla \cdot \Sigma_\varepsilon = f & \text{in } \Omega_\varepsilon \\
\nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\varepsilon
\end{cases}
\end{equation}

where the stress $\Sigma_\varepsilon$ is an $N \times N$ symmetric tensor, the velocity $u_\varepsilon$ is a vector in $\mathbb{R}^N$, the pressure $p_\varepsilon$ is a scalar, and the viscosity $\mu$ of the fluid is a strictly positive constant. The Stokes equations (2.5) are endowed with the so-called “slip” boundary condition, which, loosely speaking, allows the fluid to slip on the obstacles, but not to go through them. More precisely, the normal component of the velocity is equal to zero on the obstacles, while the tangential velocity is proportional to the tangential component of the force exerted by the flow on each point of the obstacle’s boundary. As is well known in fluid mechanics, this force is equal to the opposite of the normal stress, i.e., $-\Sigma_\varepsilon n$, where $n$ is the normal vector outward from the fluid domain. Denoting by $(\alpha, \mu)$ a sequence of strictly positive slip coefficients, the slip boundary condition on $\Gamma_\varepsilon$, is

\begin{equation}
\begin{cases}
u \cdot n = 0 & \text{on } \Gamma_\varepsilon \\
(\Sigma_\varepsilon n + \alpha, \mu u_\varepsilon) = (n \Sigma_\varepsilon n + \alpha, \mu u_\varepsilon) n & \text{on } \Gamma_\varepsilon.
\end{cases}
\end{equation}

In the end, as is well known in homogenization, the type of the homogeneous boundary condition on the “large” boundary $\partial \Omega$ is irrelevant; we therefore choose for simplicity a Dirichlet boundary condition

\begin{equation}
u_\varepsilon = 0 \quad \text{on } \partial \Omega.
\end{equation}

Regrouping (2.5) to (2.7), and taking into account that $\nabla \cdot u_\varepsilon = 0$ implies $\nabla \cdot \Sigma_\varepsilon = \mu \Delta u_\varepsilon$, we obtain the Stokes system under consideration in this paper.
We emphasize that we shall never use the Dirichlet boundary condition on $\partial \Omega$ in the technical lemmas (including Korn's and Poincaré's inequality) which prove the coercivity of this Stokes system. Thus there is no loss of generality in our choice, and all the results of this paper are also valid for any other boundary condition on $\partial \Omega$ (unless otherwise stated).

Now, we define a set $H_e$ of admissible functions for the velocity

$$H_e = \{ v \in [H^1(\Omega_c)]^N / v : n = 0 \text{ on } \Gamma_e \text{ and } v = 0 \text{ on } \partial \Omega \}.$$  

For a given force $f \in [L^2(\Omega)]^N$, the Stokes system (2.8) has the following variational formulation

$$\begin{aligned} &\text{Find } (u_e, p_e) \in H_e \times [L^2(\Omega_e)/\mathbb{R}] \text{ such that } \\
&\quad \frac{\mu}{2} \int_{\Omega_e} e(u_e) : e(v) - \int_{\Omega_e} p_e \nabla \cdot v + \mu \alpha_e \int_{\Gamma_e} u_e \cdot v = \int_{\Omega_e} f \cdot v \\
&\quad \int_{\Omega_e} q \nabla \cdot u_e = 0 \\&\text{for each } v \in H_e \\
&\text{for each } q \in L^2(\Omega_e)/\mathbb{R}. \end{aligned}$$

Remark 2.1. In [11] the Stokes equations (2.5) were considered with a Dirichlet boundary condition on the obstacles

$$u_e = 0 \quad \text{on } \partial \Omega_e.$$  

In that case, because (2.11) does not involve the stress $\Sigma_e$, the Stokes system (2.5)–(2.11) is equivalent to the following variational formulation

$$\begin{aligned} &\text{Find } (u_e, p_e) \in [H^1_0(\Omega_e)]^N \times [L^2(\Omega_e)/\mathbb{R}] \text{ such that } \\
&\quad \mu \int_{\Omega_e} \nabla u_e \cdot \nabla v - \int_{\Omega_e} p_e \nabla \cdot v = \int_{\Omega_e} f \cdot v \quad \text{for each } v \in [H^1_0(\Omega_e)]^N. \\
&\quad \int_{\Omega_e} q \nabla \cdot u_e = 0 \quad \text{for each } q \in L^2(\Omega_e)/\mathbb{R}. \end{aligned}$$
The fundamental difference between (2.10) and (2.12) is the presence in (2.10) of the symmetric stress tensor in the boundary condition. As a consequence, in this paper there are two additional difficulties, beyond those of [1]. First, the extension of the velocity is no longer obvious: in [1] we extended it by zero in the holes, and this extension was continuous because of the Dirichlet boundary condition. Here, we need another kind of extension (see Lemma 2.2). Second, the coercivity of the variational formulation (2.10) (and thus a uniform estimate of the solution) is proved here through a Korn-type inequality (see Lemma 2.4).

**Proposition 2.1.** If the slip coefficient $\alpha_e$ is strictly positive, there exists a unique solution of the variational formulation (2.10), and thus of the equivalent Stokes system (2.8). (Moreover this result does not use the boundary condition on $\partial \Omega$).

Proposition 2.1 is proved at the end of this section. Before that we prove several technical lemmas which will also be used in Sections 3 and 4. We recall that, in the present section, the hole size is not specified. First, let us introduce an extension operator for the velocity.

**Lemma 2.2.** There exists a continuous linear map $E$, from $[H^1(\Omega_e)]^N$ into $[H^1(\Omega)]^N$, such that, for each $v \in [H^1(\Omega_e)]^N$, we have

\[(i) \quad E_v = v \quad \text{in} \quad \Omega_e\]
\[(ii) \quad \|e(E_v)\|_{L^2(\Omega)} \leq C \|e(v)\|_{L^2(\Omega_e)}\]

where $C$ is a positive constant, which depends only on $\Omega$ and $T$, and not on $\varepsilon$.

Furthermore, if $v$ is defined in the whole $\Omega$, i.e., $v \in [H^1(\Omega)]^N$, then $e(E_v)$ converges strongly to $e(v)$ in $[L^2(\Omega)]^{N^2}$.

Proof: This lemma is classical in the homogenization literature (see Conca [11], Cioranescu and Saint Jean Paulin [10], Tartar [27]), so we only sketch its proof. Each hole $T^\varepsilon_i$ is included in a ball $B_\varepsilon$ of radius $\varepsilon$. We define the closed set

\[W(B_\varepsilon^\varepsilon - T^\varepsilon_i) = \{ v \in [H^1(B_\varepsilon^\varepsilon - T^\varepsilon_i)]^N / e(v) = 0 \in B_\varepsilon^\varepsilon - T^\varepsilon_i \}. \]

It is easy to check that $W(B_\varepsilon^\varepsilon - T^\varepsilon_i)$ is also equal to the set $\{ v = Ax + b \text{ with } 'A = -A \}$. Each $v \in [H^1(B_\varepsilon^\varepsilon - T^\varepsilon_i)]^N$ can be decomposed in

\[v = v_1 + v_2 \quad \text{with} \quad v_1 \in W(B_\varepsilon^\varepsilon - T^\varepsilon_i) \quad \text{and} \quad v_2 \in W(B_\varepsilon^\varepsilon - T^\varepsilon_i) \perp.\]

The function $v_1$ is continuously extended in $T^\varepsilon_i$ by its linear value $Ax + b$. Now, let $F$ be any continuous extension from $[H^1(B_1 - T)]^N$ to $[H^1(B_1)]^N$. By rescaling $F$, we obtain an extension $F^\varepsilon_i$ operating on $(B_\varepsilon^\varepsilon - T^\varepsilon_i)$. Finally we define the extension $E_\varepsilon$ by

\[E_\varepsilon v = v_1 + F^\varepsilon_i v_2 \quad \text{in each} \quad B_\varepsilon^\varepsilon \]
\[E_\varepsilon v = v \quad \text{elsewhere in} \quad \Omega_e.\]
NAVIER-STOKES EQUATIONS

Because \( \|v\|_{H^1(B_1)} \) and \( \|v\|_{L^2(B_1)} \) are two equivalent norms in \( W'(B_1) \), and because the set \( B_1 - T \) is connected, we obtain the desired estimate (2.13) for \( E_\varepsilon \).

Before proving the last statement of Lemma 2.2, let us explain what it means: if \( v \) happens to be defined in the whole \( \Omega \), then the extension \( E_\varepsilon v \) is indeed very "close" to \( v \) in the sense that the sequence \( e(E_\varepsilon v) \) converges strongly to \( e(v) \) in \([L^2(\Omega)]^N\), as \( \varepsilon \) goes to zero. To see that, we write

\[
\|e(v) - e(E_\varepsilon v)\|_{L^2(\Omega)}^2 \leq \sum_{i=1}^{N(\varepsilon)} \|e(v)\|_{L^2(T_i')}^2 + \|e(E_\varepsilon v)\|_{L^2(T_i')}^2
\]

(2.14)

where \( \chi_\varepsilon \) is the characteristic function of the set \( \bigcup_{i=1}^{N(\varepsilon)} B_{\varepsilon i}^\varepsilon \), which has the property that its measure tends to zero. Then, applying the Lebesgue theorem of dominated convergence, the right-hand side of (2.14) goes to zero with \( \varepsilon \), and the lemma is proved.

The next lemma is about a trace result on the boundary \( \Gamma_\varepsilon \).

**Lemma 2.3.** There exists a constant \( C \) which does not depend on \( \varepsilon \) such that, for any \( v \in H^1(\Omega_\varepsilon) \), we have

\[
\frac{\varepsilon^N}{a_\varepsilon^{N-1}} \|v\|_{L^2(\Gamma_\varepsilon)}^2 \leq C \left[ \|v\|_{L^2(\Omega_\varepsilon)}^2 + \sigma^2 \|\nabla v\|_{L^2(\Omega_\varepsilon)}^2 \right]
\]

where \( \sigma \) is defined in (2.2).

**Proof:** Let \( v \in D(\bar{\Omega}_\varepsilon) \). As the model hole \( T \) is included in the unit ball \( B_1 \), each hole \( T_i' \) is also included in a ball \( B_{\varepsilon i}^\varepsilon \) of radius \( a_\varepsilon \). In a first step, by using radial coordinates, we establish an estimate for the trace of \( v \) on \( \partial B_{\varepsilon i}^\varepsilon \). In a second step, we take the trace of \( v \) on \( \partial T_i' \), by using a simple trace lemma in \( B_{\varepsilon i}^\varepsilon - T_i' \).

Let \( B_i' \) be the ball of radius \( \varepsilon \) with the same center as \( B_{\varepsilon i}^\varepsilon \). Let \( r \) be the distance between the center of \( B_i' \) and a point \( x \in B_i' \), and let \( e_r \) be the associated unit radial vector. In \( B_i' - B_{\varepsilon i}^\varepsilon \) we have

\[
v(x) - v(x + (a_\varepsilon - r)e_r) = \int_0^r \frac{\partial v}{\partial r} [x + (t - r)e_r] \, dt.
\]
Then
\[ \frac{\epsilon^N}{a^N} \|v\|_{L^2(\partial B_T^\gamma)} \leq \frac{\epsilon^N}{a^N} \|v\|_{L^2(\Omega_t)} + \left\| \left[ \int_{\partial B_T^\gamma} \frac{\partial v}{\partial r}[x + (t - r)e_r] \, dr \right] \right\|_{L^2(\partial B_T^\gamma)}.
\]

But the Schwarz inequality gives
\[ \left( \int_{\partial B_T^\gamma} \frac{\partial v}{\partial r}[x + (t - r)e_r] \, dr \right)^2 \leq \left( \int_{\partial B_T^\gamma} \left[ \frac{\partial v}{\partial r}[x + (t - r)e_r] \right]^2 \, dr \right) \int_{\partial B_T^\gamma} \, dr \leq \sigma^2 \|\nabla v\|_{L^2(\partial B_T^\gamma)}.
\]

Thus
\[ \left\| \left[ \int_{\partial B_T^\gamma} \frac{\partial v}{\partial r}[x + (t - r)e_r] \, dr \right] \right\|^2_{L^2(\partial B_T^\gamma)} \leq \left( \int_{\partial B_T^\gamma} \left[ \frac{\partial v}{\partial r}[x + (t - r)e_r] \right]^2 \, dr \right) \int_{\partial B_T^\gamma} \, dr \leq \sigma^2 \|\nabla v\|_{L^2(\partial B_T^\gamma)}.
\]

Hence
\[ (2.15) \quad \frac{\epsilon^N}{a^N} \|v\|_{L^2(\partial B_T^\gamma)} \leq \frac{\epsilon^N}{a^N} \|v\|_{L^2(B_T^\gamma)} + \sigma^2 \|\nabla v\|_{L^2(\partial B_T^\gamma)}.
\]

Now, consider the following trace estimate in \(B_1 - T\), which obviously holds true because \(B_1 - T\) is a smooth connected set
\[ \|v\|_{L^2(B_1 - T)} \leq C \left[ \|v\|_{L^2(B_1)} + \|\nabla v\|_{L^2(B_1 - T)} \right].
\]

Rescaling it yields
\[ \|v\|_{L^2(B_1 - T)} \leq C \left[ \|v\|_{L^2(B_1)} + \|\nabla v\|_{L^2(B_1 - T)} \right].
\]

Recalling that \(a_\epsilon \ll \epsilon \ll \sigma_\epsilon\), and combining the above estimate with (2.15), we obtain
\[ (2.16) \quad \frac{\epsilon^N}{a^N} \|v\|_{L^2(\partial T_\gamma)} \leq C \left[ \|v\|_{L^2(B_1 - T)} + \|\nabla v\|_{L^2(B_1 - T)} \right].
\]

Summing (2.16) for \(i = 1\) to \(N(\epsilon)\) leads to the desired result for any function \(v \in D(\tilde{\Omega}_\epsilon)\). By the density of \(D(\tilde{\Omega}_\epsilon)\) in \(H^1(\Omega_\epsilon)\), the result holds true also in \(H^1(\Omega_\epsilon)\).

The following lemma is concerned with a Korn-type inequality which allows us to prove the coercivity of the variational formulation (see the proof of Proposition
2.1 below) and to establish an a priori estimate for the velocity in Sections 3 and 4.

**Lemma 2.4.** There exists a positive constant $C$, which depends only on $\Omega$ and $T$, such that for each $v \in [H^1(\Omega)]^N$

$$
\|\nabla v\|_{L^2(\Omega)} \leq C \left[ (1 + \sigma_v) \|e(v)\|_{L^2(\Omega)} + \left[ \frac{\varepsilon^N}{a^{N-1}_\varepsilon} \right]^{1/2} \|v\|_{L^2(\Omega)} \right].
$$

Proof: Let $W(\Omega)$ be the closed set $\{ v \in [H^1(\Omega)]^N / e(v) = 0 \text{ in } \Omega \}$, which is also equal to $\{ v = Ax + b \text{ with } \vA = -A \}$. We denote by $W(\Omega)^\perp$ its orthogonal complement with respect to the usual inner product in $[H^1(\Omega)]^N$. Each $v \in [H^1(\Omega)]^N$ can be decomposed in

$$
v = v_1 + v_2 \quad \text{with} \quad v_1 \in W(\Omega) \quad \text{and} \quad v_2 \in W(\Omega)^\perp.
$$

It is well known that $\|\nabla v\|_{L^2(\Omega)}$ and $\|e(v)\|_{L^2(\Omega)}$ are two equivalent norms in $W(\Omega)^\perp$. Thus, for $v_2 \in W(\Omega)^\perp$, we have

$$
(2.17) \quad \|\nabla v_2\|_{L^2(\Omega)} \leq C \|e(v_2)\|_{L^2(\Omega)},
$$

where the constant $C$ depends only on $\Omega$. For $v_1 = (Ax + b)$, with $\vA = -A$, we compute

$$
\|\nabla v_1\|_{L^2(\Omega)} = |A|^2 |\Omega|
$$

$$
\|v_1\|_{L^2(\Omega)}^2 = \frac{a^{N-1}_\varepsilon |\partial \Omega|}{(2\varepsilon)^N} \|Ax + b\|_{L^2(\Omega)}^2 [1 + o(1)].
$$

Thus

$$
(2.18) \quad \|\nabla v_1\|_{L^2(\Omega)} \leq C \frac{\varepsilon^N}{a^{N-1}_\varepsilon} \|v_1\|_{L^2(\Omega)},
$$

where the constant $C$ does not depend on $\varepsilon$, nor on $v_1$. Summing (2.17) and (2.18) leads to

$$
(2.19) \quad \|\nabla v\|_{L^2(\Omega)} \leq C \left[ \|e(v)\|_{L^2(\Omega)} + \frac{\varepsilon^N}{a^{N-1}_\varepsilon} \|v_1\|_{L^2(\Omega)} \right]
$$

$$
\leq C \left[ \|e(v)\|_{L^2(\Omega)} + \frac{\varepsilon^N}{a^{N-1}_\varepsilon} \|v\|_{L^2(\Omega)} + \frac{\varepsilon^N}{a^{N-1}_\varepsilon} \|v_2\|_{L^2(\Omega)} \right].
$$
Using the trace Lemma 2.3 we get

\begin{equation}
\frac{\epsilon^N}{a_{\epsilon}^{N-1}} \|v_2\|_{L^2(\Gamma)}^2 \leq \|v_2\|_{L^2(\Omega)}^2 + \sigma_\epsilon^2 \|\nabla v_2\|_{L^2(\Omega)}^2.
\end{equation}

Because $v_2$ belongs to the orthogonal complement of $W(\Omega)$ (which contains the constants), its average over $\Omega$ is equal to zero. Thus, the Poincaré-Wirtinger inequality gives

\[ \|v_2\|_{L^2(\Omega)} \leq C \|\nabla v_2\|_{L^2(\Omega)}. \]

Thanks to (2.17) we deduce from (2.20)

\[ \frac{\epsilon^N}{a_{\epsilon}^{N-1}} \|v_2\|_{L^2(\Gamma)}^2 \leq (1 + \sigma_\epsilon^2) \|e(v_2)\|_{L^2(\Omega)}. \]

Plugging the above estimate in (2.19) concludes the proof.

**Remark 2.2.** Lemmas 2.2, 2.3, and 2.4 do not use any assumptions on the type of the boundary condition on $\Gamma_\epsilon$ and on $\partial \Omega$. In particular, the fact that the normal component of the velocity is equal to zero on $\Gamma_\epsilon$ is never used. Therefore those lemmas hold true in a more general context. However, Lemma 2.5 below relies upon our choice of the boundary condition on $\Gamma_\epsilon$ for the velocity.

Now, we establish a Poincaré inequality in $H_\epsilon$, which will be used in Sections 3 and 4 for obtaining an a priori estimate for the velocity.

**Lemma 2.5.** There exists a positive constant $C$, which depends only on $\Omega$, such that

(i) for each $v \in [H^1(\Omega)]^N$, satisfying $v \cdot n = 0$ on $\Gamma_\epsilon$

\[ \|v\|_{L^2(\Omega)} \leq C\epsilon \|\nabla v\|_{L^2(\Omega)}. \]

(ii) for each $v \in [H^1(\Omega)]^N$, satisfying $v \cdot n = 0$ on $\Gamma_\epsilon$

\[ \|v\|_{L^2(\Omega)} \leq C\epsilon \|\nabla v\|_{L^2(\Omega)}. \]

**Proof:** The same arguments as in the previous Lemma 2.3 lead to an inequality similar to (2.15)

\[ \|v\|_{L^2(\Omega)} \leq \frac{\epsilon^N}{a_{\epsilon}^{N-1}} \|v\|_{L^2(\omega_{\epsilon})} + \sigma_\epsilon^2 \|\nabla v\|_{L^2(\omega_{\epsilon})}. \]
Recall that the model hole $T$ contains a ball $B_{0}$ (with radius $r_{0} > 0$). Thus, the following Poincaré inequality holds for any $v \in [H^{1}(B_{1} - T)]^{N}$ such that $v \cdot n = 0$ on $\partial T$

$$\|v\|_{L^{2}(B_{1} - T)} \leq C\|\nabla v\|_{L^{2}(B_{1} - T)}.$$ 

Besides, we also have the following trace property for any $v \in [H^{1}(B_{1} - T)]^{N}$ such that $v \cdot n = 0$ on $\partial T$

$$\|v\|_{L^{2}(\partial B_{1})} \leq C\|\nabla v\|_{L^{2}(B_{1} - T)}.$$ 

Rescaling to size $a$, the two previous inequalities give

$$\|v\|_{L^{2}(aB_{1} - T')} \leq Ca^{\frac{3}{2}}\|\nabla v\|_{L^{2}(aB_{1} - T')}$$

(2.24) $$\|v\|_{L^{2}(\partial aB_{1})} \leq Ca\|\nabla v\|_{L^{2}(aB_{1} - T')}$$

for any $v \in [H^{1}(B_{1} - T')]^{N}$ such that $v \cdot n = 0$ on $\partial T'$.

Recalling that $a_{\epsilon} \leq \epsilon \leq \sigma_{\epsilon}$, and combining (2.23) and (2.24) yields

(2.25) $$\|v\|_{L^{2}(a_{\epsilon}B_{1} - T')} \leq C\sigma_{\epsilon}^{2}\|\nabla v\|_{L^{2}(a_{\epsilon}B_{1} - T')}.$$ 

From (2.25), it is not difficult to obtain an equivalent inequality in $P_{\epsilon} - T'$. Then, summing these inequalities for all cells $P_{\epsilon}$ gives inequality (2.21). In order to obtain (2.22), we just need to add to the above ingredients the following Poincaré inequality in $T$, which holds for any $v \in [H^{1}(T)]^{N}$ such that $v \cdot n = 0$ on $\partial T$

$$\|v\|_{L^{2}(T)} \leq \|\nabla v\|_{L^{2}(T)}.$$ 

Rescaling it to size $a_{\epsilon}$, and adding it to (2.25) gives (2.22).
LEMMA 2.7. Let \( p_e \) be a function in \( L^2(\Omega_e) / \mathbb{R} \). There exists a unique function \( P_e \in L^2(\Omega_e) / \mathbb{R} \) which satisfies the following equality

\[
(2.26) \quad \langle \nabla P_e, v \rangle_{H^{-1},H(\Omega)} = \langle \nabla p_e, R_e v \rangle_{H^{-1},H(\Omega)} \quad \text{for each} \quad v \in [H_0^1(\Omega)]^N.
\]

Thanks to Lemma 2.6, \( P_e \) is an extension of \( p_e \), but in addition, because of the explicit construction of \( R_e \), we obtain the precise value of \( P_e \) in the holes

\[
(2.27) \quad P_e = p_e \quad \text{in} \ \Omega_e \quad \text{and} \quad P_e = \frac{1}{|C_i|} \int_{C_i} p_e \quad \text{in each hole } T_i,
\]

where \( C_i \) is a control volume around the hole \( T_i \), defined as the part outside \( T_i \) of the ball of radius \( r \) with same center as \( T_i \).

(See Section 2.2 and Proposition 3.4.10 in [1] for proofs of Lemmas 2.6 and 2.7.)

Proof of Proposition 2.1: Thanks to the Korn-type inequality obtained in Lemma 2.4, the variational formulation (2.10) is coercive as soon as the slip coefficient \( \alpha_e \) is strictly positive

\[
K_e \|
abla u_e \|^2_{L^2(\Omega_e)} \leq K_e \|
abla (e(u_e)) \|^2_{L^2(\Omega_e)} \leq C[\|
abla e(u_e) \|^2_{L^2(\Omega_e)} + \alpha_e \|
abla u_e \|^2_{L^2(\Omega_e)}]
\]

with

\[
K_e = \min \left[ \frac{1}{1 + \sigma^2}, \frac{\alpha_e \alpha^{-1}}{e^N} \right].
\]

Then a variant of the Lax-Milgram theorem asserts that the variational formulation (2.10) has a unique solution (see, e.g., Girault Raviart [14]). Finally, integrating by parts in (2.10) it is easy to show that a solution of (2.10) is also a weak solution of the Stokes system (2.8), and vice-versa.

3. The Critical Scaling: Main Results

In this section we prove the convergence of the homogenization process, when the hole size and the slip coefficient scale as follows:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \alpha_e & = \alpha \\
\lim_{\varepsilon \to 0} \sigma & = \sigma,
\end{align*}
\]

where \( \alpha \) and \( \sigma \) are strictly positive constants. In the sequel, we will refer to those relations (3.1) as the critical scaling. For example, the latter relation in (3.1) is
satisfied by a hole size \( a \), equal to \( C_0 e^{N/(N-2)} \) for \( N \geq 3 \), or \( \exp\left\{-C_0/\epsilon^2\right\} \) for \( N = 2 \) with \( C_0 > 0 \), and \( \sigma \) given by \( \sigma = 1/C_0^{(N-1)/2} \) for \( N \geq 3 \), or \( \sigma = C_0^{1/2} \) for \( N = 2 \).

Using the lemmas of the first section, we obtain some a priori estimates for the solution of the variational formulation (2.10), without making essential use of the Dirichlet boundary condition on the “large” boundary \( \partial \Omega \).

**Proposition 3.1.** Let \((u_\epsilon, p_\epsilon)\) be the solution of (2.10), and \((E_\epsilon(u_\epsilon), P_\epsilon)\) its extension in \( \Omega \), as defined in Lemmas 2.2 and 2.7. Then we have

\[
\|E_\epsilon u_\epsilon\|_{L^2(\Omega)} + \|\nabla(E_\epsilon u_\epsilon)\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}
\]

\[
\|P_\epsilon\|_{L^2(\Omega)^{N\times N}} \leq C \|f\|_{L^2(\Omega)}
\]

where the constant \( C \) depends only on \( \Omega \) and \( T \) (and not on \( \epsilon \)).

**Proof:** Introducing \( v = u_\epsilon \) in (2.10) leads to

\[
\frac{\mu}{2} \int_{\Omega_\epsilon} |e(u_\epsilon)|^2 + \mu \alpha_\epsilon \int_{\Gamma_\epsilon} |u_\epsilon|^2 = \int_{\Omega_\epsilon} f \cdot u_\epsilon.
\]

Because \( u_\epsilon \in [H^1_0(\Omega_\epsilon)]^N \) and satisfies \( u_\epsilon \cdot n = 0 \) on \( \Gamma_\epsilon \), Poincaré’s inequality (2.21) (see Lemma 2.5) furnishes an upper bound of the right-hand side of (3.2)

\[
\int_{\Omega_\epsilon} f \cdot u_\epsilon \leq C \|f\|_{L^2(\Omega)} \|\nabla u_\epsilon\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|\nabla(E_\epsilon u_\epsilon)\|_{L^2(\Omega)}.
\]

Thanks to the properties of the extension \( E_\epsilon \) and to the scaling (3.1), the Korn-type inequality of Lemma 2.4 gives a lower bound of the left-hand side of (3.2)

\[
\frac{\mu}{2} \int_{\Omega_\epsilon} |e(u_\epsilon)|^2 + \mu \alpha_\epsilon \int_{\Gamma_\epsilon} |u_\epsilon|^2 \\
\geq C \left[ \int_{\Omega} |e(E_\epsilon u_\epsilon)|^2 + \alpha_\epsilon \int_{\Gamma_\epsilon} |E_\epsilon(u_\epsilon)|^2 \right] \geq C \|\nabla(E_\epsilon u_\epsilon)\|_{L^2(\Omega)}.
\]

Finally, (3.3) and (3.4) yield

\[
\|\nabla(E_\epsilon u_\epsilon)\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)},
\]

and Poincaré’s inequality (2.22) gives

\[
\|E_\epsilon u_\epsilon\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
\]
Now we turn to an estimate for the pressure. Recall the definition (2.26) of the extension $P_e$

$$
\langle \nabla P_e, v \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \nabla P_e, R_v \rangle_{H^{-1}, H_0^1(\Omega)} \text{ for each } v \in [H_0^1(\Omega)]^N.
$$

Introducing the Stokes equation, and integrating by parts, leads to

$$
|\langle \nabla P_e, v \rangle| \leq \|\nabla u_u\|_{L^2(\Omega)} \|\nabla (R_v)\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|R_v\|_{L^2(\Omega)}.
$$

Thanks to the properties of $R$, in Lemma 2.6, and to the previous estimate on the velocity $u_u$, we obtain

$$
(3.5) \quad \|\nabla P_e\|_{H^{-1}(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
$$

But a well-known lemma of functional analysis claims that, because of (3.5), $P_e$ belongs to $L^2(\Omega)$, and we have

$$
\|P_e\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla P_e\|_{H^{-1}(\Omega)}.
$$

Hence (3.5) gives the result.

Because of the estimates in Proposition 3.1, one can extract a subsequence of extensions of the solutions, still denoted $(E, u_u, P_e)_{e>0}$ and there exists some $(u, p) \in [H_0^1(\Omega)]^N \times L^2(\Omega)/\mathbb{R}$ such that $(E, u_u, P_e)$ converges weakly to $(u, p)$ in $[H_0^1(\Omega)]^N \times L^2(\Omega)/\mathbb{R}$. The problem is now to find which equation is satisfied by the limit (or homogenized) velocity $u$ and pressure $p$. For that purpose we use the energy method introduced by L. Tartar in [27] (see also F. Murat [22]). The main idea is to compute in a unit cell the solutions of the “local” problem corresponding to the Stokes system (2.8). Then, by rescaling and extending by periodicity these local solutions, we construct so-called test functions (which can be interpreted as boundary layers around the holes $T_1$). After multiplication by a fixed smooth function, they are introduced in the variational formulation. Finally, integrating by parts, we can pass to the limit in the variational formulation, and obtain the limit equation. The following lemma gives the so-called local problem.

**Lemma 3.2.** For $N \geq 3$ the local problem is

$$
(3.6) \quad \begin{cases}
\nabla q_k - \Delta w_k = 0 & \text{in } \mathbb{R}^N - T \\
\nabla \cdot w_k = 0 & \text{in } \mathbb{R}^N - T \\
w_k \cdot n = 0 & \text{on } \partial T \\
\alpha w_k = 2\left(\frac{\partial w_k}{\partial n}\right)n - e(w_k)n & \text{on } \partial T \\
w_k = e_k & \text{at } \infty.
\end{cases}
$$
There exists a unique solution of (3.6) such that $\|q_k\|_{L^2(\mathbb{R}^N - T)} < +\infty$ and $\|\nabla w_k\|_{L^2(\mathbb{R}^N - T)} < +\infty$. Moreover, denoting by $F_k$ the drag force applied on $T$, i.e., $F_k = \int_{\partial T} (q_k n - e(w_k)n)$, where $n$ is the interior normal vector of $\partial T$, we have

$$F_k \cdot e_i = \left[ \frac{1}{2} \int_{\mathbb{R}^N - T} e(w_k) \cdot e(w_i) + \alpha \int_{\partial T} w_k \cdot w_i \right] \text{ for each } i \in [1; N].$$

For $N = 2$ the local problem is

$$\begin{cases}
\nabla q_k - \Delta w_k = 0 & \text{in } \mathbb{R}^N - T \\
\nabla \cdot w_k = 0 & \text{in } \mathbb{R}^N - T \\
w_k \cdot n = 0 & \text{on } \partial T \\
\alpha w_k = 2 \left( \frac{\partial w_k}{\partial n} \right) n - e(w_k)n & \text{on } \partial T \\
w_k = (\log r)e_k & \text{at } \infty.
\end{cases}$$

There exists a unique solution of (3.8), and still denoting by $F_k = \int_{\partial T} (q_k n - e(w_k)n)$ the drag force applied on $T$, we obtain a paradoxical result: whatever the size and the shape of $T$, and for any value of the slip coefficient $\alpha$, we have

$$F_k = \frac{4\pi e_k}{\alpha} \text{ for } k = 1, 2.$$  

(See Section 5 for the proof.)

**Definition 3.1.** Let us define a matrix $M$ by

$$Me_k = \frac{1}{2N} F_k,$$

where $F_k$ is the drag force introduced in the above Lemma 3.2. In the two-dimensional case, we always have $M = \pi Id$, but in the other cases ($N \geq 3$), the matrix $M$ is given by formula (3.7) and depends on the slip coefficient $\alpha$. In any case, $M$ is a symmetric positive definite matrix.

**Remark 3.1.** There are two fundamental differences between the local problem in two and three (or more) dimensions. First, the profile of the velocity at infinity is uniform for the three (or more) dimensional case, while it is logarithmic for the two-dimensional case. Actually the celebrated Stokes paradox asserts that, in the plane, there is no solution of (3.8) which remains bounded at infinity, except the zero solution (see, e.g., Finn [12]). That is why there is logarithmic growth at infinity in (3.8). Second, although in the three (or more) dimensional case the drag force $F_k$ heavily depends both on the size and shape of the obstacle $T$, and on the slip coefficient $\alpha$, in two dimensions it does not, and actually is equal to the...
drag force obtained with a Dirichlet boundary condition. This paradox has already been pointed out by R. Finn and D. Smith in [13].

**Proposition 3.3.** With the help of the local solutions \((w_k, q_k)_{1 \leq k \leq N}\), we can construct test functions \((w^e_k, q^e_k)\), which satisfy the following properties:

(i) \(q^e_k \to 0\) in \(L^2(\Omega) / \mathbb{R}\) weakly.
(ii) \(w^e_k \to e_k\) in \(H^1(\Omega))^N\) weakly, and \(\nabla \cdot w^e_k = 0\) in \(\Omega\).
(iii) \(w^e_k \cdot n = 0\) and \((\alpha / a_e)w^e_k = 2[(\partial w^e_k / \partial n) \cdot n] - e(w^e_k) n\) on \(\Gamma_e\).
(iv) Let \(\mu^e_k = \nabla q^e_k - \Delta w^e_k \in [H^{-1}(\Omega)]^N\). Then \(\mu^e_k\) has compact support in \(\Omega_e\).

Moreover, extending \(\mu^e_k\) by 0 in \(\Omega - \Omega_e\), we have

\[
\mu^e_k \to \frac{1}{\pi} \text{Me}_k \quad \text{in} \quad [H^{-1}(\Omega)]^N \quad \text{strongly}.
\]

**Remark 3.2.** Let us explain in more detail property (iv) of Proposition 2.5. A priori the distribution \((\nabla q^e_k - \Delta w^e_k)\) is defined in the whole of \(\Omega\); because of (i) and (ii) it is bounded (but not necessarily compact) in \([H^{-1}(\Omega)]^N\), and its weak limit is zero. But if we restrict it to \([H^{-1}(\Omega_e)]^N\), that is, if we skip the contributions coming from the obstacles and their boundaries, then it becomes a compact sequence with a non-zero limit. From a technical viewpoint, in the process of extending \(\mu^e_k\) by 0 in \(\Omega - \Omega_e\), the main ingredient is that it has compact support in \(\Omega_e\), i.e., it is equal to zero in a vicinity of the boundary \(\partial \Omega_e\).

Proposition 3.3 is proved in Section 5. Let us explain roughly what is the connection between Lemma 3.2 and Proposition 3.3. The test functions \((w^e_k, q^e_k)_{1 \leq k \leq N}\) are obtained by rescaling the local solutions \((w_k, q_k)_{1 \leq k \leq N}\) in each cell \(P^\varepsilon_i\). Let \(C^\varepsilon_i\) be the ball of radius \(\varepsilon / 2\) whose center coincides with that of \(P^\varepsilon_i\). In \(C^\varepsilon_i\) we define

\[
\begin{align*}
\{ \begin{array}{l}
w^e_k(x) = w_k \left( \frac{x}{a_e} \right) \quad \text{and} \quad q^e_k = \frac{1}{a_e} q_k \left( \frac{x}{a_e} \right) \quad \text{for} \quad N \geq 3 \\
w^e_k(x) = \frac{1}{\log \frac{e}{2a_e}} w_k \left( \frac{x}{a_e} \right) \quad \text{and} \quad q^e_k = \frac{1}{a_e \log \left( \frac{e}{2a_e} \right)} q_k \left( \frac{x}{a_e} \right) \quad \text{for} \quad N = 2.
\end{array} \}
\end{align*}
\]

In the remaining part \(P^\varepsilon_i - C^\varepsilon_i\) of each cell, we match the above expression with the value \(w^e_k = e_k\) on the boundary \(\partial P^\varepsilon_i\), in order to insure that \(w^e_k\) belongs to \([H^1(\Omega)]^N\).

We are now in a position to state our main result

**Theorem 3.4.** Let \(M\) be the matrix defined in Definition 3.1. Let \((u_e, p_e)\) be the unique solution of the Stokes system (2.8). Then its extension \((E u_e, P_e)\) converges
weakly in $[H^1_0(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$ to $(u, p)$, which is the unique solution of the homogenized system (3.10)

\[
\begin{align*}
\text{Find } (u, p) &\in [H^1_0(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \text{ such that:} \\
\n\n&\nabla \cdot u = 0 \quad \text{in } \Omega \\
\n&\nabla p - \mu \Delta u + \frac{\mu}{\sigma^2} Mu = f \quad \text{in } \Omega \\
\n\end{align*}
\]

(3.10)

Remark 3.3. The homogenized system (3.10) is a law of Brinkman type (see the original paper of Brinkman [8]). The new term $Mu$ in (3.10) expresses the presence of the holes which have otherwise disappeared after passing to the limit. We already obtained in [1] a Brinkman type law as the limit system of the Stokes equations with a Dirichlet boundary condition. Although the size of the holes is the same in both cases, the local problem and therefore the matrix $M$ are different, except in two dimensions. The dependence of the matrix $M$ on $\alpha$ is not obvious. However, we can compare the value of $M$ for two different values of $\alpha$

\[
0 < \alpha_1 \leq \alpha_2 \Rightarrow \langle \xi M(\alpha_1) \xi \rangle \leq \langle \xi M(\alpha_2) \xi \rangle \quad \text{for any } \xi \in \mathbb{R}^N.
\]

Moreover, the limit of $M(\alpha)$, when $\alpha$ tends to infinity, is the matrix $M$ obtained with a Dirichlet boundary condition (see Section 5). Finally, in order to illustrate the dependence of $\alpha$ on $M$, we give its value when the obstacle $T$ is the unit ball

\[
M = \frac{S_n}{2^N} \frac{N(N-2)}{N-1} \frac{2 + \alpha}{N + \alpha} \text{Id} \quad \text{for } N \geq 3
\]

($S_n$ is the area of the unit sphere in $\mathbb{R}^N$).

Proof: We apply the energy method (introduced by L. Tartar in [27], and adapted to the case of domains perforated with tiny holes by D. Cioranescu and F. Murat in [9]). For any fixed $\phi \in D(\Omega)$, we introduce in the variational formulation (2.10) the test functions

\[(\phi w_k) \in H_{\varepsilon} \quad \text{and} \quad (\phi q_k) \in L^2(\Omega_\varepsilon)/\mathbb{R}.
\]

We obtain

\[
\begin{align*}
\frac{\mu}{2} \int_{\Omega_\varepsilon} e(u_\varepsilon) : e(\phi w_k) - \int_{\Omega_\varepsilon} p_\varepsilon \nabla \cdot (\phi w_k) + \mu \alpha_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon \cdot (\phi w_k) = \int_{\Omega_\varepsilon} f \cdot (\phi w_k) \\
\int_{\Omega_\varepsilon} (\phi q_k) \nabla \cdot u_\varepsilon = 0.
\end{align*}
\]

(3.11)
Expanding (3.11), and using that \( w_k^i \) is divergence-free, gives

\[
\begin{aligned}
\frac{\mu}{2} \int_{\Omega_i} \phi e(u_\varepsilon) : e(w_k^i) + \frac{\mu}{2} \int_{\Omega_i} e(u_\varepsilon) : (\nabla \phi \otimes w_k^i + w_k^i \otimes \nabla \phi) \\
- \int_{\Omega_i} p_k w_k^i \cdot \nabla \phi + \mu \alpha_\varepsilon \int_{\Gamma_i} \phi u_\varepsilon \cdot w_k^i = \int_{\Omega_i} \phi f \cdot w_k^i \\
\int_{\Omega_i} \phi q_k^i \nabla \cdot u_\varepsilon = 0.
\end{aligned}
\]

(3.12)

We integrate by parts in the first equation of (3.12)

\[
\begin{aligned}
\int_{\Omega_i} \phi e(u_\varepsilon) : e(w_k^i) = \int_{\Omega_i} e(\phi u_\varepsilon) : e(w_k^i) \\
- \int_{\Omega_i} e(w_k^i) : (\nabla \phi \otimes u_\varepsilon + u_\varepsilon \otimes \nabla \phi),
\end{aligned}
\]

(3.13)

and

\[
\begin{aligned}
\frac{\mu}{2} \int_{\Omega_i} e(\phi u_\varepsilon) : e(w_k^i) = -\mu \int_{\Omega_i} \phi u_\varepsilon \cdot \Delta w_k^i + \mu \int_{\Gamma_i} \phi u_\varepsilon \left( \frac{\partial w_k^i}{\partial n} + \nabla w_k^i \cdot n \right).
\end{aligned}
\]

(3.14)

An integration by parts in the second equation of (3.12) gives

\[
\begin{aligned}
\int_{\Omega_i} \phi q_k^i \nabla \cdot u_\varepsilon = -\int_{\Omega_i} \phi u_\varepsilon \cdot \nabla q_k^i - \int_{\Omega_i} q_k^i u_\varepsilon \cdot \nabla \phi + \int_{\Gamma_i} \phi q_k^i u_\varepsilon \cdot n.
\end{aligned}
\]

(3.15)

Summing the two equations in (3.12), and combining (3.13) with (3.15) yields

\[
\begin{aligned}
\mu \int_{\Omega_i} \phi u_\varepsilon \cdot (\nabla q_k^i - \Delta w_k^i) + \frac{\mu}{2} \int_{\Omega_i} e(u_\varepsilon) : (\nabla \phi \otimes w_k^i + w_k^i \otimes \nabla \phi) \\
- \frac{\mu}{2} \int_{\Omega_i} e(w_k^i) : (\nabla \phi \otimes u_\varepsilon + u_\varepsilon \otimes \nabla \phi) + \mu \int_{\Omega_i} q_k^i u_\varepsilon \cdot \nabla \phi \\
- \int_{\Omega_i} p_k w_k^i \cdot \nabla \phi + \mu \int_{\Gamma_i} \phi u_\varepsilon \left( \frac{\partial w_k^i}{\partial n} + \nabla w_k^i \cdot n - q_k^i n + \alpha_\varepsilon w_k^i \right) = \int_{\Omega_i} \phi f \cdot w_k^i.
\end{aligned}
\]

(3.16)

Denote by \( 1_{\Omega_\varepsilon} \) the characteristic function of the set \( \Omega_\varepsilon \), which is equal to 1 in \( \Omega_\varepsilon \) and 0 elsewhere. Introducing the extensions of \((u_\varepsilon, p_\varepsilon)\), and using the properties of \((w_k^i, q_k^i)\) (see Proposition 3.3), we obtain from (3.16)
In order to pass to the limit in (3.17), we note that, if a sequence \(u_i\) converges strongly to a limit \(v\) in \(L^2(\Omega)\), then the sequence \(I_\Omega u_i\) converges strongly to the same limit \(v\) in \(L^2(\Omega)\) (because the measure of \(\Omega - \Omega_i\) tends to zero). Thus, using the estimates of Propositions 3.1 and 3.3, and Rellich’s theorem, we obtain

\[
\frac{\mu}{2} \int_{\Omega} 1_\Omega e(u_i) : (\nabla \phi \otimes \nabla w_i) + \mu \left( \frac{\alpha}{\alpha_c} \right) \int_{\Gamma_e} \phi u_i \cdot w_i \\
+ \mu \int_{\Omega} 1_\Omega q_i \nabla E_e(u_i) \cdot \nabla \phi - \int_{\Omega} \nabla \cdot (1_\Omega P_i w_i) \cdot \nabla \phi = \int_{\Omega} 1_\Omega \phi f \cdot w_i.
\]

Integrating (3.18) by parts, and using the symmetry of \(M\), gives

\[
\frac{\mu}{\sigma^2} \langle Mu, \phi e_k \rangle + \mu \langle \Delta u, \phi e_k \rangle + \langle \nabla p, \phi e_k \rangle = \langle f, \phi e_k \rangle \quad \text{for each} \quad k \in [1; N].
\]

Thus

(3.19) \quad \nabla p - \mu \Delta u + \frac{\mu}{\sigma^2} Mu = f \quad \text{in } [D'(\Omega)]^N.

Besides, we know that \(\nabla \cdot (u_i) = 0\) in \(\Omega_i\), with \(u_i \cdot n = 0\) on \(\partial \Omega_e\), and that \(1_\Omega u_i\) converges strongly to \(u\) in \([L^2(\Omega)]^N\). Passing to the limit yields

(3.20) \quad \nabla \cdot u = 0 \quad \text{in } \Omega.

Regrouping (3.19) and (3.20) we obtain the following homogenized system

\[
\begin{align*}
\text{Find } (u, p) & \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \quad \text{such that} \\
\nabla p - \mu \Delta u + \frac{\mu}{\sigma^2} Mu = f & \quad \text{in } \Omega \\
\nabla \cdot u = 0 & \quad \text{in } \Omega.
\end{align*}
\]

Because \(M\) is a positive matrix, there exists a unique solution of (3.21). Then, all the subsequences of \((E_{i}, u_{i}, P_{i})\) converge to the same limit. So the entire sequence converges.
Remark 3.4. When the space dimension is $N = 2$ or $3$, Theorem 3.4 can be easily generalized to the Navier-Stokes equations

\[
\begin{cases}
\text{Find } (u_c, p_c) \in H_c \times [L^2(\Omega_c)/\mathbb{R}] \quad \text{such that} \\
\nabla p_c + u_c \cdot \nabla u_c - \mu \Delta u_c = f \quad \text{in } \Omega_c \\
\nabla \cdot u_c = 0 \quad \text{in } \Omega_c \\
u_c \cdot n = 0 \quad \text{on } \Gamma_c \\
\alpha_c u_c = 2 \left( \frac{\partial u_c}{\partial n} \right) n - e(u_c) n \quad \text{on } \Gamma_c.
\end{cases}
\]

(3.22)

For $N = 2$ or $3$, it is well known that there always exists a solution of system (3.22), which is unique at least for small values of $\|f\|_{L^2(\Omega)}$. For such a force $f$, the extension of the solution $(E_c u_c, p_c)$ converges weakly in $[H^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$ to $(u, p)$, which is the unique solution of the homogenized system (3.23)

\[
\begin{cases}
\text{Find } (u, p) \in [H^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \quad \text{such that} \\
\nabla p + u \cdot \nabla u - \mu \Delta u + \frac{\mu}{\sigma^2} M u = f \quad \text{in } \Omega \\
\nabla \cdot u = 0 \quad \text{in } \Omega.
\end{cases}
\]

(3.23)

More precisely, because the sequence $E_c u_c$ converges weakly to $u$ in $[H^1(\Omega)]^N$, the non-linear term $E_c u_c \cdot \nabla (E_c u_c)$ converges strongly to $u \cdot \nabla u$ in $[H^{-1}(\Omega)]^N$, for $N = 2$ or $3$. It is worth noticing that the local problem, the functions $(w_k^i, q_k^i)_{1 \leq k \leq N}$, introduced in Proposition 3.3, and therefore the matrix $M$, are exactly the same for Stokes and Navier-Stokes homogenization.

Remark 3.5. In [1] we gave several other theorems, including correctors for both the velocity and the pressure, and error estimates. We also generalized Theorem 3.4 to the case of a surface distribution of the holes. All those results still hold here, with slight changes in the proofs due to the difference between the Dirichlet boundary condition in [1] and the slip boundary condition under consideration in this paper. For brevity we do not repeat those proofs, and we refer to [1] for further details.

4. Other Scalings

In this section we consider scalings of the hole size and of the slip coefficient different from the critical one (3.1). First, we keep the size of the holes critical, and we examine the full range of values of the slip coefficient. Second, we briefly describe what happens for holes larger or smaller than critical, which is a mere reproduction of the situation described in Part II of [1].
PROPOSITION 4.1. Assume that the following scaling holds

\[
\begin{align*}
\lim_{\varepsilon \to 0} \alpha \beta \varepsilon &= +\infty \\
\lim_{\varepsilon \to 0} \sigma \varepsilon &= \sigma
\end{align*}
\]

where \( \sigma \) is a strictly positive constant. Then Theorem 3.4 is still true, and the matrix \( M \) in the homogenized system (3.10) is obtained through a local problem with a Dirichlet boundary condition, which corresponds in Lemma 3.2 to the limit value \( \alpha = +\infty \).

Proof: Because \( \alpha \varepsilon \) is larger than \( 1/\alpha \varepsilon \), we can still derive from Lemma 2.4 the uniform coerciveness of the variational formulation (2.10)

\[
\| \nabla (E \varepsilon u \varepsilon) \|_{L^2(\Omega)} \leq C \left[ \frac{1}{2} \int_{\Omega_\varepsilon} |e(u \varepsilon)|^2 + \alpha \varepsilon \int_{\Gamma_\varepsilon} |u \varepsilon|^2 \right].
\]

Thus, we obtain the same estimates, as in Proposition 3.1, for the velocity and the pressure.

On the other hand, we may define local problems as in Lemma 3.2, but with a Dirichlet boundary condition on \( \partial T \). Then, in the same way we can construct test functions \( (w_k^\varepsilon, q_k^\varepsilon) \) which satisfy the properties of Proposition 3.3, except that \( w_k^\varepsilon \cdot \mathbf{n} = 0 \) on \( \Gamma \varepsilon \) is replaced by \( w_k^\varepsilon = 0 \) on \( \Gamma \varepsilon \). In this case the matrix \( M \) corresponds to an infinite value of \( \alpha \). We repeat the proof of Theorem 3.4 with these new functions, and the only difference from Section 3 comes from the boundary contribution

\[
\int_{\Gamma_\varepsilon} \phi (u \varepsilon \cdot \left( \frac{\partial w_k^\varepsilon}{\partial n} + \nabla w_k^\varepsilon \mathbf{n} \right)) .
\]

In order to pass to the limit in this new term (4.3) we point out that, because the boundary \( \partial T \) is smooth, a standard regularity result implies that the solution \( w_k \) of the local problem (3.6) or (3.8) satisfies

\[
\left( \frac{\partial w_k}{\partial n} + \nabla w_k \mathbf{n} \right) \in L^2(\partial T).
\]

Rescaling (4.4) (see definitions (5.19) and (5.20) for the correct scaling of \( w_k^\varepsilon \) in terms of \( w_k \)) and summing over all the holes yields

\[
\left\| \left( \frac{\partial w_k^\varepsilon}{\partial n} + \nabla w_k^\varepsilon \mathbf{n} \right) \right\|_{L^2(\Gamma_\varepsilon)}^2 \leq \frac{C}{\alpha \varepsilon}.
\]
In the meantime from (4.2) we obtain

\[(4.6) \quad \alpha \varepsilon \int_{\Gamma^e} |u| \leq C \|f\|_{L^2(\Omega)}. \]

Combining (4.5) and (4.6) we can bound (4.3)

\[(4.7) \quad \left| \int_{\Gamma^e} \phi u \cdot \left( \frac{\partial w^k}{\partial n} + \nabla w^k \cdot n \right) \right| \leq \frac{C}{(a \varepsilon)^{1/2}}. \]

Thanks to the scaling (4.1) the upper bound in (4.7) tends to zero, so we are still able to pass to the limit and to obtain the homogenized system (3.10).

**Proposition 4.2.** Assume that the following scaling holds

\[(4.8) \quad \begin{cases} \lim_{\varepsilon \to 0} \alpha \varepsilon = 0 \\ \lim_{\varepsilon \to 0} \sigma = \sigma \end{cases} \]

where \(\sigma\) is a strictly positive constant. Moreover, we explicitly assume that we have a Dirichlet boundary condition on the large boundary \(\partial \Omega\) (but not on the boundaries of the holes \(\Gamma_j\)). Then Theorem 3.4 is still true, and the matrix \(M\) in the homogenized system (3.10) is obtained through a local problem with a mixed Dirichlet-Neumann boundary condition which corresponds in Lemma 3.2 to the limit value \(\alpha = 0\).

**Proof:** Because \(\alpha\) is smaller than \(1/\alpha\), we can no longer derive from Lemma 2.4 the uniform estimates for the velocity and the pressure. However, in the presence of a Dirichlet boundary condition on \(\partial \Omega\), these estimates are still valid. Indeed, since \(E, u\) belongs to \([H_0^1(\Omega)]^N\), a standard Korn inequality in \(\Omega\) gives

\[(4.9) \quad \|\nabla (E, u)\|_{L^2(\Omega)} \leq C \int_{\Omega} |e(E, u)|^2. \]

It is easy to see, using (4.9), that the conclusion of Proposition 3.1 still holds.

On the other hand, Lemma 3.2, which furnishes the form of the local problems, is still valid if the slip coefficient \(\alpha\) is equal to zero. Then, as in Proposition 3.3, we construct test functions \((w^k, q^k)\) (note that the matrix \(M\) is defined for \(\alpha = 0\)). We repeat the proof of Theorem 3.4 with these new functions, and the only difference from Section 3 comes from the boundary contribution

\[(4.10) \quad \alpha \varepsilon \int_{\Gamma^e} \phi u \cdot w^k. \]
If the spatial dimension is $N \geq 3$, then using the trace Lemma 2.3 we bound (4.10)

$$\left| \alpha_e \int_{\Gamma_e} \phi u_e \cdot w^e_k \right| \leq C \frac{\alpha_e a_e^{N-1}}{e^N} \| w^e_k \|_{H^1(\Omega_e)} \| u_e \|_{H^1(\Omega_e)}$$

(4.11)

$$\leq C \alpha_e a_e.$$

Thanks to the scaling (4.8) the upper bound in (4.11) tends to zero, so we are still able to pass to the limit and to obtain the homogenized system (3.10). In the two-dimensional case $N = 2$, the definition of $\sigma$, is slightly different (see (2.2)) and we cannot bound (4.10) by using the trace Lemma 2.3. Remark that Lemma 2.3 is not optimal here because it does not use the fact that both $u_e$ and $w^e_k$ have zero normal component on $\Gamma_e$. Nevertheless we can still prove that (4.10) tends to zero, and here is the trick. Note that, in $B_1 - T$, the following trace inequality holds for any $w \in [H^1(B_1 - T)]^N$, such that $w \cdot n = 0$ on $\partial T$

$$\| w \|_{L^2(\partial T)} \leq C(T) \| \nabla w \|_{L^2(B_1 - T)}.$$  

(4.12)

Then, rescaling (4.12), we obtain

$$\| w^e_k \|_{L^2(\partial \Gamma_e')} \leq C a_e \| \nabla w^e_k \|_{L^2(\partial \gamma'})$$

and

$$\| u_e \|_{L^2(\partial \Gamma_e')} \leq C a_e \| \nabla u_e \|_{L^2(\partial \gamma')}$$

which implies

$$\| w^e_k \|_{L^2(\Gamma_e)} \leq C a_e \quad \text{and} \quad \| u_e \|_{L^2(\Gamma_e)} \leq C a_e \| f \|_{L^2(\Omega)}.$$ 

With these estimates, we obtain the same bound for (4.10) as in the case $N \geq 3$.

**PROPOSITION 4.3.** Assume that the size of the holes is smaller than the critical one, i.e., the ratio $\alpha_e$, defined in (2.2), tends to infinity

$$\lim_{e \to 0} \sigma_e = +\infty.$$ 

(4.13)

We emphasize that the scaling of the slip coefficient is free, but we explicitly assume that we have a Dirichlet boundary condition on the large boundary $\partial \Omega$ (not on $\Gamma_e$). Let $(u_0, p_0)$ be the unique solution of the Stokes system (2.8). Then its extension
\((E, u_\varepsilon, P_\varepsilon)\) converges strongly in \([H^1_\Omega)^N \times [L^2(\Omega)/\mathbb{R}]\) to \((u, p)\), which is the unique solution of the Stokes system

\[
\begin{aligned}
\text{Find } (u, p) \in [H^1_\Omega)^N \times [L^2(\Omega)/\mathbb{R}] & \quad \text{such that} \\
\nabla p - \mu \Delta u &= f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega.
\end{aligned}
\]

(4.14)

Proof: Because of the scaling (4.13), we can no longer derive uniformly bounded estimates of the solution with the help of Lemmas 2.4 and 2.5. Therefore, we need the Dirichlet boundary condition on \(\partial \Omega\) in order to establish both Poincaré's and Korn's inequality with a uniform constant. For each \(v \in [H^1_\Omega)^N\), they give

\[
\|v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)} \quad \text{and} \quad \|\nabla v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega)}.
\]

(4.15)

Using (4.15) for \(E, u_\varepsilon\), it is easy to see that the conclusion of Proposition 3.1 still holds.

On the other hand, because of the small size of the holes (4.13), an easy but tedious computation (see [1]) shows that, for any positive value \((\pm 0, \pm \infty)\) of the slip coefficient \(\alpha\), the convergence of the test functions \((w_k, q_k)\), defined in Proposition 3.3, is actually strong in \([H^1(\Omega)^N \times L^2(\Omega)]\). Thus, the function \(\mu_k\), defined as \((\nabla q_k - \Delta w_k)\) in \(\Omega\), and zero elsewhere, converges strongly to zero in \([H^{-1}(\Omega)]^N\). This implies that no matrix \(M\) appears when passing to the limit, and the homogenized system reduces to the Stokes equations (4.14). This fact can be checked straightforwardly (i.e., without the tedious computation of the test functions) as follows. In each cell \(P^i\), let us define a function \(\theta_e\)

\[
\begin{aligned}
\theta_e(r) &= 0 \quad \text{for } 0 \leq r \leq a_e, & \quad \theta_e(r) &= \frac{r}{a_e} - 1 \quad \text{for } a_e \leq r \leq 2a_e, \\
\theta_e(r) &= 1 \quad \text{elsewhere, for } N \geq 3 \\
\theta_e(r) &= 0 \quad \text{for } 0 \leq r \leq a_e, & \quad \theta_e(r) &= \frac{\log r - \log a_e}{\log \epsilon - \log a_e} \quad \text{for } a_e \leq r \leq \epsilon, \\
\theta_e(r) &= 1 \quad \text{elsewhere, for } N = 2.
\end{aligned}
\]

Then, for any \(\phi \in D(\Omega)\), we have

\[
\theta_e \phi \in H^1_\Omega(\Omega_e), \quad \text{and} \quad \|\theta_e \phi - \phi\|_{H^1_\Omega(\Omega)} \leq \frac{C}{\sigma_e}.
\]

Using such test functions in the variational formulation (2.10), it is easy to see that the limit of (2.10) is a Stokes problem (without any additional term), because the sequence \(\theta_e \phi\) converges strongly to \(\phi\) in \(H^1_\Omega(\Omega)\).
To prove that the solutions converge strongly, we observe that

\begin{equation}
\frac{1}{2} \int_{\Omega_\epsilon} |e(u_{\epsilon})|^2 + \alpha_\epsilon \int_{\Gamma_\epsilon} |u_{\epsilon}|^2 = \int_{\Omega} f \cdot u_{\epsilon} \rightarrow \int_{\Omega} f \cdot u = \int_{\Omega} |\nabla u|^2.
\end{equation}

Because \( \nabla \cdot u = 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), an integration by parts shows that

\begin{equation}
\frac{1}{2} \int_{\Omega} |e(u)|^2 = \int_{\Omega} |\nabla u|^2.
\end{equation}

Let \( \chi_\epsilon \) be the characteristic function of the set \( \Omega_\epsilon \) (i.e., \( \chi_\epsilon = 1 \) in \( \Omega_\epsilon \) and 0 elsewhere). We know that the sequence \( e(E_u_{\epsilon}) \) converges weakly to \( e(u) \) in \( [L^2(\Omega)]^{N^2} \). Because \( \chi_\epsilon \) converges almost everywhere in \( \Omega \) to 1, the sequence \( \chi_\epsilon e(E_u_{\epsilon}) \) converges also weakly to \( e(u) \) in \( [L^2(\Omega)]^{N^2} \). Then, using the semilower continuity of the weak convergence, it follows from (4.16) and (4.17) that

\begin{equation}
\lim_{\epsilon \to 0} \int_{\Omega} |\chi_\epsilon e(E_u_{\epsilon})|^2 = \int_{\Omega} |e(u)|^2 \quad \text{and} \quad \lim_{\epsilon \to 0} \alpha_\epsilon \int_{\Gamma_\epsilon} |u_{\epsilon}|^2 = 0.
\end{equation}

The first limit in (4.18) implies the strong convergence of \( \chi_\epsilon e(E_u_{\epsilon}) \) to \( e(u) \) in \( [L^2(\Omega)]^{N^2} \). To obtain the same result for \( e(E_u_{\epsilon}) \) (without \( \chi_\epsilon \)), we bound

\[ \|e(u) - e(E_u_{\epsilon})\|_{L^2(\Omega)} \leq \|e(u) - e(E_u_{\epsilon})\|_{L^2(\Omega)} + \|e(E_u_{\epsilon}) - e(E_u_{\epsilon})\|_{L^2(\Omega)}. \]

Thanks to Lemma 2.2 on the extension operator \( E_{\epsilon} \), we obtain

\begin{align*}
\|e(u) - e(E_u_{\epsilon})\|_{L^2(\Omega)} & \leq \|e(u) - e(E_u_{\epsilon})\|_{L^2(\Omega)} + C\|e(u) - e(E_u_{\epsilon})\|_{L^2(\Omega)} \\
& \leq \|e(u) - e(E_u_{\epsilon})\|_{L^2(\Omega)} + C\|\chi_\epsilon e(u) - \chi_\epsilon e(E_u_{\epsilon})\|_{L^2(\Omega)} \\
& + \|e(u) - \chi_\epsilon e(E_u_{\epsilon})\|_{L^2(\Omega)}. 
\end{align*}

The three terms in the right-hand side of (4.19) go to zero, as a consequence of Lemma 2.2 for the first one, the Lebesgue theorem of dominated convergence for the second one, and (4.18) for the last one. Hence we deduce the strong convergence of \( e(E_u_{\epsilon}) \) to \( e(u) \) in \( [L^2(\Omega)]^{N^2} \). Combined with the strong one in \( [L^2(\Omega)]^{N} \), it yields the strong convergence of \( E_u_{\epsilon} \) to \( u \) in \( [H^1(\Omega)]^{N} \). Then, using Lemma 2.7 (which defines the extension of the pressure), it is easy to conclude that \( p_{\epsilon} \) converges strongly to \( p \) in \( L^2(\Omega)/\mathbb{R} \).

**Proposition 4.4.** Assume that the size of the holes is larger than the critical one, i.e., the ratio \( \sigma_{\epsilon} \), defined in (2.2), tends to zero

\begin{equation}
\lim_{\epsilon \to 0} \sigma_{\epsilon} = 0.
\end{equation}
Let \((u, p)\) be the unique solution of the Stokes system (2.8). After rescaling the velocity, the extension \((E, u_{cs}, P)\) converges strongly in \([L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]\) to \((u, p)\), which is the unique solution of Darcy's law

\[
\begin{align*}
\text{Find } (u, p) \in [L^2(\Omega)]^N \times [H^1(\Omega)/\mathbb{R}] \\
u = \frac{M^{-1}}{\mu} (f - \nabla p) & \quad \text{in } \Omega \\
\nabla \cdot u = 0 & \quad \text{in } \Omega \\
u \cdot n = 0 & \quad \text{on } \partial \Omega,
\end{align*}
\]

where the matrix \(M\) is still given by Definition 3.1, and depends on the slip coefficient \(\alpha\) obtained through the following scaling

\[
\alpha = \lim_{\epsilon \to 0} a_{\epsilon} \alpha_{\epsilon}.
\]

If \(\lim_{\epsilon \to 0} (a_{\epsilon}^{-1} \alpha_{\epsilon})/\epsilon^N = 0\), then the present proposition holds only if there is a Dirichlet boundary condition on \(\partial \Omega\).

Proof: For the case of a Dirichlet boundary condition on \(\Gamma_s\), this proposition is proved in [1] (see Section 3.4). Using the ideas of the present paper, the proof for the slip boundary condition is a mere repetition of that in [1]. For the sake of brevity we do not repeat it here. However, let us emphasize that, even though the local problem is still given by Lemma 3.2, the test functions \((w^\epsilon, q^\epsilon)\), involved in the proof, satisfy similar but weaker properties than that of Proposition 3.3. Furthermore, if \(\lim_{\epsilon \to 0} (a_{\epsilon}^{-1} \alpha_{\epsilon})/\epsilon^N = 0\), the Korn-type inequality of Lemma 2.4 fails to provide a uniform coercive estimate. In that case we need a Dirichlet boundary condition on \(\partial \Omega\), in order to use the Korn inequality in \([H^1(\Omega)]^N\) (as in Proposition 4.3).

5. Construction of the Test Functions

This section is devoted to the proof of Lemma 3.2 and Proposition 3.3. We emphasize that, in this section, we consider only the critical scaling defined in (3.1). Actually, we focus on what is new compared to [1], where all the omitted details can be found. The treatment of the two-dimensional case is especially different from [1] in order clearly to show the source of the paradoxical result (3.9).

Proof of Lemma 3.2:

\* \(N \geq 3\)

For a fixed \(\xi\) in \(\mathbb{R}^N\), consider the Stokes system (5.1)
NAVIER-STOKES EQUATIONS

\[
\begin{aligned}
\nabla q_\xi - \Delta w_\xi &= 0 \quad \text{in } \mathbb{R}^N - T \\
\nabla \cdot w_\xi &= 0 \quad \text{in } \mathbb{R}^N - T \\
w_\xi \cdot n &= 0 \quad \text{on } \partial T \\
\alpha w_\xi &= 2 \left( \frac{\partial w_\xi}{\partial n} \right) n - e(w_\xi)n \quad \text{on } \partial T \\
w_\xi &= \xi \quad \text{at } \infty.
\end{aligned}
\]

(5.1)

In order to prove the existence and uniqueness of a solution we seek an equivalent minimization problem for (5.1). Let \( D^{1,2}(\mathbb{R}^N) \) be the completion, with respect to the \( L^2 \) norm of the gradient, of the space of all smooth functions with compact support.

\[
D^{1,2}(\mathbb{R}^N) = \overline{D(\mathbb{R}^N)^{\nabla|\nabla s}}.
\]

(5.2)

If \( N \geq 3 \), the following continuous embeddings hold (see, e.g., Ladyzhenskaya [16] for the first one, and Lemma 1.1, Chapter 4, in Lax-Phillips [17] for the second one):

\[
D^{1,2}(\mathbb{R}^N) \subset L^{2N/(N-2)}(\mathbb{R}^N) \quad \text{and} \quad D^{1,2}(\mathbb{R}^N) \subset L^2_{\text{loc}}(\mathbb{R}^N).
\]

(5.3)

Let \( H_\xi = \{ w \in [D^{1,2}(\mathbb{R}^N)]^N/\nabla \cdot w = 0 \text{ in } \mathbb{R}^N \text{ and } (w + \xi) \cdot n = 0 \text{ on } \partial T \} \), and, for positive \( \alpha \), consider the following minimization problem

\[
\text{Find } w'_\xi \text{ which achieves }
\]

\[
\inf_{w \in H_\xi} \left( \frac{1}{2} \int_{\mathbb{R}^N - T} |e(w)|^2 + \alpha \int_{\partial T} |w + \xi|^2 \right).
\]

(5.4)

Because of (5.3), the functions in \( H_\xi \) are equal to zero at infinity in the \( L^{2N/(N-2)} \) sense. Then, it is easy to check that (5.4) is equivalent to the Stokes problem (5.1): indeed, a coercive and strictly convex function on a closed, convex set admits a unique minimizer, and it is easy to see that \( w_\xi = w'_\xi + \xi \) is the unique solution of (5.1). In [1] we also proved that there exists a unique pressure \( q_\xi \) solving (5.1) in \( L^2(\mathbb{R}^N) \). Furthermore, an integration by parts shows that the drag force \( F_\xi \) exerted on \( T \) satisfies

\[
F_\xi \cdot \xi = \int_{\partial T} (q_\xi n - e(w_\xi)n) \cdot \xi = \frac{1}{2} \int_{\mathbb{R}^N - T} |e(w_\xi)|^2 + \alpha \int_{\partial T} |w_\xi|^2.
\]
Decomposing $\xi$ in $\sum_{i=1}^N \xi_i e_i$, we obtain

(5.5) \[ 2^N M_\xi \xi = \sum_{1 \leq i, j \leq N} \xi_i \xi_j F_{ij} \cdot e_j = \frac{1}{2} \int_{\mathbb{R}^N - T} |e(w_\xi)|^2 + \alpha \int_{\partial T} |w_\xi|^2. \]

Now, if we derive (5.5) with respect to $\alpha$, no contribution comes from $w_\xi$ because it achieves the minimum, and the derivative is positive, equal to $\int_{\partial T} |w_\xi|^2$. This gives the proof that $\alpha_1 \leq \alpha_2$ implies $M(\alpha_1) \leq M(\alpha_2)$. Going back to (5.4) it is easy to see that the limit case when $\alpha$ tends to infinity corresponds to a Dirichlet boundary condition on $\partial T$. As in [1], using the fundamental solution of the Stokes operator in $\mathbb{R}^N$ (see [12] and [16]), we obtain asymptotic expansions at infinity of the solution of (5.1)

\[
\begin{aligned}
  w_\xi &= \xi - \frac{1}{2 S_N r^{N-2}} \left[ \frac{F_\xi}{N-2} + (F_\xi \cdot e_\xi) e_\xi \right] + O\left( \frac{1}{r^{N-1}} \right) \\
  q_\xi &= -\frac{1}{S_N r^{N-1}} (F_\xi \cdot e_\xi) + O\left( \frac{1}{r^N} \right) \\
  \nabla w_\xi &= O\left( \frac{1}{r^{N-1}} \right) \\
  \frac{\partial w_\xi}{\partial r} - q_\xi e_\xi &= \frac{1}{2 S_N r^{N-1}} [F_\xi + N(F_\xi \cdot e_\xi) e_\xi] + O\left( \frac{1}{r^N} \right).
\end{aligned}
\]

These expansions are accurate if the drag force $F_\xi$ is non-zero. That is always the case: by comparison with the small ball $B_0$ included in $T$, it is not difficult to show that $F_\xi \cdot \xi > 0$ for any non-zero $\xi$.

- $N = 2$

This case is completely different from the previous one, because there is no solution of (5.1) in two dimensions (this is the well-known Stokes paradox). That is why we seek a solution of the Stokes equations with a logarithmic growth at infinity:

\[
\begin{aligned}
  \nabla q_\xi - \Delta w_\xi &= 0 \quad \text{in } \mathbb{R}^2 - T \\
  \nabla \cdot w_\xi &= 0 \quad \text{in } \mathbb{R}^2 - T \\
  w_\xi \cdot n &= 0 \quad \text{on } \partial T \\
  \alpha w_\xi &= 2 \left( \frac{\partial w_\xi}{\partial n} \right) n - e(w_\xi) n \quad \text{on } \partial T \\
  w_\xi &= (\log r) \xi \quad \text{at } \infty.
\end{aligned}
\]

If $T$ is a ball $B_0$ of radius $\alpha$, we can compute an exact solution of (5.7), denoted by $(q_\xi^0, w_\xi^0)$.
\[
\begin{align*}
\mathbf{M}^0 = \frac{1}{r^2} \left( -1 + \frac{A^2}{r^2} \right) \\
\mathbf{N}^0 = \log r - \frac{A}{2r^2} + B \\
h(r) = -\frac{2}{r^2}
\end{align*}
\]

Moreover, the drag force \( F^0_z \) is equal to \( 4\pi \xi \), for any value of \( \alpha \) and \( a \). In order to extend that result to an arbitrary shape of the obstacle \( T \), we seek a solution of (5.7) in the form \( w^r = w_0^r + w^r_1 \) and \( q^r = q_0^r + q_1^r \) where, from now on, \( (q_0^r, w_0^r) \) denotes the Stokes solution for the ball \( B_1 \) with a Dirichlet boundary condition (i.e., \( a = +\infty \) in formulae (5.10)). Let \( \delta_{B_1} \) be the homogeneous, unit mass, surface measure concentrated on \( \partial B_1 \). An easy computation shows that \( q_0^r e_r - (\partial w_0^r / \partial r) \) is equal to \( -2\xi \) on the circle \( \partial B_1 \). By subtraction \( (q_1^r, w_1^r) \) is a solution of the following system

\[
\begin{align*}
\nabla q_1^r - \Delta w_1^r &= -\left( q_0^r e_r - \frac{\partial w_0^r}{\partial r} \right) \delta_{B_1} = 2\xi \delta_{B_1} \quad \text{in } \mathbb{R}^2 - T \\
\nabla \cdot w_1^r &= 0 \quad \text{in } \mathbb{R}^2 - T \\
w_1^r \cdot n &= 0 \quad \text{on } \partial T \\
\alpha w_1^r &= 2 \left( \frac{\partial w_1^r}{\partial n} , n \right) n - e(w_1^r) n \quad \text{on } \partial T \\
w_1^r &= o(\log r) \quad \text{at } \infty \quad (\text{i.e., } \frac{|w_1^r|}{\log r} \to 0). 
\end{align*}
\]

We shall give a minimization formulation to system (5.11) as we did for (5.1). Let \( D^{1,2}(\mathbb{R}^2) \) be the completion, with respect to the \( L^2 \) norm of the gradient, of the space of all smooth functions with compact support in \( \mathbb{R}^2 \). Unfortunately, there is no Sobolev embedding for \( D^{1,2}(\mathbb{R}^2) \) (an important consequence of that fact is that functions of \( D^{1,2}(\mathbb{R}^2) \) do not go to zero at infinity; for example, constants belong...
to $D^{1,2}(\mathbb{R}^2))$. Nevertheless, O. A. Ladyzhenskaya in [16] proved that any smooth function $\phi$ with compact support in $\mathbb{R}^2 - B_1$ satisfies

$$\left\| \frac{\phi}{r \log r} \right\|_{L^2(\mathbb{R}^2 - B_1)} \leq 4 \| \nabla \phi \|_{L^2(\mathbb{R}^2 - B_1)}.$$

Thus, we can deduce a continuous embedding, weaker than (5.3)

$$(5.12) \quad D^{1,2}(\mathbb{R}^2) \subset \left\{ \phi \in L^2_{\text{loc}}(\mathbb{R}^2) : \frac{\phi}{(r + 1) \log(r + 2)} \in L^2(\mathbb{R}^2) \right\}.$$  

Functions of $D^{1,2}(\mathbb{R}^2)$ behave like $o(\log r)$ at infinity, in the sense of (5.12), because $\log r$ does not belong to the space on the right-hand side of (5.12). Let $H_t = \{ w \in [D^{1,2}(\mathbb{R}^2)]' : \nabla \cdot w = 0 \text{ in } \mathbb{R}^2 \text{ and } w \cdot n = 0 \text{ on } \partial T \}$.

Consider now the following minimization problem

Find $w'_t$ which achieves

$$(5.13) \quad \inf_{w \in H_t} \left\{ \frac{1}{4} \int_{\mathbb{R}^2 - T} |e(w)|^2 + \frac{\alpha}{2} \int_{\partial T} |w|^2 - 2 \int_{\partial B} w \cdot \xi \right\}.$$  

It is easy to check that (5.13) is equivalent to the Stokes system (5.11). As (5.13) admits a unique minimizer $w'_t$, we conclude that $w_t = w'_t + w_t^0$ is the unique solution of (5.7). Now it remains to show that the drag force $F_t$ is always equal to $4\pi \xi$ (this paradoxical result has already been pointed out by R. Finn and D. Smith in [13]). We have

$$F_t = \int_{\partial T} \left( q_t n - \frac{\partial w'_t}{\partial n} \right) = \int_{\partial B} \left( \frac{\partial w'_t}{\partial r} - q'_t \epsilon_r \right) \quad \text{for any} \quad R \geq 1.$$  

Decomposing $w_t$ as $w^0_t + w'_t$ we obtain

$$(5.14) \quad F_t = 4\pi \xi + \int_{\partial B} \left( \frac{\partial w^0_t}{\partial r} - q'_t \epsilon_r \right) \quad \text{for any} \quad R \geq 1.$$  

Let $w_t^0$ be the same function as $w'_t$, but for $\xi' \neq \xi$. From (5.8)–(5.10) we know that $w_t^0$ is equal to 0 on $\partial B_1$ and to $\log r[1 + o(1)]\xi'$ at infinity. Multiplying equation (5.11) by $w_t^0/\log R$ and integrating by parts over $B_R - B_1$ yields

$$(5.15) \quad \frac{1}{\log R} \int_{B_R - B_1} \nabla w^0_t \cdot \nabla w'_t = \left[ \int_{\partial B} \left( \frac{\partial w'_t}{\partial n} - q'_t \epsilon_r \right) \cdot \xi' \right] [1 + o(1)].$$
For any $\xi'$ we can bound (5.15)
\[
\left| \int_{\partial B_R} \left( \frac{\partial w_{\xi'}}{\partial n} - q_{\xi'} \xi' \right) \cdot \xi \right| \leq \frac{1}{\log R} \| \nabla w_{\xi'} \|_{L^2(B_R - B_i)} \| \nabla w_{\xi} \|_{L^2(\mathbb{R}^2 - T)}
\]
\[(5.16)\]
\[
\leq \frac{C}{(\log R)^{1/2}}.
\]
When $R$ tends to infinity, we deduce from (5.14) and (5.16)
\[
F_{\xi} = 4\pi \xi.
\]

We now turn to the construction of the test functions, which, roughly speaking, are obtained by rescaling the solutions of the local problems, and expanding them by $P_i$-periodicity in the whole domain $\Omega$.

Proof of Proposition 3.3: We decompose each cell $P_i$ in smaller subdomains. We set
\[
\tilde{P}_i = T_i \cup C_i \cup \bar{D}_i \cup \bar{K}_i
\]
where $C_i$ is the open ball of radius $\varepsilon/2$ centered in $P_i$, and perforated by $T_i$, $D_i$ is equal to $B_i$ perforated by $\bar{C}_i \cup T_i$, and $K_i$ is the remainder, i.e., the corners of $P_i$.

Let $(w_k, q_k)_{1 \leq k \leq N}$ be the solutions of the local problem defined in Lemma 3.2. We define the functions $(w_k, q_k)_{1 \leq k \leq N}$ by
- For each cube $P_i$ which meets $\partial \Omega$
  \[
  \begin{cases}
    w_k = e_k & \text{in } P_i \cap \Omega \\
    q_k = 0 & \text{in } P_i
  \end{cases}
\]
- For each cube $P_i$ entirely included in $\Omega$
  \[
  \begin{cases}
    w_k = e_k & \text{in } K_i \\
    q_k = 0 & \text{in } K_i \\
    \nabla q_k - \Delta w_k = 0 & \text{in } D_i \\
    \nabla \cdot w_k = 0 & \text{in } D_i
  \end{cases}
\]
\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{w_k(x)}{a_e} = \frac{x}{a} & \text{in } C_i^\varepsilon \quad \text{for } N \geq 3 \\
\frac{1}{a_e} q_k(x) = \frac{x}{a} & \text{in } C_i^\varepsilon \quad \text{for } N = 2
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{1}{\log 2a_e} w_k(x) = \frac{x}{a} & \text{in } C_i^\varepsilon \quad \text{for } N \geq 3 \\
\frac{1}{a_e \log \left( \frac{e}{2a_e} \right)} q_k(x) = \frac{x}{a} & \text{in } C_i^\varepsilon \quad \text{for } N = 2
\end{array} \right.
\end{align*}
\]

where \((w_k^\varepsilon, q_k^\varepsilon) \in [H^1(P_i)]^N \times L^2(P_i^\varepsilon).\)

Using the properties of the local solutions (including the asymptotic expansions in the case \(N \geq 3\), and the decomposition in a “spherical” part and a small remainder in the case \(N = 2\)), we can easily prove, as we did in \([1]\), that

(i) \(q_k \to 0\) in \(L^2(\Omega)/\mathbb{R}\) weakly.

(ii) \(w_k \to e_k\) in \([H^1(\Omega)]^N\) weakly, and \(\nabla \cdot w_k^\varepsilon = 0\) in \(\Omega\).

(iii) \(w_k^\varepsilon \cdot n = 0\) and \((\alpha/a_e) w_k^\varepsilon = 2[\partial w_k^\varepsilon / \partial s] \cdot n = e(w_k^\varepsilon)n\) on \(\Gamma_e\).

Basically the main fact is to remark that

\[
\begin{align*}
\left\{ \begin{array}{ll}
\|\nabla w_k\|_{L^2(\mathbb{R}^N)} < \infty & \text{for } N \geq 3 \\
\|\nabla w_k\|_{L^2(\mathbb{R}^N)}^2 = 2\pi \log R[1 + o(1)] & \text{for } N = 2.
\end{array} \right.
\end{align*}
\]

Then by rescaling (5.20), and summing over cells \(P_i^\varepsilon\) (the number of which is of order \(e^{-N}\)), we obtain

\[
\begin{align*}
\|\nabla w_k\|_{L^2(\Omega)}^2 \leq C \frac{a_e^{N-2}}{e^N} \|\nabla w_k\|_{L^2(P_i)}^2 \leq C \frac{a_e^{N-2}}{e^N} \|\nabla w_k\|_{L^2(\mathbb{R}^N)}^2 \leq C & \quad \text{for } N \geq 3, \\
\|\nabla w_k\|_{L^2(\Omega)}^2 \leq C \frac{e}{e^2 \log \left( \frac{e}{2a_e} \right)} \|\nabla w_k\|_{L^2(P_i)}^2 \leq C & \quad \text{for } N = 2.
\end{align*}
\]

For the sake of simplicity we omitted in (5.21) the estimates of \(\nabla w_k^\varepsilon\) in \(D_i^\varepsilon\) and in \(T_i^\varepsilon\), which anyway turn out to be similar (and even stronger) to those in \(C_i^\varepsilon\) (see [1] for details).
The main point to prove in Proposition 3.3 is the last statement (iv), namely

(iv) Let \( \mu_k = \nabla q_k^\varepsilon - \Delta w_k^\varepsilon \in [H^{-1}(\Omega_\varepsilon)]^N \). Then \( \mu_k \) has compact support in \( \Omega_\varepsilon \). Moreover, extending \( \mu_k \) by 0 in \( \Omega - \Omega_\varepsilon \) we have

\[
\mu_k \rightarrow \frac{1}{\sigma^2} Me_k \quad \text{in} \quad [H^{-1}(\Omega)]^N \quad \text{strongly}.
\]

First, thanks to the definitions (5.18) and (5.19), the function \((\nabla q_k^\varepsilon - \Delta w_k^\varepsilon)\) is equal to zero everywhere in \(\Omega_\varepsilon\) except on the boundaries of \(C_\varepsilon^i\) and \(D_\varepsilon^i\). Thus \(\mu_k\) has compact support in \(\Omega_\varepsilon\), as claimed in (iv). Second, we shall prove that \(\mu_k\) converges strongly to \((1/\sigma^2 2^N) F_k\) in \([H^{-1}(\Omega)]^N\). However we only sketch this proof, because it is a mere repetition of Lemmas 2.3.3 and 2.3.7 in [1]. A tedious computation yields the strong \(L^2\) convergence of \((q_k^\varepsilon, \nabla w_k^\varepsilon)\) to zero in the union of the \(D_\varepsilon^i\). Thus the part of \(\nabla q_k^\varepsilon - \Delta w_k^\varepsilon\) concentrated on the boundaries of the \(D_\varepsilon^i\) converges strongly to zero in \([H^{-1}(\Omega)]^N\). On the other hand, because of the definition of the function \(\mu_k\) we may skip the contribution of \(\nabla q_k^\varepsilon - \Delta w_k^\varepsilon\) on the boundaries \(\partial C_\varepsilon^i \cap \partial T_\varepsilon^i\). It only remains to prove that

\[
\sum_{i=1}^{N(\varepsilon)} \left( \frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_i^\varepsilon \right) \delta_i^\varepsilon / 2
\]

(where \(\delta_i^\varepsilon / 2\) is the unit mass concentrated on the sphere \(\partial C_\varepsilon^i \cap \partial T_\varepsilon^i\)) converges strongly to \((1/\sigma^2 2^N) F_k\) in \([H^{-1}(\Omega)]^N\). Introducing the solutions of the local problem leads to

\[
\sum_{i=1}^{N(\varepsilon)} \left( \frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_i^\varepsilon \right) \delta_i^\varepsilon / 2
\]

\[
= \begin{cases} 
\sum_{i=1}^{N(\varepsilon)} \frac{2^N}{4\sigma^2 S_N} [F_k + N(F_k \cdot e_i^\varepsilon) e_i^\varepsilon [1 + o(1)]] & \text{for } N \geq 3 \\
\sum_{i=1}^{N(\varepsilon)} \frac{2}{\sigma^2} e_\varepsilon \delta_i^\varepsilon / 2 [1 + o(1)] & \text{for } N = 2.
\end{cases}
\]

Then using a lemma of D. Cioranescu and F. Murat (see [9]), which asserts that

\[
\sum_{i=1}^{N(\varepsilon)} \varepsilon \delta_i^\varepsilon \rightarrow \frac{S_N}{2^N} \quad \text{in} \quad H^{-1}(\Omega) \quad \text{strongly}
\]

\[
\sum_{i=1}^{N(\varepsilon)} \varepsilon \delta_i^\varepsilon (e_k \cdot e_i^\varepsilon) e_i^\varepsilon \rightarrow \frac{S_N}{N 2^N} e_k \quad \text{in} \quad [H^{-1}(\Omega)]^N \quad \text{strongly},
\]
we prove the desired convergence (see [1] for details)

$$\sum_{i=1}^{N(e)} \left( \frac{\partial w_{k}^{i}}{\partial r_{i}} - q_{k} e_{i}^{i} \right) \delta_{i/2} \rightarrow \frac{1}{\sigma^{2} N} F_{k}.$$  

Acknowledgments. Part of this research was supported by DARPA Grant No. F49620-87-C-0065. The author wishes to thank the Courant Institute of Mathematical Sciences for its hospitality. The author is also indebted to R. V. Kohn, F. Murat, and J. P. Puel for their unfailing help throughout this work.

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Received June 1990.