ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Correction of exercise 6

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $d \geq 1$. Let $\alpha > 0$ be a constant and $g : \mathbb{R} \mapsto \mathbb{R}$ a C^1 function which has at most linear growth at infinity, in the sense that there exists $M > 0$ and $C > 0$ such that, if $|s| > M$, then

$$0 \leq g(s)s \leq Cs^2 \quad \text{and } |g'(s)| \leq C. \tag{1}$$

For given $f \in L^2(\Omega)$, consider the following non-linear model

$$\begin{cases} -\Delta u + \alpha\rho g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In (2) $\rho \in \mathcal{U}_{ad}$ is an optimization variable which, for $\rho_{min}, \rho_{max} \in \mathbb{R}^+$, belongs to the admissible set

$$\mathcal{U}_{ad} = \left\{ \rho \in L^2(\Omega) \ , \quad \rho_{max} \geq \rho(x) \geq \rho_{min} \geq 0 \text{ a.e. in } \Omega \right\}.$$

For a given target field $u_0 \in H^1_0(\Omega)$, we consider the optimization problem

$$\inf_{\rho \in \mathcal{U}_{ad}} \left\{ J(\rho) = \frac{1}{2} \int_\Omega |u(x) - u_0(x)|^2 dx \right\}, \tag{3}$$

where u is the solution of (2). This is an inverse problem where we want to reconstruct the coefficient ρ in (2).

1. Prove that the boundary value problem (2) admits at least one solution in $H^1_0(\Omega)$.

   Consider the minimization in $H^1_0(\Omega)$ of the energy

   $$E(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \alpha \int_\Omega \rho G(u) dx - \int_\Omega f u \, dx$$

   where G is a primitive of g. From the assumption (1) one can check that the term $\int_\Omega \rho G(u)dx$ is uniformly bounded from below and has at most quadratic growth. Then taking a minimizing sequence, since $H^1_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, one can pass to the limit, up to a subsequence, and deduce the existence of at least one minimizer of $E(u)$. The Euler optimality condition yields a solution of (2).

2. Prove that, if $\alpha > 0$ is small enough, then there exists at most one solution of (2) in $H_0^1(\Omega)$.

   Take two solutions u_1 and u_2 and multiply by $(u_2 - u_1)$ the difference of their equations to get

$$0 \geq \int_\Omega |\nabla(u_2 - u_1)|^2 dx - C\alpha\rho_{max} \int_\Omega (u_2 - u_1)^2 dx,$$

   because $g'(s)$ is uniformly bounded by $C > 0$. Applying Poincaré inequality to the first term, we deduce that $(u_2 - u_1) = 0$ for α small enough.

3. From now on we assume that $\alpha = 1$ and that $s \mapsto g(s)$ is non-decreasing. Prove there exists at most one solution of (2) in $H_0^1(\Omega)$.

   The fact that g is non-decreasing implies that G is convex, so $E(u)$ is strongly convex, which yields the result.

4. Prove that there exists at least one minimizer for (3).

   Take a minimizing sequence ρ_n of (3) and denote u_n the corresponding solution of (2). From the energy minimization of $E(u)$ we deduce that the sequence u_n is bounded in $H_0^1(\Omega)$. Then, by the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ we pass to the limit both in (2) and (3).

5. Check that the map

$$\begin{array}{rcl} L^2(\Omega) & \mapsto & H_0^1(\Omega) \\ \rho & \mapsto & u \text{ solution of (2)} \end{array}$$

   is Fréchet differentiable and compute its directional derivative in a direction w.

   The Fréchet differentiability can be established by the implicit function theorem, cf. the course. Computing the directional derivative $v = \langle u'(\rho), w\rangle$ in a direction w is a simple computation

$$\begin{cases} -\Delta v + \rho g'(u) v = -w g(u) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

6. Find the Lagrangian of the problem and deduce the adjoint state.

   The Lagrangian is defined for $(\rho, u, p) \in L^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$\mathcal{L}(\rho, u, p) = \frac{1}{2} \int_\Omega |u - u_0|^2 dx + \int_\Omega (\nabla u \cdot \nabla p + \rho g(u) p - f p)\, dx.$$

The adjoint is defined by $\langle \frac{\partial \mathcal{L}}{\partial u}, \phi \rangle = 0$ *for any* $\phi \in H_0^1(\Omega)$*, which yields*

$$\begin{cases} -\Delta p + \rho g'(u)p = -(u - u_0) & \textit{in } \Omega, \\ p = 0 & \textit{on } \partial\Omega. \end{cases}$$

7. Compute the derivative with respect to ρ of the objective function.

   The derivative is given by

$$J'(\rho) = \frac{\partial \mathcal{L}}{\partial \rho}(\rho, u, p) = g(u)p,$$

   where u *is the solution of (2) and* p *is the solution of the adjoint equation.*

8. Suggest and describe a numerical algorithm to solve (3).

   One can use a projected gradient algorithm.