ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Exercise 7

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $d \geq 1$. We consider the following coupled model (which is a simplification of a thermo-elasticity model) for two unknowns u and v. For given $u_0(x) \in L^2(\Omega)$ and $g(t) \in L^2(0,T)$, consider the parabolic equation

$$\left\{ \begin{array}{ll} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega. \end{array} \right. \tag{1}$$

Note that $g(t)$ can be seen as a uniform in space, varying in time, heat flux at the boundary. Then, for $\alpha \in \mathbb{R}, \alpha \neq 0$ and $f(t,x) \in L^2((0,T) \times \Omega)$ for a.e. $t \in (0,T)$ consider the elliptic equation

$$\left\{ \begin{array}{ll} -\Delta v = f + \alpha u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2}$$

In (1) $g \in \mathcal{U}_{ad}$ is an optimization variable which, for $g_{min} < g_{max} \in \mathbb{R}$, belongs to the admissible set

$$\mathcal{U}_{ad} = \left\{ g \in L^2(0,T) \ , \quad g_{max} \geq g(t) \geq g_{min} \text{ a.e. in } (0,T) \right\}.$$

For a given target field $v_0 \in H^1_0(\Omega)$, we consider the optimization problem

$$\inf_{g \in \mathcal{U}_{ad}} \left\{ J(g) = \frac{1}{2} \int_0^T \int_\Omega |v(t,x) - v_0(x)|^2 dx dt \right\}, \tag{3}$$

where v is the solution of (2).

1. Assuming that (1) has a unique solution $u \in L^2((0,T); H^1(\Omega)) \cap C([0,T]; L^2(\Omega))$, prove that the boundary value problem (2) admits a unique solution v in $L^2((0,T); H^1_0(\Omega))$.
2. Prove that there exists at least one minimizer of (3) in $\mathcal{U}_{ad}$.
3. Check that the map

$$\begin{array}{rcl} L^2(0,T) & \mapsto & L^2((0,T); H^1_0(\Omega)) \\ g & \mapsto & v \text{ solution of (2)} \end{array}$$

   is Fréchet differentiable and compute its directional derivative in a direction h.
4. Find the Lagrangian of the problem and deduce the adjoint state.
5. Compute the derivative with respect to g of the objective function.
6. Suggest and describe a numerical algorithm to solve (3).