

*Homogenization of the Navier-Stokes Equations
in Open Sets Perforated with Tiny Holes
I. Abstract Framework,
a Volume Distribution of Holes*

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Abstract

This paper treats the homogenization of the Stokes or Navier-Stokes equations with a Dirichlet boundary condition in a domain containing many tiny solid obstacles, periodically distributed in each direction of the axes. (For example, in the three-dimensional case, the obstacles have a size of ε^3 and are located at the nodes of a regular mesh of size ε .) A suitable extension of the pressure is used to prove the convergence of the homogenization process to a Brinkman-type law (in which a linear zero-order term for the velocity is added to a Stokes or Navier-Stokes equation).

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Introduction

This two-part paper is devoted to the homogenization of the Stokes or Navier-Stokes equations, with a Dirichlet boundary condition, in open sets perforated with tiny holes. Many physical phenomena involve viscous fluid flow past an array of fixed solid obstacles. Such flows are governed by the Stokes or Navier-Stokes equations with a no-slip (Dirichlet) boundary condition on the obstacles, and the fluid domain is mathematically represented by an open set perforated with holes (*i.e.*, obstacles). As the number of holes increases, the flow tends to the solution of certain effective or “homogenized” equations which are homogeneous in form (*i.e.*, without obstacles). Homogenization is a mathematical method that provides such effective models (see, *e.g.*, [6] and [25] for a general introduction to this topic).

In the sequel we pay particular attention to two different kind of flows: in porous media, and through mixing grids. For flow in a porous medium it has been proved that the homogenization of the Stokes equations leads to the well-known Darcy law if the medium is represented as the periodic repetition of an elementary cell of size ε , in which the solid obstacle is also of size ε . (See, *e.g.*, [16], [20], and [25] for two-scale methods, and [28] for the proof of convergence; see also [2] for a generalization of [28] to the case of connected solid obstacles.) Beside Darcy’s law, other equations describe fluid flows in porous media: For example, in the late 1940’s H. BRINKMAN [8] introduced a new set of equations, intermediate between the Darcy and Stokes equations. The so-called Brinkman’s law is obtained from the Stokes equations by adding to the momentum equation a term proportional to the velocity. In this paper we prove the convergence of the solutions of the Stokes equations to the solution of Brinkman’s law when a porous medium is modeled as the periodic repetition of an elementary cell of size ε , in which the solid obstacle is of size ε^3 (in the three-dimensional case). Furthermore, if the size of the holes is asymptotically larger than this critical size, then we establish that the homogenized problem is governed by Darcy’s law; if the size of the holes is asymptotically smaller than the critical size, then we obtain the Stokes equations as the homogenized problem.

Consider now fluid flow through a mixing grid. C. CONCA [10] and E. SANCHEZ-PALENCIA [26] dealt with Stokes flows through periodic sieves, in which the holes have the same size as the period, and obtained an effective model, roughly speaking, equivalent to Darcy’s law. E. SANCHEZ-PALENCIA [27] also studied ideal fluid flows through perforated walls, but he was not concerned with the Stokes equations, because ideal fluid flows are governed by a Laplace equation for the potential of the velocity. Here we propose a mathematical model for fluid flows through mixing grids, which is based on a particular form of Brinkman’s law (*i.e.*, the additional term is concentrated on the plane of the grid). This model is obtained through homogenization of the Stokes equations in a domain containing a mixing grid, which is represented by its vanes of size ε^2 (in the three-dimensional case) periodically distributed at the nodes of a regular mesh of size ε . (We neglect the lattice which support the vanes.)

Although the distribution and the size of the holes are very different in each example, the underlying idea of the convergence proof are the same. For this

reason we begin by introducing an abstract framework (including both cases) which allows us to prove general theorems under theoretical assumptions on the hole distribution. The first part of this paper is devoted to this abstract framework, and to the derivation of Brinkman's law in the case of a volume distribution of holes of the critical size. The second part deals with a volume distribution of holes, having a size different from the critical one, and with the case of a surface distribution of holes (leading to our model of fluid flows through mixing grids).

Since the original paper of H. BRINKMAN [8], the derivation and justification of Brinkman's law from the Stokes equations has been extensively studied. V. A. MARČENKO & E. JA. HRUSLOV [22] were the first to prove that Brinkman's law describe the limiting behavior of Stokes flow in a periodically perforated domain for a particular scaling of the holes. A similar result was obtained by A. BRILLARD [7] by using the framework of epi-convergence. E. SANCHEZ-PALENCIA [24] and T. LÉVY [19] also derived Brinkman's law by means of a three-scale expansion method. Besides these works, which are concerned with periodic homogenization, J. RUBINSTEIN [23] dealt with the case of a random array of spheres in a three-dimensional domain. Using probabilistic methods, he proved that Brinkman's law describes the effective behavior in this context. Like [7], [19], [22], and [24], we focus here on obstacles with spatial periodicity rather than with a random distribution. Besides recovering the previous results from a new perspective, we obtain a number of physically significant new results. First, in the two-dimensional setting we show that the limiting Brinkman-type law is independent of the shape of the holes (see Proposition 2.1.6). This is due to a version of the Stokes paradox. Second, for holes that are too large to give a Brinkman law but still smaller than the inter-hole distance, we show that the limiting behavior is described by a Darcy-type law (see Theorem 3.4.4). This situation is not the same as that studied by E. SANCHEZ-PALENCIA [25] and L. TARTAR [28], although it leads to the same type of effective equations. Third, we give effective equations associated with holes distributed on a hypersurface rather than throughout the volume of the fluid (see Theorem 4.1.3). Finally, from a theoretical point of view, the major novelty of our analysis is the optimal L^2 -estimate of the pressure, which leads to a very simple proof of the convergence and gives new results, including correctors and error estimates.

We turn now to a more detailed introduction of this first part of the paper. For a given force $f \in [L^2(\Omega)]^N$, consider the Stokes equations (with a Dirichlet boundary condition) in a domain Ω_ε obtained by removing from a smooth open set Ω , included in \mathbb{R}^N , a collection of holes $(T_i^\varepsilon)_{1 \leq i \leq N(\varepsilon)}$,

$$(S_\varepsilon) \quad \begin{cases} \text{Find } (u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)]/\mathbb{R} \text{ such that} \\ \nabla p_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon. \end{cases}$$

Following an idea of D. CIORANESCU & F. MURAT [9], we introduce, in the first section, an abstract framework of Hypotheses (H1) to (H6) on the holes T_i^ε . This allows us to construct an extension P_ε of the pressure (see Proposition 1.1.4), and to pass to the limit, when ε tends to 0, with the help of the energy method

due to L. TARTAR [29]. It turns out (see Theorem 1.1.8) that the homogenized problem in Ω is governed by a Brinkman law:

$$(S_0) \quad \begin{cases} \text{Find } (u, p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \text{ such that} \\ \nabla p - \Delta u + Mu = f \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in } \Omega \end{cases}$$

where M is a positive and symmetric matrix that depends neither on the force f nor on the solution (u, p) (see Proposition 1.1.2). We summarize these results in the following

Theorem. *Let $(u_\varepsilon, p_\varepsilon)$ be the unique solution of (S_ε) . Let \tilde{u}_ε be the extension by 0 in the holes (T_i^ε) of the velocity u_ε . Then $(\tilde{u}_\varepsilon, P_\varepsilon)$ converges weakly to (u, p) in $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u, p) is the unique solution of Brinkman's law (S_0) .*

We also prove various results concerning first-order correctors (see Theorems 1.2.3 and 1.2.4), and error estimates (see Proposition 1.2.5).

In the second section of this paper we check the Hypotheses (H1)–(H6) when the holes T_i^ε are periodically distributed in each direction of the axes with period 2ε , and each T_i^ε is similar to the same model hole T scaled to size a_ε (see Figure 1). The size a_ε is assumed to be critical, typically $a_\varepsilon = \varepsilon^3$ for $N = 3$, and $a_\varepsilon = \exp(-1/\varepsilon^2)$ for $N = 2$. For $N \geq 3$ we can calculate the matrix M through a local computation of a Stokes flow in \mathbb{R}^N past the model hole T (see Proposition 2.1.4). For $N = 2$, because of the Stokes paradox, the matrix M is always a scalar matrix that does not depend on the choice of the model hole T (see Proposition 2.1.6). We also obtain precise bounds for the errors (see Theorem 2.1.9), and following an idea of R. LIPTON & M. AVELLANEDA [21], we can make explicit the extension of the pressure (see Proposition 2.1.2). We summarize the results of the second section in the following

Theorem. *Let the hole size a_ε satisfy*

$$\lim_{\varepsilon \rightarrow 0} \frac{a_\varepsilon}{\varepsilon^{N/(N-2)}} = C_0 \quad \text{for } N \geq 3 \quad \text{or} \quad \lim_{\varepsilon \rightarrow 0} -\varepsilon^2 \text{Log}(a_\varepsilon) = C_0 \quad \text{for } N = 2$$

where C_0 is a strictly positive constant ($0 < C_0 < +\infty$). Then Hypotheses (H1)–(H6) are fulfilled, and all the previous results of the first section hold. Moreover, the extension P_ε of the pressure turns out to be equal to

$$P_\varepsilon = p_\varepsilon \text{ in } \Omega_\varepsilon \quad \text{and} \quad P_\varepsilon = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} p_\varepsilon \text{ in each hole } T_i^\varepsilon,$$

where C_i^ε is a “control” volume around the hole T_i^ε : C_i^ε is that part of the ball of radius ε with the same center as T_i^ε which is outside T_i^ε .

If $N = 2$, then $M = \frac{\pi}{C_0} Id$, whatever the shape of the model hole T . If $N \geq 3$, then ${}^t e_i M e_k = \frac{C_0^{N-2}}{2^N} \int_{\mathbb{R}^{N-T}} \nabla w_k : \nabla w_i$ where, for $1 \leq k \leq N$, e_k is the k^{th}

unit basis vector in \mathbb{R}^N , and w_k is the solution of the following Stokes system

$$\begin{aligned} \nabla q_k - \Delta w_k &= 0 && \text{in } \mathbb{R}^N - T, \\ \nabla \cdot w_k &= 0 && \text{in } \mathbb{R}^N - T, \\ w_k &= 0 && \text{on } \partial T, \\ w_k &= e_k && \text{at infinity.} \end{aligned}$$

It is only for the sake of simplicity that we restrict ourselves to the Stokes equations. The same theorems hold (with obvious slight changes) for the stationary Navier-Stokes equations, because, in this framework the non-linear term is a compact perturbation of (S_ε) (the matrix M is the same for Stokes or Navier-Stokes homogenization).

The present paper deals exclusively with Dirichlet boundary conditions. In a forthcoming paper [4] we generalize our results to the case of a "slip" boundary condition consisting of $u_\varepsilon \cdot n = 0$ and an additional condition for the tangential component of the normal stress on the boundary. The present results have been previously announced in [1] and [3].

Notation. Throughout this paper, C denotes various real positive constants independent of ε . The duality products between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and between $[H_0^1(\Omega)]^N$ and $[H^{-1}(\Omega)]^N$, are both denoted by $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1(\Omega)}$. $(e_k)_{1 \leq k \leq N}$ is the canonical basis of \mathbb{R}^N .

1. Abstract Framework

1.1 Formulation of the problem and the convergence theorem

Let Ω be a bounded connected open set in \mathbb{R}^N ($N \geq 2$), with Lipschitz boundary $\partial\Omega$, Ω being locally located on one side of its boundary. Let ε be a sequence of strictly positive real numbers which tends to zero. For each ε we consider a family of closed sets $(T_i^\varepsilon)_{1 \leq i \leq N(\varepsilon)}$ (the holes), and we define a perforated open set Ω_ε by

$$\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} T_i^\varepsilon.$$

We assume that Ω_ε is also a bounded connected open set in \mathbb{R}^N ($N \geq 2$), locally located on one side of its Lipschitz boundary $\partial\Omega_\varepsilon$. The flow of an incompressible viscous fluid in the domain Ω_ε under the action of an exterior force $f \in [L^2(\Omega)]^N$, with a no-slip (Dirichlet) boundary condition, is described by the following problem for the Stokes equations (see Remark 1.1.10 for the case of Navier-Stokes equations), where u_ε is the velocity, and p_ε the pressure of the fluid

$$\begin{aligned} \text{Find } (u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}] \quad \text{such that} \\ \nabla p_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon. \end{aligned} \tag{1.1.1}$$

The viscosity and density of the fluid have been set equal to 1. As is well known, the Stokes system (1.1.1) is equivalent to the following variational formulation, which has a unique solution

$$\begin{aligned} &\text{Find } (u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}] \quad \text{such that} \\ &\int_{\Omega_\varepsilon} \nabla u_\varepsilon : \nabla v - \int_{\Omega_\varepsilon} p_\varepsilon \nabla \cdot v = \int_{\Omega_\varepsilon} f \cdot v \quad \text{for each } v \in [H_0^1(\Omega_\varepsilon)]^N, \quad (1.1.2) \\ &\int_{\Omega_\varepsilon} q \nabla \cdot u_\varepsilon = 0 \quad \text{for each } q \in L^2(\Omega_\varepsilon)/\mathbb{R}. \end{aligned}$$

Introducing $v = u_\varepsilon$ in (1.1.2) leads to

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon.$$

Let us denote by $\tilde{\cdot}$ the extension operator from $H_0^1(\Omega_\varepsilon)$ into $H_0^1(\Omega)$ defined by

$$\text{for any } \phi \in H_0^1(\Omega_\varepsilon), \quad \tilde{\phi} = \phi \text{ in } \Omega_\varepsilon \quad \text{and} \quad \tilde{\phi} = 0 \text{ in } \Omega - \Omega_\varepsilon. \quad (1.1.3)$$

With the help of the Poincaré inequality in Ω , it is easy to see that

$$\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \quad (1.1.4)$$

where the constant C depends only on Ω (and not on ε). Consequently the sequence $(\tilde{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $[H_0^1(\Omega)]^N$. Thus there exists a subsequence, still denoted $(\tilde{u}_\varepsilon)_{\varepsilon>0}$, and there exists a $u \in [H_0^1(\Omega)]^N$ such that \tilde{u}_ε converges weakly to u in $[H_0^1(\Omega)]^N$. Note that this result is obtained without any assumptions on the holes $(T_i^\varepsilon)_{1 \leq i \leq N(\varepsilon)}$. The main problem is now to find an a priori estimate for the pressure p_ε , which yields the existence of a limit pressure p , and to see which homogenized equations are satisfied by the limit (u, p) . But, while the velocity u_ε can be naturally continued by zero in $\Omega - \Omega_\varepsilon$, it is not obvious how to construct an extension of the pressure p_ε that is bounded in $L^2(\Omega)/\mathbb{R}$. For that purpose, we now introduce an abstract framework of hypotheses on the holes, which allows us to prove the convergence of the homogenization process in general. Of course these assumptions will be verified in the other sections of this paper for the typical cases of hole distributions described in the introduction.

Hypotheses (H1)–(H6). Let us assume that the holes T_i^ε are such that there exist functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k^\varepsilon)_{1 \leq k \leq N}$ and a linear map R_ε such that

- (H1) $w_k^\varepsilon \in [H^1(\Omega)]^N, q_k^\varepsilon \in L^2(\Omega)$,
- (H2) $\nabla \cdot w_k^\varepsilon = 0$ in Ω and $w_k^\varepsilon = 0$ on the holes T_i^ε ,
- (H3) $w_k^\varepsilon \rightharpoonup e_k$ in $[H^1(\Omega)]^N$ weakly, $q_k^\varepsilon \rightharpoonup 0$ in $L^2(\Omega)/\mathbb{R}$ weakly,
- (H4) $\mu_k^\varepsilon \in [W^{-1,\infty}(\Omega)]^N$,
- (H5) For each sequence v_ε , for each v such that

$$v_\varepsilon \rightharpoonup v \text{ in } [H^1(\Omega)]^N \text{ weakly, } v_\varepsilon = 0 \text{ on the holes } T_i^\varepsilon$$

and for each $\phi \in D(\Omega)$ it follows that

$$\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \rightarrow \langle \mu_k, \phi v \rangle_{H^{-1}, H_0^1(\Omega)},$$

$$(H6) \quad \begin{cases} R_\varepsilon \in L([H_0^1(\Omega)]^N; [H_0^1(\Omega_\varepsilon)]^N), \\ \text{If } u \in [H_0^1(\Omega_\varepsilon)]^N, \text{ then } R_\varepsilon \tilde{u} = u \text{ in } \Omega_\varepsilon, \\ \text{If } \nabla \cdot u = 0 \text{ in } \Omega, \text{ then } \nabla \cdot (R_\varepsilon u) = 0 \text{ in } \Omega_\varepsilon, \\ \|R_\varepsilon u\|_{H_0^1(\Omega_\varepsilon)} \leq C \|u\|_{H_0^1(\Omega)} \text{ and } C \text{ does not depend on } \varepsilon. \end{cases}$$

Remark 1.1.1. The functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ involved in Hypotheses (H1)–(H5) seem, at first sight, rather mysterious. Actually they are the test functions that will be used in the energy method in order to prove the convergence of the homogenization process. Moreover, from a physical point of view, they turn out to be the velocity and the pressure of viscous layers around the holes (see Remark 2.1.5). The idea of such an abstract framework has been introduced by D. CIORANESCU & F. MURAT for the Laplacian (see [9]), but here, there is a new hypothesis (H6) which is crucial for the construction of a bounded extension of the pressure. The Hypotheses (H1)–(H6) have also direct consequences for the distribution and geometry of the holes: From (H3) and Rellich’s theorem one can deduce that the sequence w_k^ε converges to e_k in $[L^2(\Omega)]^N$ strongly, while being equal to zero on the holes T_i^ε . This implies that the measure of Ω_ε in \mathbb{R}^N tends to the measure of Ω , i.e., the holes are very small and disappear in the limit. Moreover, for a given family of holes $(T_i^\varepsilon)_{1 \leq i \leq N(\varepsilon)}$ the functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ which satisfy hypotheses (H1)–(H5) are “quasi-unique” (see Proposition I.2.9 in [1] for more details).

Now we give some properties of the functions $(\mu_k)_{1 \leq k \leq N}$.

Proposition 1.1.2. *Let $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ be functions that satisfy Hypotheses (H1)–(H5). Let M be the matrix defined by its columns $(\mu_k)_{1 \leq k \leq N}$, i.e., by its entries $(\mu_k^i)_{1 \leq k, i \leq N}$ given by $\mu_k^i = \mu_k \cdot e_i$. Then for each $\phi \in D(\Omega)$ we have*

$$\langle \mu_k^i, \phi \rangle_{D, D(\Omega)} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi \nabla w_k^\varepsilon : \nabla w_i^\varepsilon. \tag{1.1.5}$$

Thus M is a symmetric matrix, which is positive in the following sense

$$\langle M\Phi, \Phi \rangle_{H^{-1}, H_0^1(\Omega)} \geq 0 \quad \text{for each } \Phi \in [D(\Omega)]^N.$$

Proof. Take $v_\varepsilon = w_i^\varepsilon$ and $v = e_i$ in Hypothesis (H5). Then

$$\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi w_i^\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \rightarrow \langle \mu_k, \phi e_i \rangle_{H^{-1}, H_0^1(\Omega)} \quad \text{for each } \phi \in D(\Omega). \tag{1.1.6}$$

Integrating this expression by parts and using (H2), we reduce the left-hand side of (1.1.6) to

$$\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi w_i^\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = - \int_{\Omega} q_k^\varepsilon w_i^\varepsilon \cdot \nabla \phi + \int_{\Omega} \nabla w_k^\varepsilon : w_i^\varepsilon \nabla \phi + \int_{\Omega} \phi \nabla w_k^\varepsilon : \nabla w_i^\varepsilon.$$

Combining (H3) and Rellich’s theorem in the above equation gives

$$\lim_{\varepsilon \rightarrow 0} \langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi w_i^\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi \nabla w_k^\varepsilon : \nabla w_i^\varepsilon.$$

Therefore (1.1.6) is equivalent to

$$\int_{\Omega} \phi \nabla w_k^\varepsilon : \nabla w_i^\varepsilon \rightarrow \langle \mu_k, \phi e_i \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \mu_k^i, \phi \rangle_{H^{-1}, H_0^1(\Omega)}.$$

Thus (1.1.5) is proved. Moreover M is a symmetric matrix, being the limit of a sequence of symmetric matrices $(\nabla w_k^\varepsilon : \nabla w_i^\varepsilon)_{1 \leq i, k \leq N}$. On the other hand, for each $\Phi \in [D(\Omega)]^N$

$$\langle M\Phi, \Phi \rangle_{H^{-1}, H_0^1(\Omega)} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \sum_{k=1}^N \phi_k \nabla w_k^\varepsilon \right|^2 \geq 0. \tag{1.1.7}$$

Thus M is positive. Q.E.D.

Remark 1.1.3. From equality (1.1.7) we deduce the following equivalence:

$$\mu_k \equiv 0 \quad \text{if and only if} \quad w_k^\varepsilon \rightarrow e_k \text{ in } [H^1(\Omega)]^N \text{ strongly.}$$

Consequently, the mere weak convergence of w_k^ε is required to obtain non-zero functions μ_k (corresponding to the interesting cases in the convergence Theorem 1.1.8).

Following TARTAR's idea (see [28]), we construct an extension operator for the pressure under hypothesis (H6).

Proposition 1.1.4. *If there exists a linear operator R_ε satisfying (H6), then the operator P_ε defined by*

$$\langle \nabla [P_\varepsilon(q_\varepsilon)], u \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon u \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)}, \quad \text{for each } u \in [H_0^1(\Omega)]^N, \tag{1.1.8}$$

is a linear continuous extension operator from $L^2(\Omega_\varepsilon)/\mathbb{R}$ into $L^2(\Omega)/\mathbb{R}$ such that the following conditions hold for each $q_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$:

- (i) $P_\varepsilon(q_\varepsilon) = q_\varepsilon$ in $L^2(\Omega_\varepsilon)/\mathbb{R}$,
- (ii) $\|P_\varepsilon(q_\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \leq C \|q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}}$,
- (iii) $\|\nabla [P_\varepsilon(q_\varepsilon)]\|_{H^{-1}(\Omega)} \leq C \|\nabla q_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)}$

where C is a constant independent of q_ε and ε .

Before proving Proposition 1.1.4, we need the following three lemmas.

Lemma 1.1.5. *Let ω be a bounded connected open set in \mathbb{R}^N , with Lipschitz boundary $\partial\omega$, ω being locally located on one side of its boundary. Let p be a distribution on ω such that $\nabla p \in [H^{-1}(\omega)]^N$. Then $p \in L^2(\omega)/\mathbb{R}$, and*

$$\|p\|_{L^2(\omega)/\mathbb{R}} \leq C \|\nabla p\|_{H^{-1}(\omega)}.$$

where the constant C depends only on ω .

Lemma 1.1.6. *Let ω be a bounded connected open set in \mathbb{R}^N , with Lipschitz boundary $\partial\omega$, ω being locally located on one side of its boundary. Let $f \in [H^{-1}(\omega)]^N$ be such*

that

$$\langle f, u \rangle_{H^{-1}, H_0^1(\omega)} = 0 \quad \text{for each } u \in [H_0^1(\omega)]^N \text{ with } \nabla \cdot u = 0 \text{ in } \omega.$$

Then there exists a $p \in L^2(\omega)/\mathbb{R}$ such that $f = \nabla p$.

Lemma 1.1.7. *Let ω be a bounded connected open set in \mathbb{R}^N , with Lipschitz boundary $\partial\omega$, ω being locally located on one side of its boundary. For each $f \in L_0^2(\omega)$, i.e., for each $f \in L^2(\omega)$ such that $\int_{\omega} f = 0$, there exists $u \in [H_0^1(\omega)]^N$ satisfying*

- (i) $\nabla \cdot u = f$ in ω ,
- (ii) the map $f \rightarrow u$ is linear, and

$$\|u\|_{[H_0^1(\omega)]^N} \leq C \|f\|_{L^2(\omega)}$$

where the constant C depends only on ω .

The proofs of these lemmas are classical and may be found, e.g., in [30] or in [1] (with the references to the original papers).

Proof of Proposition 1.1.4. Let $q_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$. Because there exists a linear map R_ε satisfying (H6), we may define a functional F_ε on $[H_0^1(\Omega)]^N$ by

$$\langle F_\varepsilon, u \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon u \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} \quad \text{for each } u \in [H_0^1(\Omega)]^N. \quad (1.1.9)$$

Using the estimate of $R_\varepsilon u$ provided by (H6), we obtain

$$\|F_\varepsilon\|_{H^{-1}(\Omega)} \leq C \|\nabla q_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)}. \quad (1.1.10)$$

Thus $F_\varepsilon \in [H^{-1}(\Omega)]^N$. Furthermore, integrating (1.1.9) by parts, we get

$$\langle F_\varepsilon, u \rangle_{H^{-1}, H_0^1(\Omega)} = - \int_{\Omega_\varepsilon} q_\varepsilon \nabla \cdot (R_\varepsilon u).$$

According to (H6), $\nabla \cdot u = 0$ in Ω implies that $\nabla \cdot (R_\varepsilon u) = 0$ in Ω_ε . Thus

$$\langle F_\varepsilon, u \rangle_{H^{-1}, H_0^1(\Omega)} = 0 \quad \text{for each } u \in [H_0^1(\Omega)]^N \text{ with } \nabla \cdot u = 0 \text{ in } \Omega. \quad (1.1.11)$$

Applying Lemma 1.1.6 we deduce from (1.1.11) the existence of $Q_\varepsilon \in L^2(\Omega)/\mathbb{R}$ such that

$$F_\varepsilon = \nabla Q_\varepsilon \quad \text{in } \Omega.$$

Then we define the operator P_ε by $P_\varepsilon(q_\varepsilon) = Q_\varepsilon$. It is clear that P_ε is a linear continuous operator from $L^2(\Omega_\varepsilon)/\mathbb{R}$ into $L^2(\Omega)/\mathbb{R}$. Let us prove that $P_\varepsilon(q_\varepsilon) \equiv q_\varepsilon$ in $L^2(\Omega_\varepsilon)/\mathbb{R}$. Integrating (1.1.9) by parts, we obtain

$$\int_{\Omega} Q_\varepsilon \nabla \cdot u = \int_{\Omega_\varepsilon} q_\varepsilon \nabla \cdot (R_\varepsilon u).$$

According to (H6), if $u \in [H_0^1(\Omega_\varepsilon)]^N$, then $R_\varepsilon \tilde{u} \equiv u$ in Ω_ε . Thus, for each $u \in [H_0^1(\Omega_\varepsilon)]^N$,

$$\int_{\Omega_\varepsilon} (Q_\varepsilon - q_\varepsilon) \nabla \cdot u = 0. \quad (1.1.12)$$

Applying Lemma 1.1.7, we reduce (1.1.12) to $\int_{\Omega} (Q_\varepsilon - q_\varepsilon) f = 0$ for each $f \in L^2_0(\Omega_\varepsilon)$. In other words,

$$Q_\varepsilon - q_\varepsilon \equiv 0 \quad \text{in } L^2(\Omega_\varepsilon)/\mathbb{R}.$$

In the same way one can obtain estimate (ii), and (1.1.10) is just estimate (iii).
 Q.E.D.

Now, we are able to state and prove the main theorem about the convergence of the homogenization process.

Theorem 1.1.8. *Let Hypotheses (H1)–(H6) hold, and denote by M the matrix defined in Proposition 1.1.2. Let $(u_\varepsilon, p_\varepsilon)$ be the unique solution of the Stokes system (1.1.1). Let \tilde{u}_ε be the extension of the velocity by 0 in $\Omega - \Omega_\varepsilon$. Let $P_\varepsilon(p_\varepsilon)$ be the extension of the pressure, where P_ε is the operator defined in Proposition 1.1.4. Then*

$$(\tilde{u}_\varepsilon, P_\varepsilon(p_\varepsilon)) \rightharpoonup (u, p) \quad \text{in } [H^1_0(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \text{ weakly,}$$

where (u, p) is the unique solution of the homogenized system:

Find $(u, p) \in [H^1_0(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$ such that

$$\nabla p - \Delta u + Mu = f \quad \text{in } \Omega, \tag{1.1.13}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega.$$

Remark 1.1.9. The homogenized system (1.1.13) is a law of Brinkman type (see the original paper of BRINKMAN [8]). The new term Mu in (1.1.13) expresses the presence of the holes which have disappeared after passing to the limit. For the Laplacian, the same kind of phenomenon occurs (see D. CIORANESCU & F. MURAT [9] who called this new term a “strange term”). Note that if we put the fluid’s viscosity equal to μ instead of 1, then a matrix μM would replace M in (1.1.13).

Proof of Theorem 1.1.8. The proof is divided into two parts. In the first part we show that the extension of the pressure is bounded in $L^2(\Omega)/\mathbb{R}$, and in the second part we pass to the limit with the help of the energy method, introduced by L. TARTAR [29], and adapted by D. CIORANESCU & F. MURAT [9] for the Laplacian to an abstract framework similar to ours.

(1) Recall that $P_\varepsilon(p_\varepsilon)$ is defined by (1.1.8), i.e.,

$$\langle \nabla [P_\varepsilon(p_\varepsilon)], v \rangle_{H^{-1}, H^1_0(\Omega)} = \langle \nabla p_\varepsilon, R_\varepsilon v \rangle_{H^{-1}, H^1_0(\Omega_\varepsilon)} \quad \text{for each } v \in [H^1_0(\Omega)]^N.$$

Introducing equation (1.1.1) and integrating the last equation by parts, we get

$$\langle \nabla [P_\varepsilon(p_\varepsilon)], v \rangle_{H^{-1}, H^1_0(\Omega)} = - \int_{\Omega_\varepsilon} \nabla u_\varepsilon : \nabla (R_\varepsilon v) + \int_{\Omega_\varepsilon} f \cdot R_\varepsilon v.$$

Thus

$$|\langle \nabla [P_\varepsilon(p_\varepsilon)], v \rangle| \leq \| \nabla \tilde{u}_\varepsilon \|_{L^2(\Omega)} \| \nabla (R_\varepsilon v) \|_{L^2(\Omega_\varepsilon)} + \| f \|_{L^2(\Omega)} \| R_\varepsilon v \|_{L^2(\Omega_\varepsilon)}.$$

Combining this with inequality (1.1.4) and (H6), we obtain

$$\|\nabla[P_\varepsilon(p_\varepsilon)]\|_{H^{-1}(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \quad (1.1.14)$$

According to Lemma 1.1.5, we deduce from (1.1.14)

$$\|P_\varepsilon(p_\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \leq C \|f\|_{L^2(\Omega)}$$

where the constant C depends only on Ω , and not on ε . Consequently the sequence $P_\varepsilon(p_\varepsilon)$ is bounded in $L^2(\Omega)/\mathbb{R}$: One can therefore extract a subsequence, still denoted $P_\varepsilon(p_\varepsilon)$, and there exists some $p \in L^2(\Omega)/\mathbb{R}$, such that $P_\varepsilon(p_\varepsilon)$ converges weakly to p in $L^2(\Omega)/\mathbb{R}$.

(2) Now, we apply the energy method, *i.e.*, for any fixed $\phi \in D(\Omega)$, we introduce in the variational formulation (1.1.2) the following test functions

$$v = \phi w_k^\varepsilon \in [H_0^1(\Omega_\varepsilon)]^N, \quad q = \phi q_k^\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}.$$

We obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla u_\varepsilon : \nabla(\phi w_k^\varepsilon) - \int_{\Omega_\varepsilon} p_\varepsilon \nabla \cdot (\phi w_k^\varepsilon) &= \int_{\Omega_\varepsilon} f \cdot (\phi w_k^\varepsilon), \\ \int_{\Omega_\varepsilon} (\phi q_k^\varepsilon) \nabla \cdot u_\varepsilon &= 0. \end{aligned} \quad (1.1.15)$$

Expanding (1.1.15), and using (H2) (which requires that that w_k^ε be divergence-free), gives

$$\begin{aligned} \int_{\Omega_\varepsilon} \phi \nabla u_\varepsilon : \nabla w_k^\varepsilon + \int_{\Omega_\varepsilon} \nabla u_\varepsilon : w_k^\varepsilon \nabla \phi - \int_{\Omega_\varepsilon} p_\varepsilon w_k^\varepsilon \cdot \nabla \phi &= \int_{\Omega_\varepsilon} \phi f \cdot w_k^\varepsilon, \\ \int_{\Omega_\varepsilon} \phi q_k^\varepsilon \nabla \cdot u_\varepsilon &= 0. \end{aligned} \quad (1.1.16)$$

But

$$\int_{\Omega_\varepsilon} \phi \nabla u_\varepsilon : \nabla w_k^\varepsilon = -\langle \Delta w_k^\varepsilon, \phi \tilde{u}_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} - \int_{\Omega_\varepsilon} u_\varepsilon \nabla \phi : \nabla w_k^\varepsilon. \quad (1.1.17)$$

Integrating (1.1.16) by parts, introducing (1.1.17), and adding the two equations leads to

$$\begin{aligned} \langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi \tilde{u}_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} + \int_{\Omega} q_k^\varepsilon \tilde{u}_\varepsilon \cdot \nabla \phi - \int_{\Omega} \tilde{u}_\varepsilon \nabla \phi : \nabla w_k^\varepsilon \\ + \int_{\Omega} \nabla \tilde{u}_\varepsilon : w_k^\varepsilon \nabla \phi - \int_{\Omega_\varepsilon} p_\varepsilon w_k^\varepsilon \cdot \nabla \phi &= \int_{\Omega} \phi f \cdot w_k^\varepsilon. \end{aligned} \quad (1.1.18)$$

Moreover, because $P_\varepsilon(p_\varepsilon) \equiv p_\varepsilon$ in Ω_ε and $w_k^\varepsilon = 0$ in $\Omega - \Omega_\varepsilon$, we have

$$\int_{\Omega_\varepsilon} p_\varepsilon w_k^\varepsilon \cdot \nabla \phi = \int_{\Omega} P_\varepsilon(p_\varepsilon) w_k^\varepsilon \cdot \nabla \phi.$$

Then we pass to the limit in (1.1.18) as ε tends to zero. The sequence \tilde{u}_ε fulfills the conditions of hypothesis (H5), and we obtain

$$\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi \tilde{u}_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \rightarrow \langle \mu_k, \phi u \rangle_{H^{-1}, H_0^1(\Omega)}.$$

On the other hand, recalling the following convergences

$$\begin{aligned} \tilde{u}_\varepsilon &\rightharpoonup u && \text{in } [H_0^1(\Omega)]^N \text{ weakly,} \\ w_k^\varepsilon &\rightharpoonup e_k && \text{in } [H^1(\Omega)]^N \text{ weakly,} \\ q_k^\varepsilon &\rightharpoonup 0 && \text{in } L^2(\Omega)/\mathbb{R} \text{ weakly,} \\ P_\varepsilon(p_\varepsilon) &\rightharpoonup p && \text{in } L^2(\Omega)/\mathbb{R} \text{ weakly} \end{aligned}$$

and using Rellich's Theorem, we convert (1.1.18) to

$$\langle \mu_k, \phi u \rangle_{H^{-1}, H_0^1(\Omega)} + \int_\Omega \nabla u : e_k \nabla \phi - \int_\Omega p e_k \cdot \nabla \phi = \int_\Omega \phi f \cdot e_k. \quad (1.1.19)$$

Integrating (1.1.19) by parts gives

$$\langle \mu_k, \phi u \rangle - \langle \Delta u, \phi e_k \rangle + \langle \nabla p, \phi e_k \rangle = \langle f, \phi e_k \rangle \quad \text{for each } k \in \{1, 2, \dots, N\}.$$

But M is symmetric, so that

$$\nabla p - \Delta u + Mu = f \quad \text{in } [D'(\Omega)]^N. \quad (1.1.20)$$

Furthermore, we know that $\nabla \cdot \tilde{u}_\varepsilon = 0$ in Ω , and $\tilde{u}_\varepsilon \rightharpoonup u$ in $[H_0^1(\Omega)]^N$. Passing to the limit yields

$$\nabla \cdot u = 0 \quad \text{in } \Omega. \quad (1.1.21)$$

Regrouping (1.1.20) and (1.1.21) we obtain the following homogenized problem

$$\begin{aligned} \text{Find } (u, p) &\in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \quad \text{such that} \\ \nabla p - \Delta u + Mu &= f \quad \text{in } \Omega, \\ \nabla \cdot u &= 0 \quad \text{in } \Omega. \end{aligned} \quad (1.1.22)$$

It remains to prove that (1.1.22) admits a unique solution. From Hypothesis (H4) we know that Mu belongs to $[H^{-1}(\Omega)]^N$, and from Proposition 1.1.2, that M is a positive matrix. Thus, for each $u \in H_0^1(\Omega)^N$, we have

$$\int_\Omega |\nabla u|^2 + \langle Mu, u \rangle_{H^{-1}, H_0^1(\Omega)} \geq \int_\Omega |\nabla u|^2. \quad (1.1.23)$$

From (1.1.23) we deduce the coercivity of the operator $(-\Delta + M)$, and also the existence and uniqueness of a solution of (1.1.22). Moreover, because the solution of (1.1.22) is unique, all the subsequences of $(\tilde{u}_\varepsilon, P_\varepsilon(p_\varepsilon))$ converge to the same limit. So the entire sequence converges. Q.E.D.

Remark 1.1.10. When the space dimension is $N = 2$ or 3 , Theorem 1.1.8 can be easily generalized to apply to the Navier-Stokes equations:

$$\begin{aligned} \text{Find } (u_\varepsilon, p_\varepsilon) &\in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}] \quad \text{such that} \\ \nabla p_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon - \Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon. \end{aligned} \quad (1.1.24)$$

It is well-known that there exists at least one solution of system (1.1.24), which is unique for small values of $\|f\|_{L^2(\Omega)}$ when $N = 2$ or 3 . For such f , with the same hypotheses (H1)–(H6) as for the Stokes system, we can prove the same results. More precisely, because the sequence \tilde{u}_ϵ converges weakly to u in $[H_0^1(\Omega)]^N$, the non-linear term $\tilde{u}_\epsilon \cdot \nabla \tilde{u}_\epsilon$ converges strongly to $u \cdot \nabla u$ in $[H^{-1}(\Omega)]^N$, and the homogenized problem is

$$\begin{aligned} \text{Find } (u, p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}] \quad \text{such that} \\ \nabla p + u \cdot \nabla u - \Delta u + Mu = f \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in } \Omega. \end{aligned} \tag{1.1.25}$$

It is worth noticing that the functions $(w_k^\epsilon, q_k^\epsilon, \mu_k)_{1 \leq k \leq N}$ that satisfy Hypotheses (H1)–(H5), and therefore the matrix M are exactly the same for both the Stokes and the Navier-Stokes homogenizations. For more details, see Section I.7 in [1].

Remark 1.1.11. Hypothesis (H4) can be weakened to $\mu_k \in [H^{-1}(\Omega)]^N$, and Theorem 1.1.8 still holds with a slight change in the class of solutions (see Section I.6 in [1]).

1.2. Correctors and error estimates

In this subsection we give correctors for the velocity u_ϵ and the pressure p_ϵ with the help of a weak semicontinuity result for the energy. Moreover, we give abstract error estimates which will be used in the second part of this paper in order to obtain explicit bounds for the error in concrete situations.

Proposition 1.2.1. *Let Hypotheses (H1)–(H5) hold. Then each sequence $(z_\epsilon)_{\epsilon>0}$ such that*

$$\begin{aligned} z_\epsilon \rightharpoonup z \quad & \text{in } [H_0^1(\Omega)]^N \text{ weakly,} \\ \nabla \cdot z_\epsilon \rightarrow \nabla \cdot z \quad & \text{in } L^2(\Omega) \text{ strongly,} \\ z_\epsilon = 0 \quad & \text{on the holes } T_i^\epsilon \end{aligned} \tag{1.2.1}$$

satisfies

$$\liminf_{\epsilon \rightarrow 0} \int_{\Omega} |\nabla z_\epsilon|^2 \geq \int_{\Omega} |\nabla z|^2 + \langle Mz, z \rangle_{H^{-1}, H_0^1(\Omega)}. \tag{1.2.2}$$

Proposition 1.2.2. *Let Hypotheses (H1)–(H5) hold. Then each sequence $(z_\epsilon)_{\epsilon>0}$ such that*

$$\begin{aligned} z_\epsilon \rightharpoonup z \quad & \text{in } [H_0^1(\Omega)]^N \text{ weakly,} \\ \nabla \cdot z_\epsilon \rightarrow \nabla \cdot z \quad & \text{in } L^2(\Omega) \text{ strongly,} \\ z_\epsilon = 0 \quad & \text{on the holes } T_i^\epsilon, \end{aligned} \tag{1.2.3}$$

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} |\nabla z_\epsilon|^2 = \int_{\Omega} |\nabla z|^2 + \langle Mz, z \rangle_{H^{-1}, H_0^1(\Omega)}$$

satisfies

$$(z_\varepsilon - W_\varepsilon z) \rightarrow 0 \quad \text{in } [W_0^{1,q}(\Omega)]^N \text{ strongly} \tag{1.2.4}$$

where W_ε is the matrix defined by $W_\varepsilon e_k = w_k^\varepsilon$, and $q = \frac{N}{N-1}$ if $N \geq 3$, $1 \leq q < 2$ if $N = 2$.

Theorem 1.2.3. *Let Hypotheses (H1)–(H6) hold. Then the solution u_ε of the Stokes system (1.1.1) satisfies*

$$\tilde{u}_\varepsilon = W_\varepsilon u + r_\varepsilon \tag{1.2.5}$$

where W_ε is defined by $W_\varepsilon e_k = w_k^\varepsilon$, u is the solution of the homogenized system (1.1.13), and r_ε is such

$$r_\varepsilon \rightarrow 0 \quad \text{in } [W_0^{1,q}(\Omega)]^N \text{ strongly} \tag{1.2.6}$$

with $q = \frac{N}{N-1}$ if $N \geq 3$, $1 \leq q < 2$ if $N = 2$.

Moreover, if u is smoother than $[H_0^1(\Omega)]^N$, say,

$$\begin{aligned} u &\in [W_0^{1,N}(\Omega) \cap C^0(\Omega)]^N \quad \text{if } N \geq 3, \\ u &\in [W_0^{1,2+\eta}(\Omega)]^2, \quad \text{for some } \eta > 0, \text{ if } N = 2, \end{aligned} \tag{1.2.7}$$

then (1.2.6) can be improved:

$$r_\varepsilon \rightarrow 0 \quad \text{in } [H_0^1(\Omega)]^N \text{ strongly.} \tag{1.2.8}$$

Theorem 1.2.4. *Let Hypotheses (H1)–(H6) hold. Let the solution u of the homogenized system (1.1.13) be sufficiently smooth, say,*

$$u \in [W_0^{1,N+\eta}(\Omega)]^N \quad \text{for some } \eta > 0. \tag{1.2.9}$$

Then the pressure p_ε of the Stokes system (1.1.1) satisfies

$$P_\varepsilon(p_\varepsilon - p - u \cdot Q_\varepsilon) \rightarrow 0 \quad \text{in } L^2(\Omega)/\mathbb{R} \text{ strongly,} \tag{1.2.10}$$

where Q_ε is the vector defined by $Q_\varepsilon \cdot e_k = q_k^\varepsilon$, (u, p) is the unique solution of the homogenized system (1.1.13), and P_ε is the extension operator defined in Proposition 1.1.4.

Obviously (1.2.10) implies that

$$\|p_\varepsilon - p - u \cdot Q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \rightarrow 0. \tag{1.2.11}$$

Before giving abstract error estimates in the next proposition, we replace Hypothesis (H5) by a stronger version (H5'):

$$(H5') \quad \begin{cases} \text{For each } k \in \{1, 2, \dots, N\}, \quad \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = \mu_k^\varepsilon - \gamma_k^\varepsilon \text{ in } \Omega \\ \text{with } \mu_k^\varepsilon \rightarrow \mu_k \text{ in } [H^{-1}(\Omega)]^N \text{ strongly,} \\ \text{with } \gamma_k^\varepsilon \rightharpoonup \mu_k \text{ in } [H^{-1}(\Omega)]^N \text{ weakly,} \\ \text{and with } \gamma_k^\varepsilon \equiv 0 \text{ in } [H^{-1}(\Omega_\varepsilon)]^N. \end{cases}$$

(The last equality means that, for any function $\nu \in [H_0^1(\Omega)]^N$ that satisfies $\nu = 0$ on the holes T_i^ε , we have $\langle \gamma_k^\varepsilon, \nu \rangle_{H^{-1}, H_0^1(\Omega)} = 0$.)

Obviously (H5) can be deduced from (H5') which was not introduced before because we need it only for the following proposition. Of course in the other sections of this paper, we check that (H5') is always satisfied in the examples under consideration.

Proposition 1.2.5. *Let Hypotheses (H1)–(H6) and (H5') hold. Assume that the velocity u satisfying the homogenized system (1.1.13) is smooth, say*

$$u \in [W^{2,\infty}(\Omega)]^N. \tag{1.2.12}$$

Let $\alpha_\varepsilon = p_\varepsilon - p - u \cdot Q_\varepsilon$ and $r_\varepsilon = \tilde{u}_\varepsilon - W_\varepsilon u$. Let M_ε denote the matrix defined by its columns $\mu_k^\varepsilon = M_\varepsilon e_k$. Then

$$\|\alpha_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C \|u\|_{W^{2,\infty}(\Omega)} [\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \|Q_\varepsilon\|_{H^{-1}(\Omega)}], \tag{1.2.13}$$

$$\|\nabla r_\varepsilon\|_{L^2(\Omega)} \leq C \|u\|_{W^{2,\infty}(\Omega)} [\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \|Q_\varepsilon\|_{H^{-1}(\Omega)}] \tag{1.2.14}$$

where the constant C depends only on Ω .

Remark 1.2.6. The above results on correctors and error estimates are actually generalizations to the Stokes equations of previous results obtained for the Laplacian operator. In that case, Propositions 1.2.1 and 1.2.2, and Theorem 1.2.3, have been proved by D. CIORANESCU & F. MURAT [9], while Proposition 1.2.5 (except the result for the pressure) has been proved by H. KACIMI & F. MURAT [15]. Theorem 1.2.4 is original because it is devoted to a corrector of the pressure. Furthermore, Propositions 1.2.1 and 1.2.2 correspond to the so-called Γ -convergence, introduced by E. DE GIORGI [11], [12].

Proof of Proposition 1.2.1. Let $\Phi = (\phi_1, \dots, \phi_N) \in [D(\Omega)]^N$. Consider the sequence Δ_ε of real numbers defined by

$$\Delta_\varepsilon = \int_\Omega \left| \nabla \left(z_\varepsilon - \sum_{k=1}^N \phi_k w_k^\varepsilon \right) \right|^2. \tag{1.2.15}$$

Expanding (1.2.15) gives

$$\begin{aligned} \Delta_\varepsilon &= \int_\Omega |\nabla z_\varepsilon|^2 + \sum_{1 \leq i, k \leq N} \left[\int_\Omega \phi_k \nabla w_k^\varepsilon : \phi_i \nabla w_i^\varepsilon + \int_\Omega \nabla \phi_k w_k^\varepsilon : \nabla \phi_i w_i^\varepsilon \right] \\ &\quad + 2 \sum_{1 \leq i, k \leq N} \int_\Omega \phi_k \nabla w_k^\varepsilon : \nabla \phi_i w_i^\varepsilon - 2 \sum_{1 \leq k \leq N} \int_\Omega \nabla z_\varepsilon : \nabla \phi_k w_k^\varepsilon \\ &\quad - 2 \sum_{1 \leq k \leq N} \int_\Omega \nabla z_\varepsilon : \phi_k \nabla w_k^\varepsilon. \end{aligned} \tag{1.2.16}$$

Integrating the last term in (1.2.16) by parts leads to

$$\begin{aligned} \int_{\Omega} \nabla z_{\varepsilon} : \phi_k \nabla w_k^{\varepsilon} &= - \langle \Delta w_k^{\varepsilon}, \phi_k z_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} - \int_{\Omega} z_{\varepsilon} \nabla \phi_k : \nabla w_k^{\varepsilon} \\ &= \langle \nabla q_k^{\varepsilon} - \Delta w_k^{\varepsilon}, \phi_k z_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} - \int_{\Omega} z_{\varepsilon} \nabla \phi_k : \nabla w_k^{\varepsilon} - \langle \nabla q_k^{\varepsilon}, \phi_k z_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)}. \end{aligned} \quad (1.2.17)$$

Integrating the last term in (1.2.17) by parts yields

$$\langle \nabla q_k^{\varepsilon}, \phi_k z_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} = - \int_{\Omega} q_k^{\varepsilon} (z_{\varepsilon} \cdot \nabla \phi_k + \phi_k \nabla \cdot z_{\varepsilon}). \quad (1.2.18)$$

Now we introduce (1.2.18) and (1.2.17) into (1.2.16):

$$\begin{aligned} \Delta_{\varepsilon} &= \int_{\Omega} |\nabla z_{\varepsilon}|^2 + \sum_{1 \leq i, k \leq N} \left[\int_{\Omega} \phi_k \phi_i \nabla w_k^{\varepsilon} : \nabla w_i^{\varepsilon} + \int_{\Omega} \nabla \phi_k w_k^{\varepsilon} : \nabla \phi_i w_i^{\varepsilon} \right] \\ &\quad + 2 \sum_{1 \leq i, k \leq N} \int_{\Omega} \phi_k \nabla w_k^{\varepsilon} : \nabla \phi_i w_i^{\varepsilon} \\ &\quad - 2 \sum_{1 \leq k \leq N} \left[\int_{\Omega} \nabla z_{\varepsilon} : \nabla \phi_k w_k^{\varepsilon} + \int_{\Omega} q_k^{\varepsilon} (z_{\varepsilon} \cdot \nabla \phi_k + \phi_k \nabla \cdot z_{\varepsilon}) - \int_{\Omega} z_{\varepsilon} \nabla \phi_k : \nabla w_k^{\varepsilon} \right] \\ &\quad - 2 \sum_{1 \leq k \leq N} \langle \nabla q_k^{\varepsilon} - \Delta w_k^{\varepsilon}, \phi_k z_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)}. \end{aligned} \quad (1.2.19)$$

There exists a subsequence, still denoted by z_{ε} , such that

$$\int_{\Omega} |\nabla z_{\varepsilon}|^2 \rightarrow \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla z_{\varepsilon}|^2. \quad (1.2.20)$$

Moreover, because of assumption (1.2.1), the sequence z_{ε} fulfills the conditions of (H5), and we obtain

$$\langle \nabla q_k^{\varepsilon} - \Delta w_k^{\varepsilon}, \phi_k z_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} \rightarrow \langle \mu_k, \phi_k z \rangle_{H^{-1}, H_0^1(\Omega)}. \quad (1.2.21)$$

On the other hand, Proposition 1.1.2 implies that

$$\int_{\Omega} \phi_k \phi_i \nabla w_k^{\varepsilon} : \nabla w_i^{\varepsilon} \rightarrow \langle \mu_k^i, \phi_k \phi_i \rangle_{H^{-1}, H_0^1(\Omega)}. \quad (1.2.22)$$

Recalling that $\Delta_{\varepsilon} \geq 0$, we pass to the limit in (1.2.19) with the help of (1.2.20)–(1.2.22):

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla z_{\varepsilon}|^2 + \int_{\Omega} |\nabla \Phi|^2 + \sum_{1 \leq k \leq N} \langle \mu_k, \phi_k \Phi \rangle_{H^{-1}, H_0^1(\Omega)} \\ - 2 \int_{\Omega} \nabla \Phi : \nabla z - 2 \sum_{1 \leq k \leq N} \langle \mu_k, \phi_k z \rangle_{H^{-1}, H_0^1(\Omega)} \geq 0. \end{aligned} \quad (1.2.23)$$

Because $\mu_k \in [W^{-1, \infty}(\Omega)]^N$, we can apply inequality (1.2.23) to a sequence of functions Φ that tends to z and pass to the limit. Then

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla z_{\varepsilon}|^2 \geq \int_{\Omega} |\nabla z|^2 + \langle Mz, z \rangle_{H^{-1}, H_0^1(\Omega)}. \quad \text{Q.E.D.}$$

Proof of Proposition 1.2.2. We now pass to the limit in equality (1.2.19) taking into account the new assumption on z_ε :

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon &= \int_{\Omega} |\nabla z|^2 + \langle Mz, z \rangle_{H^{-1}, H_0^1(\Omega)} + \int_{\Omega} |\nabla \Phi|^2 \\ &\quad + \sum_{1 \leq k \leq N} \langle \mu_k, \phi_k \Phi \rangle_{H^{-1}, H_0^1(\Omega)} - 2 \int_{\Omega} \nabla \Phi : \nabla z - 2 \sum_{1 \leq k \leq N} \langle \mu_k, \phi_k z \rangle_{H^{-1}, H_0^1(\Omega)} \\ &= \int_{\Omega} |\nabla(z - \Phi)|^2 + \langle M(z - \Phi), (z - \Phi) \rangle_{H^{-1}, H_0^1(\Omega)}. \end{aligned} \quad (1.2.24)$$

Let η be a strictly positive real number. Because $D(\Omega)$ is dense in $H_0^1(\Omega)$, there exists $\Phi_\eta \in [D(\Omega)]^N$ such that

$$\|z - \Phi_\eta\|_{H_0^1(\Omega)} \leq \eta. \quad (1.2.25)$$

Then we can bound (1.2.24):

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla(z_\varepsilon - W_\varepsilon \Phi_\eta)|^2 = \lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon \leq \eta^2 + \|M\|_{W^{-1}, \infty(\Omega)} \|(z - \Phi_\eta)^2\|_{W_0^{1,1}(\Omega)}.$$

But $\|(z - \Phi_\eta)^2\|_{W_0^{1,1}(\Omega)} \leq C \|z - \Phi_\eta\|_{H_0^1(\Omega)}^2$ where C depends only on Ω . Thus

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla(z_\varepsilon - W_\varepsilon \Phi_\eta)|^2 \leq (1 + C \|M\|_{W^{-1}, \infty(\Omega)}) \eta^2. \quad (1.2.26)$$

Because $z_\varepsilon - W_\varepsilon z = (z_\varepsilon - W_\varepsilon \Phi_\eta) + W_\varepsilon(\Phi_\eta - z)$, for $q \leq 2$ we have

$$\|z_\varepsilon - W_\varepsilon z\|_{W_0^{1,q}(\Omega)} \leq \|z_\varepsilon - W_\varepsilon \Phi_\eta\|_{H_0^1(\Omega)} + \|W_\varepsilon(\Phi_\eta - z)\|_{W_0^{1,q}(\Omega)}. \quad (1.2.27)$$

If $N \geq 3$, $H^1(\Omega)$ is continuously embedded in $L^{2N/(N-2)}(\Omega)$; then

$$\begin{aligned} \|W_\varepsilon(\Phi_\eta - z)\|_{W_0^{1,N/(N-1)}(\Omega)} &\leq \|W_\varepsilon\|_{L^{2N/(N-2)}(\Omega)} \|\nabla(\Phi_\eta - z)\|_{L^2(\Omega)} \\ &\quad + \|\Phi_\eta - z\|_{L^{2N/(N-2)}(\Omega)} \|\nabla W_\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

If $N = 2$, $H^1(\Omega)$ is continuously embedded in $L^p(\Omega)$ for any $p < +\infty$; then

$$\|W_\varepsilon(\Phi_\eta - z)\|_{W_0^{1,q}(\Omega)} \leq \|W_\varepsilon\|_{L^p(\Omega)} \|\nabla(\Phi_\eta - z)\|_{L^2(\Omega)} + \|\Phi_\eta - z\|_{L^p(\Omega)} \|\nabla W_\varepsilon\|_{L^2(\Omega)}$$

with $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$ (Note that if $p < \infty$, then $q < 2$). Consequently, from (1.2.27) we obtain

$$\lim_{\varepsilon \rightarrow 0} \|z_\varepsilon - W_\varepsilon z\|_{W_0^{1,q}(\Omega)} \leq C\eta \quad \text{with} \quad \begin{cases} q = \frac{N}{N-1} & \text{if } N \geq 3, \\ 1 \leq q < 2 & \text{if } N = 2. \end{cases} \quad (1.2.28)$$

Inequality (1.2.28) gives the desired result when η tends to zero. Q.E.D.

Proof of Theorem 1.2.3. We easily check that the solution \tilde{u}_ε of the Stokes system (1.1.1) satisfies the assumptions of Proposition 1.2.2. Thanks to Theorem 1.1.8 we know that

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{in } [H_0^1(\Omega)]^N \text{ weakly, } \quad \nabla \cdot \tilde{u}_\varepsilon = 0 \quad \text{in } \Omega, \quad \tilde{u}_\varepsilon = 0 \quad \text{on the holes } T_i^\varepsilon.$$

Moreover, we have

$$\int_{\Omega} |\nabla \tilde{u}_\varepsilon|^2 = \int_{\Omega} f \cdot \tilde{u}_\varepsilon \rightarrow \int_{\Omega} f \cdot u = \int_{\Omega} |\nabla u|^2 + \langle Mu, u \rangle_{H^{-1}, H_0^1(\Omega)}.$$

Thus the conclusion of Proposition 1.2.2 holds for \tilde{u}_ε , and we conclude that $\tilde{u}_\varepsilon - W_\varepsilon u = r_\varepsilon$ converges strongly to 0 in $[W_0^{1,q}(\Omega)]^N$, where $q = \frac{N}{N-1}$ if $N \geq 3$, and $1 \leq q < 2$ if $N = 2$. The proof of the improved convergence (1.2.8) of r_ε when u is smoother is not difficult, and is left to the reader (see [1] if necessary). Q.E.D.

Before proving Theorem 1.2.4, we give a generalization of Hypothesis (H5) in the following lemma, the proof of which is elementary.

Lemma 1.2.7. *Let Hypotheses (H1)–(H5) hold. Then (H5) can be generalized thus: For each sequence v_ε and for each v such that v_ε converges weakly to v in $[H^1(\Omega)]^N$ and $v_\varepsilon = 0$ on the holes T_i^ε , and for each ϕ which belongs to $W_0^{1,N}(\Omega) \cap C^0(\Omega)$ for $N \geq 3$, and to $W_0^{1,2+\eta}(\Omega)$, with $\eta > 0$, for $N = 2$, the following limit holds:*

$$\langle \nabla q_k^\varepsilon - \Delta w_k^\varepsilon, \phi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \rightarrow \langle \mu_k, \phi v \rangle_{H^{-1}, H_0^1(\Omega)}.$$

Proof of Theorem 1.2.4. First remark that $u \in [W_0^{1,N+\eta}(\Omega)]^N \subset [L^\infty(\Omega)]^N$ for $\eta > 0$; this implies that $u \cdot Q_\varepsilon \in L^2(\Omega)/\mathbb{R}$, and that $P_\varepsilon(u \cdot Q_\varepsilon)$ is meaningful. According to Lemma 1.1.5 it is equivalent to prove that

$$\nabla [P_\varepsilon(p_\varepsilon - p - u \cdot Q_\varepsilon)] \rightarrow 0 \quad \text{in } [H^{-1}(\Omega)]^N \text{ strongly.} \tag{1.2.29}$$

Let v_ε be a bounded sequence in $[H_0^1(\Omega)]^N$. We define a real sequence Δ_ε by

$$\Delta_\varepsilon = \langle \nabla [P_\varepsilon(p_\varepsilon - p - u \cdot Q_\varepsilon)], v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}. \tag{1.2.30}$$

Using Proposition 1.1.4 gives

$$\Delta_\varepsilon = \langle \nabla p_\varepsilon, R_\varepsilon v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} - \langle \nabla p, R_\varepsilon v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} - \langle \nabla(u \cdot Q_\varepsilon), R_\varepsilon v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)}. \tag{1.2.31}$$

In order to simplify the notation, from now on $R_\varepsilon v_\varepsilon$ represents both the function in $[H_0^1(\Omega_\varepsilon)]^N$ and its extension by 0 in $\Omega - \Omega_\varepsilon$. This extension belongs to $[H_0^1(\Omega)]^N$. Introducing Stokes' and Brinkman's equations in (1.2.31), and integrating it by parts leads to

$$\Delta_\varepsilon = \int_{\Omega} (\nabla u - \nabla u_\varepsilon) : \nabla (R_\varepsilon v_\varepsilon) + \langle Mu, R_\varepsilon v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} - \langle \nabla(u \cdot Q_\varepsilon), R_\varepsilon v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}. \tag{1.2.32}$$

Theorem 1.2.3 asserts that $\tilde{u}_\varepsilon = W_\varepsilon u + r_\varepsilon$ and $r_\varepsilon \rightarrow 0$ in $[H_0^1(\Omega)]^N$ strongly. Replacing \tilde{u}_ε by the above expression, and integrating (1.2.32) by parts gives

$$\begin{aligned} \Delta_\varepsilon = & \int_{\Omega} (Id - W_\varepsilon) \nabla u : \nabla (R_\varepsilon \nu_\varepsilon) - \int_{\Omega} \nabla r_\varepsilon : \nabla (R_\varepsilon \nu_\varepsilon) + \int_{\Omega} \nabla u : (R_\varepsilon \nu_\varepsilon \cdot \nabla W_\varepsilon) \\ & - \int_{\Omega} Q_\varepsilon \nabla u \cdot R_\varepsilon \nu_\varepsilon - \langle (\nabla Q_\varepsilon - \Delta W_\varepsilon) u, R_\varepsilon \nu_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} + \langle Mu, R_\varepsilon \nu_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}. \end{aligned} \quad (1.2.33)$$

From elementary arguments and Lemma 1.2.7, it follows from (1.2.33) that $\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon = 0$ for any sequence ν_ε bounded in $[H_0^1(\Omega)]^N$. This is equivalent to (1.2.29) and therefore Theorem 1.2.4 is proved. Q.E.D.

Proof of Proposition 1.2.5. Define the matrices M_ε and Γ_ε by their columns $\mu_k^\varepsilon = M_\varepsilon e_k$ and $\gamma_k^\varepsilon = \Gamma_\varepsilon e_k$. Hypothesis (H5') enables us to replace the term $(\nabla Q_\varepsilon - \Delta W_\varepsilon)$ by $(M_\varepsilon - \Gamma_\varepsilon)$ in equality (1.2.33), and we use the fact that $\gamma_k^\varepsilon \equiv 0$ in $[H^{-1}(\Omega_\varepsilon)]^N$ to obtain

$$\begin{aligned} \Delta_\varepsilon = & \int_{\Omega} (Id - W_\varepsilon) \nabla u : \nabla (R_\varepsilon \nu_\varepsilon) - \int_{\Omega} \nabla r_\varepsilon : \nabla (R_\varepsilon \nu_\varepsilon) + \int_{\Omega} \nabla u : (R_\varepsilon \nu_\varepsilon \cdot \nabla W_\varepsilon) \\ & - \int_{\Omega} Q_\varepsilon \nabla u \cdot R_\varepsilon \nu_\varepsilon + \langle (M - M_\varepsilon) u, R_\varepsilon \nu_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}. \end{aligned} \quad (1.2.34)$$

Because u is smooth we have

$$\int_{\Omega} \nabla u : (R_\varepsilon \nu_\varepsilon \cdot \nabla W_\varepsilon) = \sum_{1 \leq i, k \leq N} \langle \nabla w_{k,i}^\varepsilon \cdot \nabla u, R_\varepsilon \nu_i^\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}$$

and

$$\int_{\Omega} Q_\varepsilon \nabla u \cdot R_\varepsilon \nu_\varepsilon = \langle Q_\varepsilon \nabla u, R_\varepsilon \nu_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}.$$

Moreover, because the gradient operator is continuous from $L^2(\Omega)$ in $[H^{-1}(\Omega)]^N$, we have

$$\|\nabla W_\varepsilon\|_{H^{-1}(\Omega)} \leq C \|Id - W_\varepsilon\|_{L^2(\Omega)}.$$

Then we bound (1.2.34):

$$\begin{aligned} |\Delta_\varepsilon| \leq & \|Id - W_\varepsilon\|_{L^2(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} \|\nabla (R_\varepsilon \nu_\varepsilon)\|_{L^2(\Omega)} + \|\nabla r_\varepsilon\|_{L^2(\Omega)} \|\nabla (R_\varepsilon \nu_\varepsilon)\|_{L^2(\Omega)} \\ & + C \|\nabla u\|_{W^{1, \infty}(\Omega)} \|Id - W_\varepsilon\|_{L^2(\Omega)} \|R_\varepsilon \nu_\varepsilon\|_{H_0^1(\Omega)} \\ & + \|\nabla u\|_{W^{1, \infty}(\Omega)} \|Q_\varepsilon\|_{H^{-1}(\Omega)} \|R_\varepsilon \nu_\varepsilon\|_{H_0^1(\Omega)} \\ & + \|u\|_{W^{1, \infty}(\Omega)} \|M - M_\varepsilon\|_{H^{-1}(\Omega)} \|R_\varepsilon \nu_\varepsilon\|_{H_0^1(\Omega)}. \end{aligned} \quad (1.2.35)$$

But Hypothesis (H6) implies that $\|R_\varepsilon \nu_\varepsilon\|_{H_0^1(\Omega)} \leq C \|\nu_\varepsilon\|_{H_0^1(\Omega)}$, and Definition (1.2.30) of Δ_ε can be rewritten as

$$\Delta_\varepsilon = - \int_{\Omega} P_\varepsilon(\alpha_\varepsilon) \nabla \cdot \nu_\varepsilon \quad \text{with } \alpha_\varepsilon = p_\varepsilon - p - u \cdot Q_\varepsilon. \quad (1.2.36)$$

With the help of Lemma 1.1.7, we obtain for each $f \in L_0^2(\Omega)$ that

$$\left| \int_{\Omega} P_{\varepsilon}(\alpha_{\varepsilon}) f \right| \leq C \|\nabla r_{\varepsilon}\|_{L^2(\Omega)} \|f\|_{L_0^2(\Omega)} + C \|u\|_{W^{2,\infty}(\Omega)} \|f\|_{L_0^2(\Omega)} [\|Id - W_{\varepsilon}\|_{L^2(\Omega)} + \|M - M_{\varepsilon}\|_{H^{-1}(\Omega)} + \|Q_{\varepsilon}\|_{H^{-1}(\Omega)}]. \quad (1.2.37)$$

Because $L^2(\Omega)/\mathbb{R}$ is the dual space of $L_0^2(\Omega)$ and because $P_{\varepsilon}(\alpha_{\varepsilon}) \equiv \alpha_{\varepsilon}$ in Ω_{ε} , we conclude from (1.2.37) that

$$\|\alpha_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})/\mathbb{R}} \leq C \|\nabla r_{\varepsilon}\|_{L^2(\Omega)} + C \|u\|_{W^{2,\infty}(\Omega)} [\|M_{\varepsilon} - M\|_{H^{-1}(\Omega)} + \|Id - W_{\varepsilon}\|_{L^2(\Omega)} + \|Q_{\varepsilon}\|_{H^{-1}(\Omega)}]. \quad (1.2.38)$$

Now it remains to estimate r_{ε} in $[H_0^1(\Omega)]^N$. Following an idea of H. KACIMI & F. MURAT [15], we calculate the duality product $\langle -\Delta r_{\varepsilon}, r_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)}$ in two different ways. On the one hand we have

$$\langle -\Delta r_{\varepsilon}, r_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} = \int_{\Omega} |\nabla r_{\varepsilon}|^2.$$

On the other hand we have

$$\begin{aligned} -\Delta r_{\varepsilon} &= -\Delta(\tilde{u}_{\varepsilon} - W_{\varepsilon}u) \\ &= -\Delta\tilde{u}_{\varepsilon} + W_{\varepsilon}\Delta u + \Delta W_{\varepsilon}u + 2(\nabla W_{\varepsilon})\nabla u \\ &= -\Delta\tilde{u}_{\varepsilon} - (W_{\varepsilon} - Id)\Delta u + \Delta u - (\nabla Q_{\varepsilon} - \Delta W_{\varepsilon})u \\ &\quad + 2\nabla \cdot [(W_{\varepsilon} - Id)\nabla u] + \nabla Q_{\varepsilon}u. \end{aligned} \quad (1.2.39)$$

Introducing the Brinkman equation gives

$$\begin{aligned} -\Delta r_{\varepsilon} &= (\nabla p_{\varepsilon} - \Delta\tilde{u}_{\varepsilon} - f + \Gamma_{\varepsilon}u) + (M - M_{\varepsilon})u + 2\nabla \cdot [(W_{\varepsilon} - Id)\nabla u] \\ &\quad - (W_{\varepsilon} - Id)\Delta u - \nabla(p_{\varepsilon} - p - u \cdot Q_{\varepsilon}) - \nabla u Q_{\varepsilon}. \end{aligned} \quad (1.2.40)$$

Thanks to (H5') and the Stokes equation, the first term of the right-hand side of (1.2.40) is equal to zero in Ω_{ε} . Since $r_{\varepsilon} \equiv 0$ on the holes, integrating (1.2.40) by parts yields

$$\begin{aligned} \langle -\Delta r_{\varepsilon}, r_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} &= \langle (M - M_{\varepsilon})u, r_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} - \langle \nabla u Q_{\varepsilon}, r_{\varepsilon} \rangle_{H^{-1}, H_0^1(\Omega)} \\ &\quad - 2 \int_{\Omega} (W_{\varepsilon} - Id)\nabla u : \nabla r_{\varepsilon} - \int_{\Omega} (W_{\varepsilon} - Id)\Delta u \cdot r_{\varepsilon} \\ &\quad + \int_{\Omega} \alpha_{\varepsilon} \nabla \cdot r_{\varepsilon}. \end{aligned} \quad (1.2.41)$$

Because $u, \tilde{u}_{\varepsilon}, w_k^{\varepsilon}$ are divergence-free, we have $\nabla \cdot r_{\varepsilon} = -W_{\varepsilon} : \nabla u = (Id - W_{\varepsilon}) : \nabla u$. Then we can bound (1.2.41), and, using the Poincaré inequality, we obtain

$$\begin{aligned} \|\nabla r_{\varepsilon}\|_{L^2(\Omega)}^2 &\leq C \|u\|_{W^{2,\infty}(\Omega)} \|\nabla r_{\varepsilon}\|_{L^2(\Omega)} [\|M_{\varepsilon} - M\|_{H^{-1}(\Omega)} + \|Id - W_{\varepsilon}\|_{L^2(\Omega)} \\ &\quad + \|Q_{\varepsilon}\|_{H^{-1}(\Omega)}] + \|u\|_{W^{1,\infty}(\Omega)} \|Id - W_{\varepsilon}\|_{L^2(\Omega)} \|\alpha_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})/\mathbb{R}}. \end{aligned} \quad (1.2.42)$$

Finally, combining (1.2.38) and (1.2.42) gives the desired result (1.2.13) and (1.2.14). Q.E.D.

2. Periodically Distributed Holes in the Entire Domain

This second section is devoted to the verification of Hypotheses (H1)–(H6) in the case of identical holes of critical size, periodically distributed in Ω . This implies that all the results obtained in the abstract framework of the first section hold in the present geometrical situation. Moreover, the periodicity of the geometry yields some supplementary results, including explicit expressions for the matrix M and for the extension of the pressure, and concrete bounds for the errors.

2.1. Main results

Let Ω be a bounded connected open set in \mathbb{R}^N ($N \geq 2$), with Lipschitz boundary $\partial\Omega$, Ω being locally located on one side of its boundary. The set Ω is covered with a regular mesh of size 2ε , each cell being a cube P_i^ε , identical to $(-\varepsilon, +\varepsilon)^N$. At the center of each cube P_i^ε included in Ω there is a hole T_i^ε , each of which is similar to the same closed set T rescaled to the size a_ε . We assume that T is strictly included in the unit open ball B_1 and that $(B_1 - T)$ is a connected open set, locally located on one side of its Lipschitz boundary. Moreover, we assume that the size of the holes a_ε is critical, *i.e.*, that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{a_\varepsilon}{\varepsilon^{N/(N-2)}} &= C_0 \quad \text{for } N \geq 3, \\ \lim_{\varepsilon \rightarrow 0} -\varepsilon^2 \log(a_\varepsilon) &= C_0 \quad \text{for } N = 2 \end{aligned} \tag{2.1.1}$$

where C_0 is a strictly positive constant ($0 < C_0 < +\infty$).

Remark 2.1.1. Assumption (2.1.1) gives a unique and explicit scaling of the hole size for $N \geq 3$, but does not do so for the two-dimensional case. Actually, when $N = 2$, many different sizes of the holes satisfy (2.1.1) with the same constant C_0 . For example, $a_\varepsilon = \varepsilon^p \exp(-C_0/\varepsilon^2)$ is acceptable for any $p \in \mathbb{R}$. In any case, assumption (2.1.1) is enough for the sequel, so we do not make more precise the scaling of the holes in two dimensions. In the second part of this paper the non-critical sizes of the holes (corresponding to zero or infinite limits in (2.2.1)) are investigated, and lead to results which are completely different from those presented here.

An elementary geometrical consideration gives the number of holes

$$N(\varepsilon) = \frac{|\Omega|}{(2\varepsilon)^N} [1 + o(1)]. \tag{2.1.2}$$

The open set Ω_ε is obtained by removing from Ω all the holes $(T_i^\varepsilon)_{1 \leq i \leq N(\varepsilon)}$: $\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} T_i^\varepsilon$ (see Figure 1). Because we “perforated” only the cells entirely included in Ω , we are sure that no holes intersect the boundary $\partial\Omega$. Thus Ω_ε is also

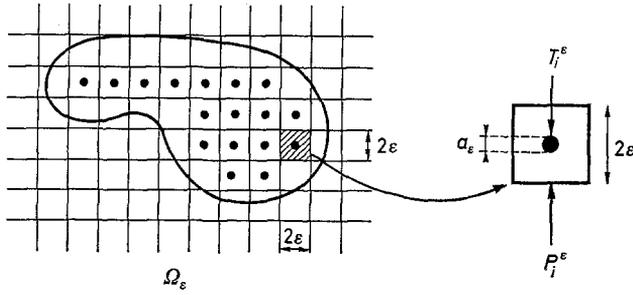


Fig. 1

a bounded connected open set, locally located on one side of its Lipschitz boundary $\partial\Omega_\epsilon$. In each cell P_i^ϵ we define B_i^ϵ as the open ball of radius ϵ included in P_i^ϵ . We also define a “control volume” C_i^ϵ around each hole by (see Figure 2)

$$C_i^\epsilon = B_i^\epsilon - T_i^\epsilon. \tag{2.1.3}$$

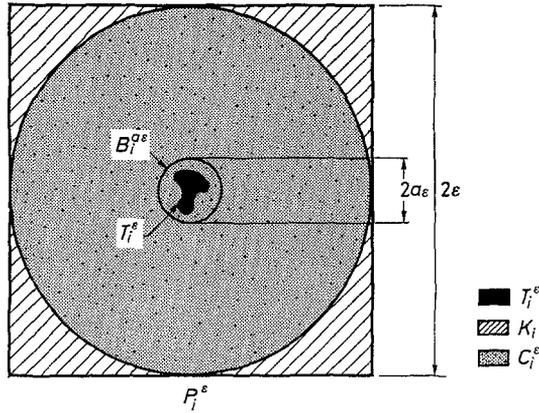


Fig. 2

Now we state the main results for such an open set Ω_ϵ , including the verification of Hypotheses (H1)–(H6). Their proofs are located in the remaining subsections 2.2, 2.3, and 2.4.

Proposition 2.1.2. *Let the hole size be critical, i.e., be given (2.1.1). Then there exists a map R_ϵ satisfying (H6). Furthermore, we construct R_ϵ such that the extension operator P_ϵ , from $L^2(\Omega_\epsilon)/\mathbb{R}$ to $L^2(\Omega)/\mathbb{R}$, defined in Proposition 1.1.4, satisfies*

$$\text{for each } q_\epsilon \in L^2(\Omega_\epsilon)/\mathbb{R} \quad \begin{cases} P_\epsilon(q_\epsilon) = q_\epsilon & \text{in } \Omega_\epsilon \\ P_\epsilon(q_\epsilon) = \frac{1}{|C_i^\epsilon|} \int_{C_i^\epsilon} q_\epsilon & \text{in each hole } T_i^\epsilon \end{cases} \tag{2.1.4}$$

where C_i^ϵ is the control volume around T_i^ϵ defined in (2.1.3).

Remark 2.1.3. Proposition 2.1.2 is actually the main new technical result in this paper (Subsection 2.2 is devoted to its proof). However, equality (2.1.4) is a generalization of a result due to R. LIPTON & M. AVELLANEDA [21]. It explains that the extension of the pressure, obtained by a duality argument from R_ε , turns out to be very simple. Nevertheless, it seems that the theoretical construction of P_ε in Proposition 1.1.4 cannot be avoided because formula (2.1.4) gives no estimate for $\nabla P_\varepsilon(p_\varepsilon)$ (as (iii) in Proposition 1.1.4), which is crucial for the proof of Theorem 1.1.8.

Before verifying Hypotheses (H1)–(H5), we introduce the so-called local problem when the space dimension is greater or equal to three. Let $N \geq 3$. For $k \in \{1, \dots, N\}$, consider the following Stokes problem:

Find (q_k, w_k) such that

$$\begin{aligned} \|q_k\|_{L^2(\mathbb{R}^{N-T})} < +\infty \quad \text{and} \quad \|\nabla w_k\|_{L^2(\mathbb{R}^{N-T})} < +\infty, \\ \nabla q_k - \Delta w_k = 0 \quad \text{in } \mathbb{R}^N - T, \\ \nabla \cdot w_k = 0 \quad \text{in } \mathbb{R}^N - T, \\ w_k = 0 \quad \text{on } \partial T, \\ w_k = e_k \quad \text{at infinity.} \end{aligned} \tag{2.1.5}$$

We prove in the Appendix that there exists a unique solution of (2.1.5). We denote by F_k the drag force applied on T by the above Stokes flow, i.e., $F_k = \int_{\partial T} \left(\frac{\partial w_k}{\partial n} - q_k n \right)$, where n is the normal exterior vector of ∂T .

In our framework, the system (2.1.5) is the local problem, around a single model obstacle, associated with the homogenization process. In the case of holes having the same size ε as the period, it is well-known (see, e.g., Section 7.2 in [25]) that the local problem holds in a unit cell, with periodic boundary conditions. But here, the hole size a_ε is asymptotically smaller than the period ε . Therefore, after a rescaling of the hole size to 1, the boundary of the cell goes to infinity, and the periodic boundary condition becomes a uniform boundary condition at infinity.

Proposition 2.1.4. *Let $N \geq 3$, and let the hole size be critical, i.e., be given by (2.1.1). Then there are functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ constructed from the solutions $(w_k, q_k)_{1 \leq k \leq N}$ of the local problem (2.1.5), that satisfy Hypotheses (H1)–(H5), and (H5').*

Moreover, the matrix M appearing in the Brinkman-type law (1.1.13) is given by

$$Me_k = \mu_k = \frac{C_0^{N-2}}{2^N} F_k \quad \text{for each } k \in \{1, 2, \dots, N\} \tag{2.1.6}$$

or, equivalently,

$${}^t e_i M e_k = \mu_k^i = \frac{C_0^{N-2}}{2^N} \int_{\mathbb{R}^{N-T}} \nabla w_k : \nabla w_i \quad \text{for each } i, k \in \{1, 2, \dots, N\}$$

or, equivalently,

$${}^t \xi M \xi = \frac{C_0^{N-2}}{2^N} \inf_{w \in E} \|\nabla w\|_{L^2(\mathbb{R}^{N-T})}^2 \tag{2.1.7}$$

with $E = \{w \in [H_{loc}^1(\mathbb{R}^N)]^N / \nabla \cdot w = 0 \text{ in } \mathbb{R}^N, w = 0 \text{ in } T, w = \xi \text{ at infinity}\}$.

Remark 2.1.5. Proposition 2.1.4 provides a very simple characterization of the matrix M which appears in Brinkman’s law. Formula (2.16) gives a physical interpretation of M : Each column of M is proportional to the drag force of a corresponding Stokes flow. This means that the new term Mu in the homogenized system (1.1.13) represents the slowing effect of the obstacles on the macroscopic flow. If the model obstacle T is not isotropic, then M may be non-scalar, and even non-diagonal; this is Brinkman’s law for an anisotropic medium. Formula (2.1.7) furnishes a mathematical interpretation of M as a “Stokes capacity” of the model obstacle T (see [5], and [15] for a similar “capacity-formula” in the case of the Laplacian operator). Roughly speaking, the functions (w_k^e, q_k^e) are constructed by rescaling (w_k, q_k) in each period. Thus they appear as the velocity and pressure of a unit boundary layer around the holes (in the e_k direction), and the matrix M may be seen as the energy of these boundary layers.

Proposition 2.1.6. *Let $N = 2$ and let T contain a small open ball. Let the hole size be critical, i.e., be given by (2.1.1). Then, there exist functions $(w_k^e, q_k^e, \mu_k)_1 \leq k \leq 2$ that satisfy Hypotheses (H1)–(H5), and (H5’).*

Moreover, whatever the shape and the size of the model hole T are, the matrix M appearing in Brinkman’s law (1.1.13) is given by

$$M = \frac{\pi}{C_0} Id. \tag{2.1.8}$$

Remark 2.1.7. In comparison with Proposition 2.1.4, the result of the above proposition is quite paradoxical. In fact, this result is close to the celebrated Stokes paradox, which asserts that the system (2.1.5) has no solution when the space dimension is $N = 2$. This result can also be connected to the fact that any two-dimensional bounded set has zero capacity.

Remark 2.1.8. From Propositions 2.1.2, 2.1.4, and 2.1.6, we know that Hypotheses (H1)–(H6) and (H5’) are satisfied by some functions $(w_k^e, q_k^e, \mu_k)_1 \leq k \leq N$ and some map R_e for any value $N \geq 2$. Of course, because of that, all the results of the first section hold, including the convergence Theorem 1.1.8, and the corrector Theorems 1.2.3 and 1.2.4.

Theorem 2.1.9. *Let the hole size be critical, i.e., be given by (2.1.1). Let the solution (u, p) satisfying Brinkman’s law (1.1.13) be smooth, say, $u \in [W^{2,\infty}(\Omega)]^N$.*

Then there exists a positive constant C that depends only on Ω and T such that

$$\begin{aligned} \|\tilde{u}_\varepsilon - W_\varepsilon u\|_{H_0^1(\Omega)} &\leq C\varepsilon \|u\|_{W^{2,\infty}(\Omega)}, \\ \|p_\varepsilon - p - u \cdot Q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} &\leq C\varepsilon \|u\|_{W^{2,\infty}(\Omega)}. \end{aligned} \tag{2.1.9}$$

Remark 2.1.10. We assume that the holes (T_i^ε) are identical; but this condition can be weakened, as will be clear from the construction of the functions $(w_k^\varepsilon, q_k^\varepsilon)$ that satisfy Hypotheses (H1)–(H5). In two dimensions, the holes may be entirely different from one another; provided that they have the required size, we still have the same results (in particular $M = \pi/C_0 Id$). In other dimensions, the hole shape may vary smoothly without interfering with the convergence of the homogenization process. (Of course the matrix M is no longer constant in Ω .)

2.2. Verification of Hypothesis (H6): Proof of Proposition 2.1.1

In this subsection we construct a linear operator R_ε that satisfies (H6), i.e., $R_\varepsilon \in L([H_0^1(\Omega)]^N; [H_0^1(\Omega_\varepsilon)]^N)$, such that

$$u \in [H_0^1(\Omega_\varepsilon)]^N \quad \text{implies that} \quad R_\varepsilon \tilde{u} = u \quad \text{in } \Omega_\varepsilon, \tag{2.2.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \quad \text{implies that} \quad \nabla \cdot (R_\varepsilon u) = 0 \quad \text{in } \Omega_\varepsilon, \tag{2.2.2}$$

$$\|R_\varepsilon u\|_{H_0^1(\Omega_\varepsilon)} \leq C \|u\|_{H_0^1(\Omega)} \quad \text{and} \quad C \text{ does not depend on } \varepsilon. \tag{2.2.3}$$

Following TARTAR’s idea [28], we easily define such an operator R_ε that satisfies conditions (2.2.1) and (2.2.2). The main difficulty consists in the verification of (2.2.3). For this purpose we introduce two technical Lemmas 2.2.3 and 2.2.4, which are the keys to the analysis of this section. For technical purposes, we decompose each cube P_i^ε entirely included in Ω by

$$\bar{P}_i^\varepsilon = T_i^\varepsilon \cup \bar{C}_i^\varepsilon \cup \bar{K}_i^\varepsilon \quad \text{with} \quad K_i^\varepsilon = P_i^\varepsilon - \bar{B}_i^\varepsilon \tag{2.2.4}$$

where T_i^ε is the hole, C_i^ε is the control volume, and K_i^ε is the remainder, i.e., the “corners” of P_i^ε (see Figure 2).

Lemma 2.2.1 *Let $u \in [H_0^1(\Omega)]^N$. For each cube P_i^ε entirely included in Ω , the following Stokes problem*

$$\begin{aligned} \text{Find } (v_i^\varepsilon, q_i^\varepsilon) &\in [H^1(C_i^\varepsilon)]^N \times [L^2(C_i^\varepsilon)/\mathbb{R}] \quad \text{such that} \\ \nabla q_i^\varepsilon - \Delta v_i^\varepsilon &= -\Delta u \quad \text{in } C_i^\varepsilon, \\ \nabla \cdot v_i^\varepsilon &= \nabla \cdot u + \frac{1}{|C_i^\varepsilon|} \int_{T_i^\varepsilon} \nabla \cdot u \quad \text{in } C_i^\varepsilon, \\ v_i^\varepsilon &= u \quad \text{on } \partial C_i^\varepsilon - \partial T_i^\varepsilon, \\ v_i^\varepsilon &= 0 \quad \text{on } \partial T_i^\varepsilon \end{aligned} \tag{2.2.5}$$

has a unique solution, depending linearly on u , such that

$$\|\nabla v_i^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leq C[\|\nabla u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2 + \|u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2] \tag{2.2.6}$$

where the constant C does not depend on ε , i , and u .

Accepting for the moment Lemma 2.2.1, we have

Proposition 2.2.2. For $u \in [H_0^1(\Omega)]^N$, let v_i^ε be the unique solution of system (2.2.5). Define an operator R_ε by:

For each cube P_i^ε entirely included in Ω ,

$$R_\varepsilon u = u \text{ in } K_i^\varepsilon, \quad R_\varepsilon u = v_i^\varepsilon \text{ in } C_i^\varepsilon, \quad R_\varepsilon u = 0 \text{ in } T_i^\varepsilon. \tag{2.2.7}$$

For each cube P_i^ε which meets $\partial\Omega$,

$$R_\varepsilon u = u \text{ in } P_i^\varepsilon \cap \Omega.$$

Then Hypothesis (H6) holds for the operator R_ε defined by (2.2.7).

Proof. It is not difficult to see that R_ε is linear and continuous from $[H_0^1(\Omega)]^N$ into $[H_0^1(\Omega_\varepsilon)]^N$, and satisfies properties (2.2.1) and (2.2.2). Moreover, summing the estimates (2.2.6) for all cubes P_i^ε , we easily obtain

$$\|\nabla R_\varepsilon u\|_{L^2(\Omega_\varepsilon)}^2 \leq C[\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2] \leq C\|u\|_{H_0^1(\Omega)}^2,$$

which is just property (2.2.3); thus R_ε satisfies (H6). Q.E.D.

Now, using the explicit Definition (2.2.7) of R_ε , we give the

Proof of Proposition 2.1.2. We have already proved in Proposition 2.2.2 that R_ε satisfies (H6). Now, following an idea of R. LIPTON & M. AVELLANEDA [21], we prove equality (2.1.4).

Let $q_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$. By Proposition 1.1.4, we already know that $P_\varepsilon(q_\varepsilon) = q_\varepsilon$ in Ω_ε . Recall property (iv) of P_ε in Proposition 1.1.4:

$$\langle \nabla [P_\varepsilon(q_\varepsilon)], w \rangle_{H^{-1}, H_0^1(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon w \rangle_{H^{-1}, H_0^1(\Omega_\varepsilon)} \quad \text{for each } w \in [H_0^1(\Omega)]^N. \tag{2.2.8}$$

In order to prove that $P_\varepsilon(q_\varepsilon)$ is a constant in each hole T_i^ε , we take $w = w_i \in [D(T_i^\varepsilon)]^N$ in formula (2.2.8), so that w_i is a smooth function with compact support in the hole T_i^ε . Using system (2.2.5), we easily check that $R_\varepsilon(w_i) \equiv 0$. Thus, from (2.2.8) we obtain

$$\langle \nabla [P_\varepsilon(q_\varepsilon)], w_i \rangle_{H^{-1}, H_0^1(\Omega)} = 0 \quad \text{for each } w_i \in [D(T_i^\varepsilon)]^N.$$

That is,

$$\nabla [P_\varepsilon(q_\varepsilon)] = 0 \text{ in } T_i^\varepsilon, \quad \text{or equivalently, } P_\varepsilon(q_\varepsilon) \text{ is constant in } T_i^\varepsilon. \tag{2.2.9}$$

In order to calculate this constant, we now take $w = v_i \in [D(B_i^\varepsilon)]^N$ in formula (2.2.8), so that v_i is a smooth function with compact support in the ball B_i^ε . Integrating (2.2.8) by parts, we obtain

$$\int_{B_i^\varepsilon} P_\varepsilon(q_\varepsilon) \nabla \cdot v_i = \int_{C_i^\varepsilon} q_\varepsilon \nabla \cdot (R_\varepsilon v_i) \quad \text{for each } v_i \in [D(B_i^\varepsilon)]^N. \quad (2.2.10)$$

Using system (2.2.5) and the fact that $P_\varepsilon(q_\varepsilon)$ is constant in T_i^ε , we compute

$$P_\varepsilon(q_\varepsilon) = \frac{1}{|C_i^\varepsilon|} \int_{C_i^\varepsilon} q_\varepsilon \quad \text{in each hole } T_i^\varepsilon. \quad \text{Q.E.D.}$$

Now we give some technical lemmas which will be crucial for proving Lemma 2.2.1. Let $\eta \in \mathbb{R}$ be such that $0 < \eta < \frac{1}{2}$. We define an open set $C_\eta = B_1 - (\eta T)$ where B_1 is the unit open ball, and (ηT) is similar to the model hole T rescaled at size η .

Lemma 2.2.3. *There exists a linear continuous operator L such that*

- (i) $L \in L[H^1(B_1); H^1(C_\eta)]$,
- (ii) $L(u) = u$ on ∂B_1 , $L(u) = 0$ on $\partial(\eta T)$ for each $u \in H^1(B_1)$,
- (iii) $\|\nabla L(u)\|_{L^2(C_\eta)} \leq C[\|\nabla u\|_{L^2(B_1)} + K_\eta \|u\|_{L^2(B_1)}]$ for each $u \in H^1(B_1)$

where $K_\eta = \eta^{\frac{N-2}{2}}$ for $N \geq 3$, $K_\eta = \frac{1}{\sqrt{|\log \eta|}}$ for $N = 2$, and C does not depend on u or η .

Proof. Define $\theta \in C^0[0; 1]$ by $\theta(r) = \begin{cases} 0 & \text{for } r \in [0; \frac{1}{2}], \\ 2r - 1 & \text{for } r \in [\frac{1}{2}; 1]. \end{cases} \quad (2.2.11)$

Define $\phi \in C^0[0; 1]$ by $\phi(r) = \begin{cases} 0 & \text{for } r \in [0, \eta], \\ 1 - \frac{\log r}{\log \eta} & \text{for } r \in [\eta; 1] \text{ and } N = 2, \\ \frac{1}{r^{N-2}} - \frac{1}{\eta^{N-2}} & \text{for } r \in [\eta; 1] \text{ and } N \geq 3. \\ 1 - \frac{1}{\eta^{N-2}} & \end{cases} \quad (2.2.12)$

Let $u \in H^1(B_1)$. Then we define the operator L by

$$L(u) = \theta(r) \left[u - \frac{1}{|B_1|} \int_{B_1} u \right] + \phi(r) \frac{1}{|B_1|} \int_{B_1} u. \quad (2.2.13)$$

We easily check properties (i) and (ii). From (2.2.13) it follows that

$$\begin{aligned} \|\nabla L(u)\|_{L^2(C_\eta)} &\leq \left\| \nabla \left(u - \frac{1}{|B_1|} \int_{B_1} u \right) \right\|_{L^2(B_1)} + 2 \left\| u - \frac{1}{|B_1|} \int_{B_1} u \right\|_{L^2(B_1)} \\ &\quad + \frac{1}{|B_1|} \int_{B_1} |u| \|\nabla \phi\|_{L^2(B_1)}. \end{aligned} \quad (2.2.14)$$

The Poincaré-Wirtinger inequality implies that

$$\left\| u - \frac{1}{|B_1|} \int_{B_1} u \right\|_{L^2(B_1)} \leq C \|\nabla u\|_{L^2(B_1)}.$$

Moreover, an elementary calculation gives

$$\|\nabla \phi\|_{L^2(B_1)} \leq CK_\eta \quad \text{where } C \text{ does not depend on } \eta.$$

Thus (2.2.14) leads to the desired property (iii). Q.E.D.

Lemma 2.2.4. *For each $f \in L^2(C_\eta)$ with $\int_{C_\eta} f = 0$ there exists $v \in [H_0^1(C_\eta)]^N$ such that*

- (i) $\nabla \cdot v = f$ in C_η ,
- (ii) *the map $f \rightarrow v$ is linear, and there exists a constant C that does not depend on η or f such that $\|v\|_{H_0^1(C_\eta)} \leq C \|f\|_{L^2(C_\eta)}$.*

Proof. Let $f \in L^2(C_\eta)$ with $\int_{C_\eta} f = 0$. We define $\tilde{f} \in L^2(B_1)$ by

$$\tilde{f} = f \quad \text{in } C_\eta, \quad \tilde{f} = 0 \quad \text{in } (\eta T). \tag{2.2.15}$$

We still have $\int_{B_1} \tilde{f} = 0$. Lemma 1.1.7 asserts that there exists a $u \in [H_0^1(B_1)]^N$ such that

$$\begin{aligned} \nabla \cdot u &= \tilde{f} \quad \text{in } B_1, \\ \|u\|_{H_0^1(B_1)} &\leq C \|\tilde{f}\|_{L^2(B_1)} \end{aligned} \tag{2.2.16}$$

where C depends only on B_1 (and not on η or \tilde{f}). We distinguish two cases according to the spatial dimension:

$N \geq 3$. We set $C_\eta = (B_1 - B_\eta) \cup \eta(B_1 - T)$, where $\eta(B_1 - T)$ denotes the set $(B_1 - T)$ rescaled to the size η . Consider the following problem in $\eta(B_1 - T)$:

Find $w \in [H^1[\eta(B_1 - T)]]^N$ such that

$$\begin{aligned} \nabla \cdot w &= f \quad \text{in } \eta(B_1 - T), \\ w &= u \quad \text{on } \partial(\eta B_1), \\ w &= 0 \quad \text{on } \partial(\eta T). \end{aligned} \tag{2.2.17}$$

Because the compatibility condition of system (2.2.17), namely $\int_{\eta(B_1 - T)} f = \int_{\partial(\eta B_1)} u \cdot n$, is satisfied, there exists a solution w . Moreover, if we assume that this solution satisfies the estimate

$$\|\nabla w\|_{L^2(\eta(B_1 - T))} \leq C \|f\|_{L^2(C_\eta)} \quad \text{where } C \text{ does not depend on } \eta, \tag{2.2.18}$$

then Lemma 2.2.4 is proved by taking v equal to

$$\begin{aligned} v &= u \quad \text{in } (B_1 - B_\eta), \\ v &= w \quad \text{in } \eta(B_1 - T). \end{aligned} \tag{2.2.19}$$

It remains to prove that estimate (2.2.18) holds for some solution w of (2.2.17). For this purpose we rescale system (2.2.17). For $y \in (B_1 - T)$, setting

$$f_0(y) = f(\eta y), \quad u_0(y) = \frac{1}{\eta} u(\eta y), \quad w_0(y) = \frac{1}{\eta} w(\eta y). \quad (2.2.20)$$

we obtain the problem

$$\begin{aligned} \text{Find } w_0 \in [H^1(B_1 - T)]^N \text{ such that} \\ \nabla \cdot w_0 = f_0 \quad \text{in } (B_1 - T), \\ w_0 = u_0 \quad \text{on } \partial B_1, \\ w_0 = 0 \quad \text{on } \partial T. \end{aligned} \quad (2.2.21)$$

Since $\int_{B_1 - T} f_0 = \int_{\partial B_1} u_0 \cdot n$, Lemma 1.1.7 implies that there exists a solution w_0 of (2.2.21), which depends linearly on u_0 and f_0 , such that

$$\|\nabla w_0\|_{L^2(B_1 - T)} \leq C[\|f_0\|_{L^2(B_1 - T)} + \|u_0\|_{L^2(B_1 - T)} + \|\nabla u_0\|_{L^2(B_1 - T)}]. \quad (2.2.22)$$

In view of (2.2.20), estimate (2.2.22) can be rewritten in the form

$$\|\nabla w\|_{L^2(\eta(B_1 - T))} \leq C \left[\|f\|_{L^2(\eta(B_1 - T))} + \|\nabla u\|_{L^2(\eta B_1)} + \frac{1}{\eta} \|u\|_{L^2(\eta B_1)} \right]. \quad (2.2.23)$$

Using the Hölder inequality in ηB_1 gives

$$\|u\|_{L^2(\eta B_1)}^2 \leq \left[\int_{\eta B_1} u^{2p} \right]^{1/p} \left[\int_{\eta B_1} 1 \right]^{1/p'} \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (2.2.24)$$

With $p = \frac{N}{N-2}$ and $p' = \frac{N}{2}$, (2.2.24) becomes

$$\|u\|_{L^2(\eta B_1)}^2 \leq C\eta^2 \|u\|_{L^{\frac{2N}{N-2}}(\eta B_1)}^2 \leq C\eta^2 \|u\|_{L^{\frac{2N}{N-2}}(B_1)}^2. \quad (2.2.25)$$

As $N \geq 3$, the following Sobolev embedding holds

$$H^1(B_1) \subset L^{\frac{2N}{N-2}}(B_1). \quad (2.2.26)$$

Then (2.2.25) becomes

$$\|u\|_{L^2(\eta B_1)}^2 \leq C\eta^2 \|u\|_{H_0^1(B_1)}^2. \quad (2.2.27)$$

Consequently, from (2.2.23), (2.2.27), and (2.2.16) we obtain the required estimate (2.2.18).

$N = 2$. This case is more complicated, and we “cut” the open set C_η into many “slices”, the number of which is asymptotically equivalent to $|\log \eta|$. Let $n \in \mathbb{N}$ be such that $\frac{1}{2^n} > \eta \geq \frac{1}{2^{n+1}}$. For any integer i , let $B_{1/2^i}$ be the disk of radius $\frac{1}{2^i}$, centered at the origin of \mathbb{R}^2 . Let C_i be the slice of C_η defined by

$$C_i = B_{1/2^{i-1}} - B_{1/2^i} \quad \text{for } 1 \leq i \leq n. \quad (2.2.28)$$

Let C_{n+1} be the smallest slice defined by

$$C_{n+1} = B_{1/2^n} - (\eta T) \quad \text{for } 1 \leq i \leq n. \quad (2.2.29)$$

We have

$$C_\eta = \bigcup_{i=1}^{n+1} C_i. \quad (2.2.30)$$

In each slice $(C_i)_{2 \leq i \leq n}$ we consider the problem

$$(S_i) \quad \begin{cases} \text{Find } v_i \in [H^1(C_i)]^2 \text{ such that} \\ \nabla \cdot v_i = f \quad \text{in } C_i, \\ v_i = u - \frac{1}{|C_{i-1} \cup C_i|} \int_{C_{i-1} \cup C_i} u \quad \text{on } \partial B_{1/2^{i-1}}, \\ v_i = u - \frac{1}{|C_i \cup C_{i+1}|} \int_{C_i \cup C_{i+1}} u \quad \text{on } \partial B_{1/2^i}; \end{cases}$$

in C_1 we consider

$$(S_1) \quad \begin{cases} \text{Find } v_1 \in [H^1(C_1)]^2 \text{ such that} \\ \nabla \cdot v_1 = f \quad \text{in } C_1, \\ v_1 = 0 \quad \text{on } \partial B_1, \\ v_1 = u - \frac{1}{|C_1 \cup C_2|} \int_{C_1 \cup C_2} u \quad \text{on } \partial B_{1/2}, \end{cases}$$

and in C_{n+1} we consider

$$(S_{n+1}) \quad \begin{cases} \text{Find } v_{n+1} \in [H^1(C_{n+1})]^2 \text{ such that} \\ \nabla \cdot v_{n+1} = f \quad \text{in } C_{n+1}, \\ v_{n+1} = u - \frac{1}{|C_n \cup C_{n+1}|} \int_{C_n \cup C_{n+1}} u \quad \text{on } \partial B_{1/2^n}, \\ v_{n+1} = 0 \quad \text{on } \partial(\eta T). \end{cases}$$

It is easy to check the compatibility conditions of these systems $(S_i)_{1 \leq i \leq n+1}$, because

$$\int_{\partial B_{1/2^i}} \left(u - \frac{1}{|C_i \cup C_{i+1}|} \int_{C_i \cup C_{i+1}} u \right) \cdot n = \int_{\partial B_{1/2^i}} u \cdot n.$$

Moreover, for $2 \leq i \leq n-1$ we rescale to the unit-size set $(C_{-1} \cup C_0 \cup C_1)$ by

$$C_{i-1} \cup C_i \cup C_{i+1} = \frac{1}{2^i} (C_{-1} \cup C_0 \cup C_1). \quad (2.2.31)$$

Consequently, each system $(S_i)_{2 \leq i \leq n-1}$ is similar to the following rescaled system

$$(S_0) \quad \begin{cases} \text{Find } v_0 \in [H^1(C_0)]^2 \text{ such that} \\ \nabla \cdot v_0 = f_0 \quad \text{in } C_0, \\ v_0 = w_0 - \frac{1}{|C_{-1} \cup C_0|} \int_{C_{-1} \cup C_0} w_0 \quad \text{on } \partial B_2, \\ v_0 = w_0 - \frac{1}{|C_0 \cup C_1|} \int_{C_0 \cup C_1} w_0 \quad \text{on } \partial B_1. \end{cases}$$

According to Lemma 1.1.7 there exists a solution of (S_0) such that

$$\begin{aligned} \|\nabla v_0\|_{L^2(C_0)} &\leq C \left[\|f_0\|_{L^2(C_0)} + \left\| w_0 - \frac{1}{|C_{-1} \cup C_0|} \int_{C_{-1} \cup C_0} w_0 \right\|_{H^1(C_0)} \right. \\ &\quad \left. + \left\| w_0 - \frac{1}{|C_0 \cup C_1|} \int_{C_0 \cup C_1} w_0 \right\|_{H^1(C_0)} \right]. \end{aligned} \quad (2.2.32)$$

But the Poincaré-Wirtinger inequality gives

$$\left\| w_0 - \frac{1}{|C_{-1} \cup C_0|} \int_{C_{-1} \cup C_0} w_0 \right\|_{H^1(C_0)} \leq C \|\nabla w_0\|_{L^2(C_{-1} \cup C_0)}. \quad (2.2.33)$$

The same inequality holds for the last term of estimate (2.2.32), which becomes

$$\|\nabla v_0\|_{L^2(C_0)} \leq C [\|f_0\|_{L^2(C_0)} + \|\nabla w_0\|_{L^2(C_{-1} \cup C_0 \cup C_1)}]. \quad (2.2.34)$$

We now apply inequality (2.2.36) to system (S_i) with $f_0(y) = f\left(\frac{y}{2^i}\right)$, $w_0(y) =$

$2^i u\left(\frac{y}{2^i}\right)$, and $v_0(y) = 2^i v_i\left(\frac{y}{2^i}\right)$ to obtain

$$\|\nabla v_i\|_{L^2(C_i)} \leq C [\|f\|_{L^2(C_i)} + \|\nabla u\|_{L^2(C_{i-1} \cup C_i \cup C_{i+1})}] \quad (2.2.35)$$

where the constant C does not depend on f , u or i . It is not difficult to get equivalent estimates for (S_1) , (S_n) , and (S_{n+1}) ; then, summing those estimates we obtain from (2.2.35) that

$$\|\nabla v\|_{L^2(C_\eta)} \leq C [\|f\|_{L^2(C_\eta)} + 3 \|\nabla u\|_{L^2(B_1)}] \leq C \|f\|_{L^2(C_\eta)},$$

where v is equal to v_i in each slice C_i . Thus Lemma 2.2.4 is proved.

Lemma 2.2.5. *Let $u \in [H^1(B_1)]^N$. Consider the non-homogeneous Stokes problem:*

Find $(v, q) \in [H^1(C_\eta)]^N \times [L^2(C_\eta)/\mathbb{R}]$ such that

$$\begin{aligned} \nabla q - \Delta v &= -\Delta u \quad \text{in } C_\eta, \\ \nabla \cdot v &= \nabla \cdot u + \frac{1}{|C_\eta|} \int_{\eta T} \nabla \cdot u \quad \text{in } C_\eta, \\ v &= u \quad \text{on } \partial B_1, \\ v &= 0 \quad \text{on } \partial(\eta T). \end{aligned} \quad (2.2.36)$$

There exists a unique solution of (2.2.36), which depends linearly on u , such that

$$\|\nabla v\|_{L^2(C_\eta)} \leq C[\|\nabla u\|_{L^2(B_1)} + K_\eta \|u\|_{L^2(B_1)}] \quad (2.2.37)$$

where $K_\eta = \eta^{\frac{N-2}{2}}$ for $N \geq 3$, and $K_\eta = \frac{1}{\sqrt{|\log \eta|}}$ for $N = 2$; C depends neither on u nor on η .

Proof. Since

$$\begin{aligned} \int_{C_\eta} \nabla \cdot v &= \int_{C_\eta} \left[\nabla \cdot u + \frac{1}{|C_\eta|} \int_{\eta T} \nabla \cdot u \right] = \int_{C_\eta} \nabla \cdot u + \int_{\eta T} \nabla \cdot u = \int_{B_1} \nabla \cdot u \\ &= \int_{\partial B_1} u \cdot n = \int_{\partial C_\eta} v \cdot n, \end{aligned}$$

the compatibility condition holds for system (2.2.36). Now using the two previous lemmas, we transform system (2.2.36) in order to have a homogeneous Dirichlet boundary condition and a divergence-free solution. Then it is easy to obtain estimate (2.2.37). Q.E.D.

Proof of Lemma 2.2.1. If we take $\eta = \frac{a_\varepsilon}{\varepsilon}$, then each control volume C_i^ε is similar to C_η rescaled at size ε . Consequently, we can apply Lemma 2.2.5 with the rescaled variables $v_i^\varepsilon(x) = \varepsilon v\left(\frac{x}{\varepsilon}\right)$. From estimate (2.2.37) we obtain

$$\|\nabla v_i^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leq C \left[\|\nabla u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2 + \frac{K_\eta^2}{\varepsilon^2} \|u\|_{L^2(C_i^\varepsilon \cup T_i^\varepsilon)}^2 \right]. \quad (2.2.38)$$

Because $\eta = \frac{a_\varepsilon}{\varepsilon}$, the quantity $\frac{K_\eta^2}{\varepsilon^2}$ is of the same order 1 in ε , and thus

$$\|\nabla v_i^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leq C \|u\|_{H^1(C_i^\varepsilon \cup T_i^\varepsilon)}^2. \quad \text{Q.E.D.}$$

2.3. Verification of Hypotheses (H1)–(H5)

This subsection is devoted to the explicit construction of functions $(w_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1 \leq k \leq N}$ that satisfy Hypotheses (H1)–(H5). These functions will be carefully defined, but many technical computations which are needed in order to verify the hypotheses will be omitted. This is done only for the sake of simplicity; no fundamental difficulties are avoided. The interested (or suspicious) reader is referred to [1] for the complete calculations. We first consider the case $N = 2$, then the case $N \geq 3$.

2.3.1. Two-dimensional case: $N = 2$ (Proof of Proposition 2.1.6)

Recall the decomposition (2.2.4) of each cube, namely $\bar{P}_i^\varepsilon = T_i^\varepsilon \cup \bar{C}_i^\varepsilon \cup \bar{K}_i^\varepsilon$. For $k = 1, 2$ we define $(w_k^\varepsilon, q_k^\varepsilon) \in [H^1(P_i^\varepsilon)]^2 \times L^2(P_i^\varepsilon)$, with $\int_{P_i^\varepsilon} q_k^\varepsilon = 0$, by:

For each cube P_i^ε that meets $\partial\Omega$

$$\begin{cases} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{cases} \quad \text{in } P_i^\varepsilon \cap \Omega.$$

For each cube P_i^ε entirely included in Ω ,

$$\begin{cases} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{cases} \text{ in } K_i^\varepsilon, \quad \begin{cases} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = 0 \\ \nabla \cdot w_k^\varepsilon = 0 \end{cases} \text{ in } C_i^\varepsilon, \quad \begin{cases} w_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{cases} \text{ in } T_i^\varepsilon. \quad (2.3.1)$$

Obviously definition (2.3.1) is meaningful, and the functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq 2}$ exist and are unique. For an arbitrary model hole T , we cannot explicitly compute these functions. However, it is possible to make them explicit when the model hole is the unit ball B_1 , and this gives an appropriate class of comparison functions. As $T \subset B_1$ let us define, for each cube P_i^ε , a ball $B_i^{a_\varepsilon}$ of radius a_ε that strictly contains the hole T_i^ε (see Figure 2).

Replacing the holes T_i^ε by the balls $B_i^{a_\varepsilon}$ in Definition (2.3.1), we obtain functions $(w_{0k}^\varepsilon, q_{0k}^\varepsilon) \in [H^1(P_i^\varepsilon)]^2 \times L^2(P_i^\varepsilon)$, with $\int_{P_i^\varepsilon} q_{0k}^\varepsilon = 0$, defined by:

For each cube P_i^ε that meets $\partial\Omega$,

$$\begin{cases} w_{0k}^\varepsilon = e_k \\ q_{0k}^\varepsilon = 0 \end{cases} \quad \text{in } P_i^\varepsilon \cap \Omega.$$

For each cube P_i^ε entirely included in Ω ,

$$\begin{cases} w_{0k}^\varepsilon = e_k \\ q_{0k}^\varepsilon = 0 \end{cases} \text{ in } K_i^\varepsilon, \quad \begin{cases} \nabla q_{0k}^\varepsilon - \Delta w_{0k}^\varepsilon = 0 \\ \nabla \cdot w_{0k}^\varepsilon = 0 \end{cases} \text{ in } C_i^\varepsilon - B_i^{a_\varepsilon}, \quad \begin{cases} w_{0k}^\varepsilon = 0 \\ q_{0k}^\varepsilon = 0 \end{cases} \text{ in } B_i^{a_\varepsilon}. \quad (2.3.2)$$

Now, we explicitly compute $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)_{1 \leq k \leq 2}$ in each $C_i^\varepsilon - B_i^{a_\varepsilon}$. Denoting by r_i and e_r^i the radial coordinate and unit vector in each $C_i^\varepsilon - B_i^{a_\varepsilon}$ we actually have

$$w_{0k}^\varepsilon = x_k r_i f(r_i) e_r^i + g(r_i) e_k, \quad q_{0k}^\varepsilon = x_k h(r_i) \quad \text{for } r_i \in [a_\varepsilon, \varepsilon], \quad (2.3.3)$$

with

$$\begin{aligned} f(r_i) &= \frac{1}{r_i^2} \left(A + \frac{B}{r_i^2} \right) + C, \\ g(r_i) &= -A \log r_i - \frac{B}{2r_i^2} - \frac{3}{2} C r_i^2 + D, \quad h(r_i) = \frac{2A}{r_i^2} - 4C, \end{aligned} \quad (2.3.4)$$

with

$$\begin{aligned}
 A &= -\frac{\varepsilon^2}{C_0} [1 + o(1)], & B &= \frac{\varepsilon^2}{C_0} e^{-\frac{2C_0}{\varepsilon^2}} [1 + o(1)], \\
 C &= \frac{1}{C_0} [1 + o(1)], & D &= 1 - \frac{\varepsilon^2 \log \varepsilon}{C_0} [1 + o(1)].
 \end{aligned}
 \tag{2.3.5}$$

Then, for $k = 1, 2$ we define the “difference” functions (w'_k, q'_k) , by

$$w'_k = w_k^\varepsilon - w_{0k}^\varepsilon, \quad q'_k = q_k^\varepsilon - q_{0k}^\varepsilon,
 \tag{2.3.6}$$

which belong to $[H_0^1(C_i^\varepsilon)]^2 \times L^2(C_i^\varepsilon)$ with $\int_{C_i^\varepsilon} q'_k = 0$, and satisfy

$$\left\{ \begin{aligned}
 \nabla q'_k - \Delta w'_k &= \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^{a_\varepsilon} \\
 \nabla \cdot w'_k &= 0
 \end{aligned} \right\} \text{ in each control volume } C_i^\varepsilon, \tag{2.3.7}$$

$$\left\{ \begin{aligned}
 w'_k &= 0 \\
 q'_k &= 0
 \end{aligned} \right\} \text{ elsewhere in } \Omega - \bigcup_{i=1}^{N(\varepsilon)} C_i^\varepsilon$$

where $\delta_i^{a_\varepsilon}$ is the measure defined as the unit mass concentrated on the sphere $\partial B_i^{a_\varepsilon}$, i.e.,

$$\langle \delta_i^{a_\varepsilon}, \phi \rangle_{D', D(\mathbb{R}^N)} = \int_{\partial B_i^{a_\varepsilon}} \phi(s) \, ds \quad \text{for any } \phi \in D(\mathbb{R}^N).$$

In the sequel we prove that the difference functions $(w'_k, q'_k)_{1 \leq k \leq 2}$ actually converge strongly to $(0, 0)$ in $[H^1(\Omega)]^2 \times L^2(\Omega)$. In the verification of Hypotheses (H1)–(H5), this means that there are almost no differences between the case of spherical holes (corresponding to the functions $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)_{1 \leq k \leq 2}$) and the general case of arbitrary holes (corresponding to the functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq 2}$).

Lemma 2.3.1. *Let the model hole T contain a small ball. Then the functions $(w'_k, q'_k)_{1 \leq k \leq 2}$ defined in (2.3.6), which belong to $[H_0^1(\Omega)]^2 \times L^2(\Omega)$, satisfy*

$$\begin{aligned}
 \|q'_k\|_{L^2(\Omega)} \leq C\varepsilon, \quad \|\nabla w'_k\|_{L^2(\Omega)} \leq C\varepsilon, \quad \text{and} \quad \|w'_k\|_{L^q(\Omega)} \leq C\varepsilon^2 \\
 \text{for } 1 \leq q < +\infty,
 \end{aligned}
 \tag{2.3.8}$$

where C does not depend on ε (but does depend on q).

Proof. A brief computation gives

$$\left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^{a_\varepsilon} = \frac{2\varepsilon^2}{C_0 a_\varepsilon} [1 + o(1)] e_k \delta_i^{a_\varepsilon}.
 \tag{2.3.9}$$

From (2.3.7) we deduce that in each C_i^ε

$$\int_{C_i^\varepsilon} |\nabla w'_k|^2 = \int_{\partial B_i^{a_\varepsilon}} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \cdot w'_k = \frac{2\varepsilon^2}{C_0 a_\varepsilon} [1 + o(1)] \int_{\partial B_i^{a_\varepsilon}} e_k \cdot w'_k.
 \tag{2.3.10}$$

Because T contains a small ball, we may use the Poincaré inequality in $B_1 - T$ for obtaining the following trace estimate:

$$\|w\|_{L^2(\partial B_1)} \leq C \|\nabla w\|_{L^2(B_1 - T)} \quad \text{for any } w \in H^1(B_1 - T) \text{ such that } w = 0 \text{ on } \partial T.$$

We rescale this estimate at size a_ε and use it for $w'_k{}^\varepsilon$. It implies that

$$\left| \int_{\partial B_i^{a_\varepsilon}} e_k \cdot w'_k{}^\varepsilon \right| \leq C a_\varepsilon \|\nabla w'_k{}^\varepsilon\|_{L^2(B_i^{a_\varepsilon} - T_i^\varepsilon)} \tag{2.3.11}$$

where the constant C does not depend on ε . As $(B_i^{a_\varepsilon} - T_i^\varepsilon) \subset C_i^\varepsilon$, it follows from (2.3.10) that

$$\|\nabla w'_k{}^\varepsilon\|_{L^2(C_i^\varepsilon)} \leq C\varepsilon^2. \tag{2.3.12}$$

Lemma 2.2.4 leads to an inequality equivalent to (2.3.12) for $q'_k{}^\varepsilon$. Recalling that the number of holes is given by (2.1.2), we obtain the desired result:

$$\begin{aligned} \|q'_k{}^\varepsilon\|_{L^2(\Omega)}^2 &= \frac{|\Omega|}{(2\varepsilon)^2} [1 + o(1)] \|q'_k{}^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leq C\varepsilon^2, \\ \|\nabla w'_k{}^\varepsilon\|_{L^2(\Omega)}^2 &= \frac{|\Omega|}{(2\varepsilon)^2} [1 + o(1)] \|\nabla w'_k{}^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 \leq C\varepsilon^2. \end{aligned}$$

Furthermore, the continuous embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ leads to the following estimate:

$$\|w'_k{}^\varepsilon\|_{L^q(\Omega)} \leq C\varepsilon \quad \text{for } 1 \leq q < +\infty.$$

We can improve this estimate by applying, in each cell, the Sobolev inequality to $w'_k{}^\varepsilon$, which belongs to $[H_0^1(P_i^\varepsilon)]^2$, for $q \geq 2$, or by applying the Hölder inequality for $q < 2$. Then, from (2.3.12) we obtain $\|w'_k{}^\varepsilon\|_{L^q(P_i^\varepsilon)}^q \leq C\varepsilon^{2q+2}$, which implies that $\|w'_k{}^\varepsilon\|_{L^q(\Omega)} \leq C\varepsilon^2$ for any $q \geq 1$. Q.E.D.

Lemma 2.3.2. *The functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq 2}$ defined in (2.3.1) satisfy Hypotheses (H1), (H2), and (H3), i.e.,*

- (H1) $w_k^\varepsilon \in [H^1(\Omega)]^2, \quad q_k^\varepsilon \in L^2(\Omega),$
- (H2) $\nabla \cdot w_k^\varepsilon = 0$ in Ω and $w_k^\varepsilon = 0$ on the holes $T_i^\varepsilon,$
- (H3) $w_k^\varepsilon \rightharpoonup e_k$ in $[H^1(\Omega)]^N$ weakly, $q_k^\varepsilon \rightharpoonup 0$ in $L^2(\Omega)/\mathbb{R}$ weakly.

Moreover,

$$\|w_k^\varepsilon - e_k\|_{L^p(\Omega)} \leq C\varepsilon^2 |\log \varepsilon| \tag{2.3.13}$$

for any $1 \leq p < +\infty$ where the constant C does not depend on ε (but does depend on p).

Proof. By their definition, the functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq 2}$ satisfy (H1) and (H2). In order to see if (H3) also holds, we remark that Lemma 2.3.1 implies that

$$w'_k{}^\varepsilon \rightarrow 0 \quad \text{in } [H_0^1(\Omega)]^2 \text{ strongly,} \quad q'_k{}^\varepsilon \rightarrow 0 \quad \text{in } L^2(\Omega) \text{ strongly.}$$

It remains to show that $(w_{0k}^\varepsilon, q_{0k}^\varepsilon)_{1 \leq k \leq 2}$ satisfy (H3). An easy but tedious computation yields

$$\|q_{0k}^\varepsilon\|_{L^2(\Omega)} \leq C, \quad \|\nabla w_{0k}^\varepsilon\|_{L^2(\Omega)} \leq C, \quad \|w_{0k}^\varepsilon - e_k\|_{L^p(\Omega)} \leq C\varepsilon^2 |\log \varepsilon|.$$

Thus w_{0k}^ε converges weakly to e_k in $[H^1(\Omega)]^2$. Because q_{0k}^ε is P_i^ε -periodic and bounded in $L^2(\Omega)$, the whole sequence converges weakly to a constant in $L^2(\Omega)$, i.e., to 0 in $L^2(\Omega)/\mathbb{R}$. Finally, the inequality $\|w_k^\varepsilon - e_k\|_{L^p(\Omega)} \leq \|w_{0k}^\varepsilon - e_k\|_{L^p(\Omega)} + \|w_k^{\prime\varepsilon}\|_{L^p(\Omega)}$ leads to (2.3.13). Q.E.D.

Before verifying that (H4), (H5), and (H5') hold, we remark that

$$\begin{aligned} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^\varepsilon + \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial r_i} - q_k^{\prime\varepsilon} e_r^i \right) \delta_i^\varepsilon \\ &\quad - \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i \right) \delta_{T_i^\varepsilon}, \end{aligned} \tag{2.3.14}$$

$$\begin{aligned} \nabla q_k^{\prime\varepsilon} - \Delta w_k^{\prime\varepsilon} &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial r_i} - q_k^{\prime\varepsilon} e_r^i \right) \delta_i^\varepsilon + \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^{\prime\varepsilon} \\ &\quad - \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial n_i} - q_k^{\prime\varepsilon} n_i \right) \delta_{T_i^\varepsilon} \end{aligned} \tag{2.3.15}$$

where δ_i^ε and $\delta_i^{\prime\varepsilon}$ are the unit masses concentrated on the spheres ∂B_i^ε and $\partial B_i^{\prime\varepsilon}$, $\delta_{T_i^\varepsilon}$ is the unit mass concentrated on the hole boundary ∂T_i^ε , and n_i is the unit exterior normal to T_i^ε . Then the functions μ_k^ε and γ_k^ε , introduced in (H5'), are defined by

$$\begin{aligned} \mu_k^\varepsilon &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^\varepsilon + \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial r_i} - q_k^{\prime\varepsilon} e_r^i \right) \delta_i^\varepsilon, \\ \gamma_k^\varepsilon &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial n_i} - q_k^\varepsilon n_i \right) \delta_{T_i^\varepsilon}. \end{aligned} \tag{2.3.16}$$

Thus $\nabla q_k^\varepsilon - \Delta w_k^\varepsilon = \mu_k^\varepsilon - \gamma_k^\varepsilon$ in $[H^{-1}(\Omega)]^2$, and $\gamma_k^\varepsilon \equiv 0$ in $[H^{-1}(\Omega_\varepsilon)]^2$ in the following sense: $\langle \gamma_k^\varepsilon, \nu \rangle_{H^{-1}, H_0^1(\Omega)} = 0$ for any $\nu \in [H_0^1(\Omega)]^2$ that satisfies $\nu = 0$ on each hole T_i^ε .

Next we give the following lemma, which immediately implies equality (2.1.8) in Proposition 2.1.6, concerning the matrix M in the Brinkman-type law.

Lemma 2.3.3. *The functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq 2}$ defined in (2.3.1) satisfy Hypotheses (H4), (H5), and (H5'), i.e.,*

$$\mu_k = \frac{\pi}{C_0} e_k \in [W^{-1, \infty}(\Omega)]^2, \quad \mu_k^\varepsilon \rightarrow \mu_k \quad \text{in } [H^{-1}(\Omega)]^2 \text{ strongly.}$$

Proof. Because μ_k is a constant vector, (H4) is obvious. Moreover we know that (H5') implies (H5), so it remains to prove that μ_k^ε converges strongly to $\frac{\pi}{C_0} e_k$.

First, we prove that

$$\sum_{i=1}^{N(\epsilon)} \left(\frac{\partial w_k^{\prime\epsilon}}{\partial r_i} - q_k^{\prime\epsilon} e_r^i \right) \delta_i^\epsilon \rightarrow 0 \quad \text{in } [H^{-1}(\Omega)]^2 \text{ strongly.} \tag{2.3.17}$$

For any sequence v_ϵ that converges weakly to a limit v in $[H_0^1(\Omega)]^2$ we define the sequence of real numbers $\Delta_\epsilon = \left\langle \sum_{i=1}^{N(\epsilon)} \left(\frac{\partial w_k^{\prime\epsilon}}{\partial r_i} - q_k^{\prime\epsilon} e_r^i \right) \delta_i^\epsilon, v_\epsilon \right\rangle_{H^{-1}, H_0^1(\Omega)}$. We now introduce the map R_ϵ , defined in Proposition 2.2.2, which satisfies (H6) not in Ω_ϵ , but in $\Omega - \bigcup_{i=1}^{N(\epsilon)} B_i^{a_\epsilon}$. Actually $R_\epsilon v_\epsilon$ belongs to $\left[H_0^1 \left(\Omega - \bigcup_{i=1}^{N(\epsilon)} B_i^{a_\epsilon} \right) \right]^2$. (Note that $\Omega - \bigcup_{i=1}^{N(\epsilon)} B_i^{a_\epsilon} \subseteq \Omega_\epsilon$.) Definition (2.2.7) implies that

$$R_\epsilon v_\epsilon = v_\epsilon \quad \text{on } \partial B_i^\epsilon. \tag{2.3.18}$$

Thus

$$\Delta_\epsilon = \left\langle \sum_{i=1}^{N(\epsilon)} \left(\frac{\partial w_k^{\prime\epsilon}}{\partial r_i} - q_k^{\prime\epsilon} e_r^i \right) \delta_i^\epsilon, R_\epsilon v_\epsilon \right\rangle_{H^{-1}, H_0^1(\Omega)}$$

Integrating equation (2.3.15) by parts, and noting that $R_\epsilon v_\epsilon \equiv 0$ in $\bigcup_{i=1}^{N(\epsilon)} B_i^{a_\epsilon}$, we obtain

$$\Delta_\epsilon = - \int_\Omega q_k^{\prime\epsilon} \nabla \cdot (R_\epsilon v_\epsilon) + \int_\Omega \nabla w_k^{\prime\epsilon} : \nabla (R_\epsilon v_\epsilon). \tag{2.3.19}$$

We bound (2.3.19) with the help of Lemma 2.3.1:

$$\|\Delta_\epsilon\| \leq C\epsilon \|R_\epsilon v_\epsilon\|_{H_0^1(\Omega)} \leq C\epsilon \|v_\epsilon\|_{H_0^1(\Omega)}, \tag{2.3.20}$$

which clearly implies (2.3.17). Now we prove that

$$\sum_{i=1}^{N(\epsilon)} \left(\frac{\partial w_{0k}^\epsilon}{\partial r_i} - q_{0k}^\epsilon e_r^i \right) \delta_i^\epsilon \rightarrow \frac{\pi}{C_0} e_k \quad \text{in } [H^{-1}(\Omega)]^2 \text{ strongly.}$$

We compute

$$\left(\frac{\partial w_{0k}^\epsilon}{\partial r_i} - q_{0k}^\epsilon e_r^i \right) \delta_i^\epsilon = \frac{2\epsilon}{C_0} [-e_k + 4(e_k \cdot e_r^i) e_r^i] [1 + o(1)] \delta_i^\epsilon. \tag{2.3.21}$$

Note that $o(1)$ in the right-hand side of (2.3.21) does not depend on the space variable x . Rather, $o(1)$ is a sequence of real numbers that tends to zero. Thus, using Lemma 2.3.4 below leads to the desired result. Q.E.D.

Lemma 2.3.4. *Let d be a fixed real number in $(0, 1]$. Let $\delta_i^{d\epsilon}$ be the unit mass concentrated on the sphere $\partial B_i^{d\epsilon}$ (of radius $d\epsilon$, and centered at the center of the cube P_i^ϵ), i.e., for each $\phi \in D(\Omega)$, $\langle \delta_i^{d\epsilon}, \phi \rangle_{D', D(\Omega)} = \int_{\partial B_i^{d\epsilon}} \phi(s) ds$. Let S_N denote the area of*

the unit sphere in \mathbb{R}^N . For $N \geq 2$:

$$\sum_{i=1}^{N(e)} d\varepsilon \delta_i^{d\varepsilon} \rightarrow \frac{S_N d^N}{2^N} \quad \text{in } H^{-1}(\Omega) \text{ strongly,} \tag{2.3.22}$$

$$\sum_{i=1}^{N(e)} d\varepsilon \delta_i^{d\varepsilon} (e_k \cdot e_r^i) e_r^i \rightarrow \frac{S_N d^N}{N 2^N} e_k \quad \text{in } [H^{-1}(\Omega)]^N \text{ strongly.} \tag{2.3.23}$$

The proof of (2.3.22) is due to D. CIORANESCU & F. MURAT, and may be found in [9]. The proof of (2.3.23) is very similar and left to the reader (see [1], if necessary).

2.3.2. Other cases: $N \geq 3$ (Proof of Proposition 2.1.4)

In this subsection we define the functions $(w_k^e, q_k^e)_{1 \leq k \leq N}$ using the solutions $(w_k, q_k)_{1 \leq k \leq N}$ of the Stokes problem (2.1.5) around the model hole T in the whole space \mathbb{R}^N . The system (2.1.5) is the local problem which furnishes the value of the Matrix M . Because we can easily get estimates and asymptotic behavior at infinity for the solutions $(w_k, q_k)_{1 \leq k \leq N}$, we can overcome the main difficulty of this paragraph which is to check Hypothesis (H5). First, we give some properties of $(w_k, q_k)_{1 \leq k \leq N}$ in the following

Lemma 2.3.5. For $k \in \{1, \dots, N\}$, the unique solution (w_k, q_k) of system (2.1.5) at infinity satisfies

$$\begin{aligned} w_k &= e_k - \frac{1}{2S_N r^{N-2}} \left[\frac{F_k}{N-2} + (F_k \cdot e_r) e_r \right] + O\left(\frac{1}{r^{N-1}}\right), \\ q_k &= -\frac{1}{S_N r^{N-1}} (F_k \cdot e_r) + O\left(\frac{1}{r^N}\right), \\ \nabla w_k &= O\left(\frac{1}{r^{N-1}}\right), \\ \frac{\partial w_k}{\partial r} - q_k e_r &= \frac{1}{2S_N r^{N-1}} [F_k + N(F_k \cdot e_r) e_r] + O\left(\frac{1}{r^N}\right) \end{aligned} \tag{2.3.25}$$

where F_k is the drag force exerted by the flow on T , i.e., $F_k = \int_{\partial T} \left(\frac{\partial w_k}{\partial n} - q_k n \right)$.
 Moreover,

$$\begin{aligned} F_k \cdot e_i &= \int_{\mathbb{R}^{N-T}} \nabla w_k : \nabla w_i \quad \text{for } i \in \{1, \dots, N\}, \\ F_k \cdot e_k &= \int_{\mathbb{R}^{N-T}} \nabla w_k : \nabla w_k = \inf_{w \in E_k} \int_{\mathbb{R}^{N-T}} \nabla w : \nabla w \end{aligned} \tag{2.3.24}$$

with

$$E_k = \{w \in [H^1_{\text{loc}}(\mathbb{R}^N)]^N / \nabla \cdot w = 0 \text{ in } \mathbb{R}^N, w = 0 \text{ in } T, \text{ and } w = e_k \text{ at infinity}\}.$$

The proof of this lemma is given in the Appendix.

In this subsection we use a decomposition of P_i^ε into smaller subdomains that differs from the one used in the two-dimensional case. We set

$$\bar{P}_i^\varepsilon = T_i^\varepsilon \cup \bar{C}_i^{\prime\varepsilon} \cup \bar{D}_i^\varepsilon \cup \bar{K}_i^\varepsilon \tag{2.3.26}$$

where $C_i^{\prime\varepsilon}$ is the open ball of radius $\varepsilon/2$ centered in P_i^ε and perforated by T_i^ε , D_i^ε is equal to B_i^ε perforated by $\bar{C}_i^{\prime\varepsilon} \cup T_i^\varepsilon$, and K_i^ε is the remainder, i.e., the corners of P_i^ε (see Figure 3). We define the functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq N}$ in each cube P_i^ε which meets $\partial\Omega$ by

$$\begin{cases} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{cases} \quad \text{in } P_i^\varepsilon \cap \Omega$$

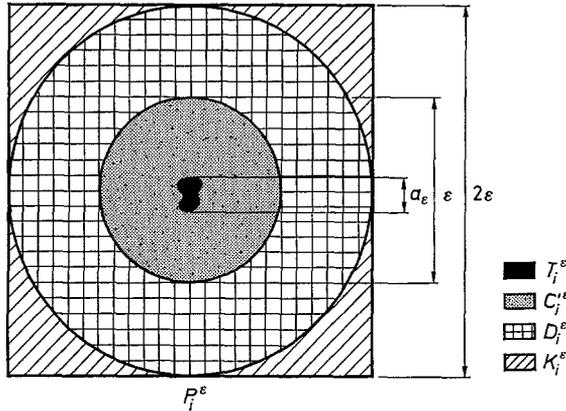


Fig. 3

and in each cube P_i^ε entirely included in Ω , by the requirement that $(w_k^\varepsilon, q_k^\varepsilon) \in [H^1(P_i^\varepsilon)]^N \times L^2(P_i^\varepsilon)$ with $\int_{D_i^\varepsilon} q_k^\varepsilon = 0$, and by

$$\begin{cases} \begin{cases} w_k^\varepsilon = e_k \\ q_k^\varepsilon = 0 \end{cases} & \text{in } K_i^\varepsilon, & \begin{cases} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon = 0 \\ \nabla \cdot w_k^\varepsilon = 0 \end{cases} & \text{in } D_i^\varepsilon, \\ \left. \begin{cases} w_k^\varepsilon = w_k\left(\frac{x}{a_\varepsilon}\right) \\ q_k^\varepsilon = \frac{1}{a_\varepsilon} q_k\left(\frac{x}{a_\varepsilon}\right) \end{cases} \right\} & \text{in } C_i^{\prime\varepsilon}, & \begin{cases} w_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{cases} & \text{in } T_i^\varepsilon. \end{cases} \tag{2.3.27}$$

Definition (2.1.5) of $(w_k, q_k)_{1 \leq k \leq N}$ implies that Definition (2.3.27) is meaningful, and that the functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq N}$ exist and are unique.

Lemma 2.3.6. *The functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq N}$ defined in (2.3.27) satisfy Hypotheses (H1), (H2), and (H3), i.e.,*

(H1) $w_k^\varepsilon \in [H^1(\Omega)]^N, \quad q_k^\varepsilon \in L^2(\Omega),$

(H2) $\nabla \cdot w_k^\varepsilon = 0$ in Ω and $w_k^\varepsilon = 0$ on the holes T_i^ε ,

(H3) $w_k^\varepsilon \rightharpoonup e_k$ in $[H^1(\Omega)]^N$ weakly,

$q_k^\varepsilon \rightharpoonup 0$ in $L^2(\Omega)/\mathbb{R}$ weakly.

Moreover,

$$\|w_k^\varepsilon - e_k\|_{L^p(\Omega)} \leq C \begin{cases} \varepsilon^2 & \text{for } 1 \leq p < \frac{N}{N-2}, \\ \varepsilon^2 |\log \varepsilon|^{\frac{N-2}{N}} & \text{for } p = \frac{N}{N-2}, \\ \varepsilon^{\frac{2N}{p(N-2)}} & \text{for } p > \frac{N}{N-2} \end{cases} \quad (2.3.28)$$

where the constant C does not depend on ε .

Proof. Hypotheses (H1) and (H2) are obviously satisfied. Let us check (H3).

Using the scaling $x \rightarrow \frac{x}{a_\varepsilon}$ and the fact that ∇w_k and q_k are bounded in $L^2(\mathbb{R}^N - T)$, we obtain

$$\begin{aligned} \|\nabla w_k^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 &\leq C a_\varepsilon^{N-2} = C' \varepsilon^N, \\ \|q_k^\varepsilon\|_{L^2(C_i^\varepsilon)}^2 &\leq C a_\varepsilon^{N-2} = C' \varepsilon^N. \end{aligned} \quad (2.3.29)$$

From estimates (2.3.25) we deduce that

$$\begin{aligned} \|w_k^\varepsilon - e_k\|_{L^p(C_i^\varepsilon)}^p &= a_\varepsilon^N \|w_k - e_k\|_{L^p(B_{\frac{\varepsilon}{2a_\varepsilon}} - T)}^p \\ &\leq C \begin{cases} \varepsilon^{N+2p} & \text{for } 1 \leq p < \frac{N}{N-2}, \\ \varepsilon^{\frac{N^2}{N-2}} |\log \varepsilon| & \text{for } p = \frac{N}{N-2}, \\ \varepsilon^{\frac{N^2}{N-2}} & \text{for } p > \frac{N}{N-2}, \end{cases} \end{aligned} \quad (2.3.30)$$

$$w_k^\varepsilon(x) = e_k + O(\varepsilon^2), \quad \nabla w_k^\varepsilon(x) = O(\varepsilon) \quad \text{on } \partial C_i^\varepsilon \cap \partial D_i^\varepsilon. \quad (2.3.31)$$

It follows from the definition of $(w_k^\varepsilon, q_k^\varepsilon)$ in D_i^ε , and from (2.3.31) that

$$\|\nabla w_k^\varepsilon\|_{L^2(D_i^\varepsilon)}^2 \leq C \varepsilon^{N+2}, \quad \|q_k^\varepsilon\|_{L^2(D_i^\varepsilon)}^2 \leq C \varepsilon^{N+2}, \quad \|w_k^\varepsilon - e_k\|_{L^p(D_i^\varepsilon)}^p \leq C \varepsilon^{N+2p}. \quad (2.3.32)$$

Noting that $\|w_k^\varepsilon - e_k\|_{L^p(T_i^\varepsilon)}^p = \|e_k\|_{L^p(T_i^\varepsilon)}^p = O(\varepsilon^{N^2/(N-2)})$, and summing (2.3.29), (2.3.30), and (2.3.32) over all the cubes P_i^ε leads to

$$\begin{aligned} \|\nabla w_k^\varepsilon\|_{L^2(\Omega)}^2 &\leq C, \quad \|q_k^\varepsilon\|_{L^2(\Omega)}^2 \leq C, \\ \|w_k^\varepsilon - e_k\|_{L^p(\Omega)}^p &\leq C \begin{cases} \varepsilon^{2p} & \text{for } 1 \leq p < \frac{N}{N-2}, \\ \varepsilon^{\frac{2N}{N-2}} |\log \varepsilon| & \text{for } p = \frac{N}{N-2}, \\ \varepsilon^{\frac{2N}{N-2}} & \text{for } p > \frac{N}{N-2}. \end{cases} \end{aligned} \quad (2.3.33)$$

From (2.3.33) we deduce the weak convergence of w_k^ε to e_k in $[H^1(\Omega)]^N$, and because q_k^ε is P_i^ε -periodic and bounded in $L^2(\Omega)$, the whole sequence weakly converges to a constant in $L^2(\Omega)$, i.e., to 0 in $L^2(\Omega)/\mathbb{R}$. Q.E.D.

Before checking Hypotheses (H4), (H5), and (H5') we remark that

$$\begin{aligned} \nabla q_k^\varepsilon - \Delta w_k^\varepsilon &= \mu_k^\varepsilon - \gamma_k^\varepsilon \text{ in } \Omega, \quad \text{with } \gamma_k^\varepsilon \equiv 0 \text{ in } [H^{-1}(\Omega_\varepsilon)]^N, \\ \mu_k^\varepsilon &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i \right) \delta_i^{\varepsilon/2} + \nabla \cdot (\chi_\varepsilon(q_k^\varepsilon \text{Id} - \nabla w_k^\varepsilon)) \end{aligned} \quad (2.3.34)$$

where $\delta_i^{\varepsilon/2}$ is the unit mass concentrated on the sphere $\partial C_i^{\varepsilon/2} \cap \partial D_i^\varepsilon$, and χ_ε is the characteristic function of $\bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon$ (which is equal to 1 on this set, and 0 elsewhere).

Note that, in the above expression for μ_k^ε , the term $\left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i \right) \delta_i^{\varepsilon/2}$ is a contribution of the inside of the set $C_i^{\varepsilon/2}$. The equality $\gamma_k^\varepsilon \equiv 0$ in $[H^{-1}(\Omega_\varepsilon)]^N$ means that $\langle \gamma_k^\varepsilon, \nu \rangle_{H^{-1}, H_0^1(\Omega)} = 0$ for any $\nu \in [H_0^1(\Omega)]^N$ that satisfies $\nu = 0$ on each hole T_i^ε .

Lemma 2.3.7. *The functions $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq N}$ defined in (2.3.27) satisfy Hypotheses (H4), (H5), and (H5'), i.e.,*

$$\mu_k = \frac{C_0^{N-2}}{2^N} F_k \in [W^{-1, \infty}(\Omega)]^N, \quad \mu_k^\varepsilon \rightarrow \mu_k \text{ in } [H^{-1}(\Omega)]^N \text{ strongly.}$$

Proof. Obviously (H4) is satisfied because μ_k is a constant vector. Furthermore, from (2.3.32) we deduce that

$$\int_\Omega \chi_\varepsilon |\nabla w_k^\varepsilon|^2 \leq C\varepsilon^2, \quad \int_\Omega \chi_\varepsilon (q_k^\varepsilon)^2 \leq C\varepsilon^2. \quad (2.3.35)$$

Thus $\nabla \cdot (\chi_\varepsilon(q_k^\varepsilon \text{Id} - \nabla w_k^\varepsilon))$ converges strongly to 0 in $[H^{-1}(\Omega)]^N$. Moreover, Lemma 2.3.5 yields

$$\left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i \right) \left(r_i = \frac{\varepsilon}{2} \right) = \frac{2^N C_0^{N-2}}{4S_N} \varepsilon [F_k + N(F_k \cdot e_r^i) e_r^i] + r_\varepsilon \quad (2.3.36)$$

where $|r_\varepsilon|_{L^\infty(\Omega)} \leq C\varepsilon^{\frac{N}{N-2}}$.

Using (2.3.36), we deduce from (2.3.34) that

$$\begin{aligned} \mu_k^\varepsilon &= \frac{2^N C_0^{N-2}}{4S_N} \sum_{i=1}^{N(\varepsilon)} [F_k + N(F_k \cdot e_r^i) e_r^i] \varepsilon \delta_i^{\varepsilon/2} + \nabla \cdot (\chi_\varepsilon(q_k^\varepsilon I d - \nabla w_k^\varepsilon)) \\ &\quad + \sum_{i=1}^{N(\varepsilon)} r_\varepsilon(x) \delta_i^{\varepsilon/2}. \end{aligned}$$

According to Lemma 2.3.4 we get

$$\sum_{i=1}^{N(\varepsilon)} [F_k + N(F_k \cdot e_r^i) e_r^i] \frac{\varepsilon}{2} \delta_i^{\varepsilon/2} \rightarrow 2 \frac{S_N}{4^N} F_k. \tag{2.3.37}$$

Thus, the strong convergence of μ_k^ε to $\mu_k = \frac{C_0^{N-2}}{2^N} F_k$ in $[H^{-1}(\Omega)]^N$ (i.e., (H5')) is achieved if we prove that $\sum_{i=1}^{N(\varepsilon)} r_\varepsilon(x) \delta_i^{\varepsilon/2}$ converges strongly to 0 in $[H^{-1}(\Omega)]^N$. For this purpose we remark that

$$-C \sum_{i=1}^{N(\varepsilon)} \varepsilon^{1+\frac{2}{N-2}} \delta_i^{\varepsilon/2} \leq \sum_{i=1}^{N(\varepsilon)} r_\varepsilon(x) \cdot e_k \delta_i^{\varepsilon/2} \leq C \sum_{i=1}^{N(\varepsilon)} \varepsilon^{1+\frac{2}{N-2}} \delta_i^{\varepsilon/2} \tag{2.3.38}$$

where, thanks to Lemma 2.3.4, the sequence $\sum_{i=1}^{N(\varepsilon)} \varepsilon^{1+\frac{2}{N-2}} \delta_i^{\varepsilon/2}$ converges strongly to 0 in $H^{-1}(\Omega)$. By adding $C \sum_{i=1}^{N(\varepsilon)} \varepsilon^{1+\frac{2}{N-2}} \delta_i^{\varepsilon/2}$ to each side of inequality (2.3.38), we can use Lemma 2.3.8 (below) to complete the proof.

Lemma 2.3.8. *Let α_ε and β_ε be two positive functions in $H^{-1}(\Omega)$ such that*

$$0 \leq \alpha_\varepsilon \leq \beta_\varepsilon. \tag{2.3.39}$$

If β_ε converges strongly to 0 in $H^{-1}(\Omega)$, then so does α_ε .

Proof. This is actually a particularly easy case of a more general lemma due to D. CIORANESCU & F. MURAT (see Lemma 2.8 in [9]). Let ϕ_ε be any weakly convergent sequence in $H_0^1(\Omega)$. We decompose each ϕ_ε into its positive and negative parts, which also belongs to $H_0^1(\Omega)$:

$$\phi_\varepsilon = \phi_\varepsilon^+ - \phi_\varepsilon^- \quad \text{with} \quad \phi_\varepsilon^+ = \sup(\phi_\varepsilon, 0) \quad \text{and} \quad \phi_\varepsilon^- = \sup(-\phi_\varepsilon, 0).$$

Then using (2.3.39) we obtain

$$\begin{aligned} \langle \alpha_\varepsilon, \phi_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} &= \langle \alpha_\varepsilon, \phi_\varepsilon^+ \rangle_{H^{-1}, H_0^1(\Omega)} - \langle \alpha_\varepsilon, \phi_\varepsilon^- \rangle_{H^{-1}, H_0^1(\Omega)} \\ &\leq \langle \beta_\varepsilon, \phi_\varepsilon^+ \rangle_{H^{-1}, H_0^1(\Omega)} - \langle \alpha_\varepsilon, \phi_\varepsilon^- \rangle_{H^{-1}, H_0^1(\Omega)} \\ &= \langle \alpha_\varepsilon, \phi_\varepsilon^- \rangle_{H^{-1}, H_0^1(\Omega)} - \langle \alpha_\varepsilon, \phi_\varepsilon^+ \rangle_{H^{-1}, H_0^1(\Omega)} \\ &\leq \langle \beta_\varepsilon, \phi_\varepsilon^- \rangle_{H^{-1}, H_0^1(\Omega)}. \end{aligned} \tag{2.3.40}$$

As ϕ_ε is a weakly convergent sequence in $H_0^1(\Omega)$, its positive and negative parts are bounded in $H_0^1(\Omega)$. Then, from (2.3.40) and the strong convergence of β_ε , it follows that

$$\langle \alpha_\varepsilon, \phi_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \rightarrow 0,$$

for any sequence ϕ_ε that converges weakly in $H_0^1(\Omega)$. Thus, we deduce that α_ε converges strongly to 0 in $H^{-1}(\Omega)$. Q.E.D.

2.4. Error estimates (Proof of Theorem 2.1.9)

In order to obtain the desired error estimates (2.1.9), we recall the results of Proposition 1.2.5:

$$\|p_\varepsilon - p - u \cdot Q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C \|u\|_{W^{2,\infty}(\Omega)} [\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \|Q_\varepsilon\|_{H^{-1}(\Omega)}],$$

$$\|\nabla(\tilde{u}_\varepsilon - W_\varepsilon u)\|_{L^2(\Omega)} \leq C \|u\|_{W^{2,\infty}(\Omega)} [\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \|Q_\varepsilon\|_{H^{-1}(\Omega)}]$$

where the constant C depends only on Ω . It only remains to prove that

$$[\|M_\varepsilon - M\|_{H^{-1}(\Omega)} + \|Id - W_\varepsilon\|_{L^2(\Omega)} + \|Q_\varepsilon\|_{H^{-1}(\Omega)}] \leq C\varepsilon.$$

Lemma 2.4.1. *Let $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq N}$ be the functions defined in (2.3.1) if $N = 2$, or in (2.3.27) if $N \geq 3$. Then*

$$\|w_k^\varepsilon - e_k\|_{L^2(\Omega)} \leq C\varepsilon, \tag{2.4.1}$$

$$\|q_k^\varepsilon\|_{H^{-1}(\Omega)} \leq C\varepsilon \tag{2.4.2}$$

where the constant C does not depend on ε .

Proof. From the previous results (2.3.13) (for $N = 2$) and (2.3.28) (for $N \geq 3$) we immediately have (2.4.1). On the other hand, from their definitions we have

$$\int_{P_i^\varepsilon} q_k^\varepsilon = 0 \quad \text{for } N = 2 \quad \text{and} \quad \int_{P_i^\varepsilon} q_k^\varepsilon = \int_{C_i^\varepsilon} q_k^\varepsilon \quad \text{for } N \geq 3. \tag{2.4.3}$$

Using estimates (2.3.25) from Lemma 2.3.5 we easily obtain $\left| \int_{C_i^\varepsilon} q_k^\varepsilon \right| \leq C\varepsilon^{N+1}$. Then for any value of the dimension N ,

$$\left| \frac{1}{|P_i^\varepsilon|} \int_{P_i^\varepsilon} q_k^\varepsilon \right| \leq C\varepsilon. \tag{2.4.4}$$

Because q_k^ε is equal to 0 in the cubes P_i^ε that intersect the boundary $\partial\Omega$, we have for any $\phi \in H_0^1(\Omega)$ that

$$\begin{aligned} \langle q_k^\varepsilon, \phi \rangle_{H^{-1}, H_0^1(\Omega)} &= \int_{\Omega} q_k^\varepsilon \phi = \sum_{i=1}^{N(\varepsilon)} \int_{P_i^\varepsilon} q_k^\varepsilon \phi \\ &= \sum_{i=1}^{N(\varepsilon)} \left[\int_{P_i^\varepsilon} q_k^\varepsilon \left(\phi - \frac{1}{|P_i^\varepsilon|} \int_{P_i^\varepsilon} \phi \right) + \left(\int_{P_i^\varepsilon} \phi \right) \frac{1}{|P_i^\varepsilon|} \int_{P_i^\varepsilon} q_k^\varepsilon \right]. \end{aligned} \tag{2.4.5}$$

But

$$\left| \int_{P_i^\varepsilon} q_k^\varepsilon \left(\phi - \frac{1}{|P_i^\varepsilon|} \int_{P_i^\varepsilon} \phi \right) \right| \leq \left\| q_k^\varepsilon \right\|_{L^2(P_i^\varepsilon)} \left\| \phi - \frac{1}{|P_i^\varepsilon|} \int_{P_i^\varepsilon} \phi \right\|_{L^2(P_i^\varepsilon)}. \tag{2.4.6}$$

Using the Poincaré-Wirtinger inequality, we convert (2.4.6) to

$$\left| \int_{P_i^\varepsilon} q_k^\varepsilon \left(\phi - \frac{1}{|P_i^\varepsilon|} \int_{P_i^\varepsilon} \phi \right) \right| \leq C\varepsilon \left\| q_k^\varepsilon \right\|_{L^2(P_i^\varepsilon)} \left\| \nabla \phi \right\|_{L^2(P_i^\varepsilon)}. \tag{2.4.7}$$

With the help of (2.4.4) and (2.4.7) we obtain from (2.4.5) that

$$\begin{aligned} |\langle q_k^\varepsilon, \phi \rangle_{H^{-1}, H_0^1(\Omega)}| &\leq C\varepsilon \sum_{i=1}^{N(\varepsilon)} \left[\left\| q_k^\varepsilon \right\|_{L^2(P_i^\varepsilon)} \left\| \nabla \phi \right\|_{L^2(P_i^\varepsilon)} + \left| \int_{P_i^\varepsilon} \phi \right| \right] \\ &\leq C\varepsilon \left[\sum_{i=1}^{N(\varepsilon)} \varepsilon^{N/2} \left\| \nabla \phi \right\|_{L^2(P_i^\varepsilon)} + \int_{\Omega} |\phi| \right] \\ &\leq C\varepsilon [\left\| \nabla \phi \right\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)}]. \end{aligned} \tag{2.4.8}$$

From (2.4.8) we deduce the desired result $\|q_k^\varepsilon\|_{H^{-1}(\Omega)} \leq C\varepsilon$. Q.E.D.

Lemma 2.4.2. *Let $H_p^1(P)$ be the space of functions belonging to $H^1(P)$ that are restrictions to P of functions belonging to $H_{loc}^1(\mathbb{R}^N)$ and have period $P = (-1, +1)^N$. Let $\langle \cdot, \cdot \rangle_p$ denote the duality product between $H_p^1(P)$ and its dual. Let h belong to the dual of $H_p^1(P)$, and be such that*

$$\langle h, 1 \rangle_p = 0. \tag{2.4.9}$$

Then there exists a unique solution v of the problem:

$$\begin{aligned} \text{Find } v \in H_p^1(P) \text{ such that} \\ -\Delta v = h \quad \text{in } P. \end{aligned} \tag{2.4.10}$$

Let h_ε be the distribution defined by

$$\langle h_\varepsilon, \phi \rangle = \varepsilon^N \langle h(x), \phi(x) \rangle \quad \text{for each } \phi \in D(\mathbb{R}^N). \tag{2.4.11}$$

Formally (2.4.11) is equivalent to $h_\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right)$. Then for each cube Q of \mathbb{R}^N we have

$$\|h_\varepsilon\|_{H^{-1}(Q)} \leq \varepsilon \left(\frac{|Q|}{2^N}\right)^{1/2} \|\nabla v\|_{L^2(P)}. \tag{2.4.12}$$

The proof of this lemma is due to R. V. KOHN & M. VOGELIUS [17], and may also be found in [15].

Lemma 2.4.3. *Let $(\mu_k^\varepsilon)_{1 \leq k \leq N}$ be the functions defined by*

$$\begin{aligned} \mu_k^\varepsilon &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i\right) \delta_i^\varepsilon + \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial r_i} - q_k^{\prime\varepsilon} e_r^i\right) \delta_i^\varepsilon \quad \text{for } N = 2, \\ \mu_k^\varepsilon &= \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right) \delta_i^{\varepsilon/2} + \nabla \cdot (\chi_\varepsilon(q_k^\varepsilon Id - \nabla w_k^\varepsilon)) \quad \text{for } N \geq 3 \end{aligned}$$

(see (2.3.16) and (2.3.34)) where δ_i^ε is the unit mass concentrated on the sphere ∂B_i^ε , $\delta_i^{\varepsilon/2}$ is the unit mass concentrated on the sphere $\partial C_i^{\prime\varepsilon} \cap \partial D_i^\varepsilon$, and χ_ε is the characteristic function of $\bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon$. Then

$$\|\mu_k^\varepsilon - \mu_k\|_{H^{-1}(\Omega)} \leq C\varepsilon \tag{2.4.13}$$

where the constant C does not depend on ε .

Proof. We begin with the case $N \geq 3$. We have

$$\begin{aligned} \|\mu_k^\varepsilon - \mu_k\|_{H^{-1}(\Omega)} &\leq \|\nabla \cdot (\chi_\varepsilon(q_k^\varepsilon Id - \nabla w_k^\varepsilon))\|_{H^{-1}(\Omega)} \\ &\quad + \left\| \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right) \delta_i^{\varepsilon/2} - \mu_k \right\|_{H^{-1}(\Omega)}. \end{aligned} \tag{2.4.14}$$

From (2.3.35) we deduce that

$$\|\nabla \cdot (\chi_\varepsilon(q_k^\varepsilon Id - \nabla w_k^\varepsilon))\|_{H^{-1}(\Omega)} \leq \|\chi_\varepsilon q_k^\varepsilon\|_{L^2(\Omega)} + \|\chi_\varepsilon \nabla w_k^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon. \tag{2.4.15}$$

Now we apply Lemma 2.4.2 in order to estimate the last term of (2.4.14). We set

$$h_\varepsilon(x) = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^\varepsilon}{\partial r_i} - q_k^\varepsilon e_r^i\right) \delta_i^{\varepsilon/2} - \mu_k. \tag{2.4.16}$$

Using the asymptotic expansion (2.3.36), we decompose h_ε into two parts $h_\varepsilon(x) = h_1^\varepsilon(x) + h_2^\varepsilon(x)$ with

$$\begin{aligned} h_1^\varepsilon(x) &= \frac{2^N C_0^{N-2}}{4S_N} \sum_{i=1}^{N(\varepsilon)} [F_k + N(F_k \cdot e_r^i) e_r^i] \varepsilon \delta_i^{\varepsilon/2} - \mu_k, \\ h_2^\varepsilon(x) &= \sum_{i=1}^{N(\varepsilon)} r_\varepsilon(x) \delta_i^{\varepsilon/2} \end{aligned}$$

with $|r_\varepsilon|_{L^\infty(\Omega)} \leq C\varepsilon^{\frac{N}{N-2}}$. By taking differences in (2.3.36), we see that the average of r_ε on each sphere $\partial C_i^\varepsilon \cap \partial D_i^\varepsilon$ is equal to zero. Following to (2.4.11), for $y \in P$, we choose

$$\begin{aligned} h_1(y) &= \frac{2^{N-2}C_0^{N-2}}{S_N} [F_k + N(F_k \cdot e_r) e_r] \delta_0^{1/2} - \mu_k, \\ h_2(y) &= r_0(y) \delta_0^{1/2} \end{aligned} \tag{2.4.17}$$

where $\delta_0^{1/2}$ is the measure defined as the unit mass on the sphere of radius $\frac{1}{2}$ centered at the origin. From the properties of r_ε , we deduce that r_0 has a zero average on the sphere of radius $\frac{1}{2}$, and that $|r_0|_{L^\infty(P)} \leq C\varepsilon^{\frac{2}{N-2}}$. Then, it is straightforward to check that both h_1 and h_2 belong to the dual of $H_p^1(P)$ and $\langle h_1, 1 \rangle_p = \langle h_2, 1 \rangle_p = 0$. Applying Lemma 2.4.2 twice we get

$$\|h_1^\varepsilon\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\nabla v_1\|_{L^2(P)} \quad \text{and} \quad \|h_2^\varepsilon\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\nabla v_2\|_{L^2(P)} \tag{2.4.18}$$

where v_1 and v_2 are defined as the solutions of (2.4.10) with h_1 and h_2 as the right-hand sides. Then summing inequalities (2.4.15) and (2.4.18) we obtain the desired result.

Now we consider the case $N = 2$. In view of (2.3.16) we have

$$\begin{aligned} \|\mu_k^\varepsilon - \mu_k\|_{H^{-1}(\Omega)} &\leq \left\| \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial r_i} - q_k^{\prime\varepsilon} e_r^i \right) \delta_i^\varepsilon \right\|_{H^{-1}(\Omega)} \\ &\quad + \left\| \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^\varepsilon - \mu_k \right\|_{H^{-1}(\Omega)}. \end{aligned}$$

From (2.3.20) we get

$$\left\| \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_k^{\prime\varepsilon}}{\partial r_i} - q_k^{\prime\varepsilon} e_r^i \right) \delta_i^\varepsilon \right\|_{H^{-1}(\Omega)} \leq C\varepsilon. \tag{2.4.19}$$

We set $h_\varepsilon(x) = \sum_{i=1}^{N(\varepsilon)} \left(\frac{\partial w_{0k}^\varepsilon}{\partial r_i} - q_{0k}^\varepsilon e_r^i \right) \delta_i^\varepsilon - \mu_k$; according to (2.4.11) we find

$$h(y) = \frac{2}{C_0} (-e_k + 4(e_k \cdot e_r) e_r) [1 + o(1)] \delta_0^1 - \frac{\pi}{C_0} e_k \quad \text{for } y \in P \tag{2.4.20}$$

where δ_0^1 is the measure defined as the unit mass on the sphere ∂B_1 , and $o(1)$ does not depend on y , as in (2.3.21). It is straightforward to check that h belongs to the dual of $H_p^1(P)$ and that $\langle h, 1 \rangle_p = 0$. Then we can apply Lemma 2.4.2, and arguing as previously for $N \geq 3$, we finally obtain $\|\mu_k^\varepsilon - \mu_k\|_{H^{-1}(\Omega)} \leq C\varepsilon$. Q.E.D.

Appendix

This appendix is devoted to the proof of Lemma 2.3.5. Throughout this discussion, we assume that the space dimension $N \geq 3$. Recall the Stokes prob-

lem (2.1.5):

Find (w_k, q_k) such that

$$\begin{aligned} \|\nabla w_k\|_{L^2(\mathbb{R}^N - T)} < +\infty \quad \text{and} \quad \|q_k\|_{L^2(\mathbb{R}^N - T)} < +\infty, \\ \nabla q_k - \Delta w_k = 0 \quad \text{in } \mathbb{R}^N - T, \\ \nabla \cdot w_k = 0 \quad \text{in } \mathbb{R}^N - T, \\ w_k = 0 \quad \text{on } \partial T, \\ w_k = e_k \quad \text{at infinity.} \end{aligned}$$

The existence and uniqueness of a solution of (2.1.5) is classical (see, e.g., Theorem 4, p. 40 in [18], or Section 2 in [13]) if one weakens the condition $q_k \in L^2(\mathbb{R}^N - T)$ into $\nabla q_k \in [H^{-1}(\mathbb{R}^N - T)]^N$. The assumption that $N \geq 3$ is essential, because existence of a solution of (2.1.5) fails if $N = 2$ (this is the well-known Stokes paradox). To the best of my knowledge the only way previously known to prove that q_k actually belongs to $L^2(\mathbb{R}^N - T)$ is to use the asymptotic behavior (2.3.25). We give here a new proof of this fact, relying only upon variational arguments, and more precisely upon Lemma 2.2.4.

For any real $R \geq 1$, let B_R be the open ball of radius R , centered at the origin. Let $\phi \in [H_0^1(B_R - T)]^N$. Multiplying the equation of system (2.3.24) by ϕ , and integrating the product by parts, we find that

$$\int_{B_R - T} q_k \nabla \cdot \phi = \int_{B_R - T} \nabla w_k : \nabla \phi. \tag{A.1}$$

Now we apply Lemma 2.2.4 in the open set $B_1 - \frac{1}{R}T = \frac{1}{R}(B_R - T)$. After rescaling we obtain that for each $f \in L^2(B_R - T)$ with $\int_{B_R - T} f = 0$, there exists a $\phi \in [H_0^1(B_R - T)]^N$ such that $\nabla \cdot \phi = f$ in $B_R - T$ and $\|\nabla \phi\|_{L^2(B_R - T)} \leq C \|f\|_{L^2(B_R - T)}$ where the constant C does not depend on R . Then we deduce from (A.1) that $\|q_k\|_{L^2(B_R - T)/\mathbb{R}} \leq C \|\nabla w_k\|_{L^2(B_R - T)} \leq C$ where the constant C does not depend on R . Letting $R \rightarrow +\infty$, we easily see that q_k belongs to $L^2(\mathbb{R}^N - T)/\mathbb{R}$. *A priori* the pressure q_k is defined up to a constant (that is why the space L^2 is factored by the set \mathbb{R} of real constants). However, because in the present case constants (except 0) do not belong to $L^2(\mathbb{R}^N - T)$, there exists only one representative of the class q_k in $L^2(\mathbb{R}^N - T)/\mathbb{R}$ that belongs to $L^2(\mathbb{R}^N - T)$.

Next, we seek the pointwise estimates (2.3.25), namely

$$\begin{aligned} w_k &= e_k - \frac{1}{2S_N r^{N-2}} \left[\frac{F_k}{N-2} + (F_k \cdot e_r) e_r \right] + O\left(\frac{1}{r^{N-1}}\right), \\ q_k &= -\frac{1}{S_N r^{N-1}} (F_k \cdot e_r) + O\left(\frac{1}{r^N}\right), \\ \nabla w_k &= O\left(\frac{1}{r^{N-1}}\right), \\ \frac{\partial w_k}{\partial r} - q_k e_r &= \frac{1}{2S_N r^{N-1}} [F_k + N(F_k \cdot e_r) e_r] + O\left(\frac{1}{r^N}\right) \end{aligned}$$

where F_k is the drag force, i.e., $F_k = \int_{\partial T} \left(\frac{\partial w_k}{\partial n} - q_k n \right)$ with n the unit exterior normal to T . For this, we use the theory of hydrodynamical potentials due to B. K. G. ODQVIST & L. LICHTENSTEIN (see [13] and [18] for details and references). Consider the fundamental singular solutions of the Stokes equations

$$\begin{aligned}
 p_k &= \frac{x_k - y_k}{S_N |x - y|^N}, \\
 u_j^k &= \frac{1}{2S_N} \left[\frac{\delta_j^k}{(N-2) |x - y|^{N-2}} + \frac{(x_k - y_k)(x_j - y_j)}{|x - y|^N} \right]
 \end{aligned}
 \tag{A.2}$$

which satisfy

$$\nabla p_k - \Delta u_k = \delta(x - y) e_k, \quad \nabla \cdot u_k = 0 \quad \text{in } \mathbb{R}^N,
 \tag{A.3}$$

and

$$p = \delta(x - y), \quad u_j = \frac{x_j - y_j}{S_N |x - y|^N}
 \tag{A.4}$$

which satisfy

$$\nabla p - \Delta u = 0, \quad \nabla \cdot u = \delta(x - y) \quad \text{in } \mathbb{R}^N.
 \tag{A.5}$$

By convolving these fundamental singular solutions (A.2) and (A.4) with the source terms (provided that they are smooth enough), we obtain solutions of the Stokes systems in the whole space.

Let $\theta \in D(B_1)$ such that $\theta \equiv 1$ in the vicinity of T . We set

$$\begin{aligned}
 \left\{ \begin{aligned} \tilde{q}_k &= (1 - \theta) q_k \\ \tilde{w}_k &= (1 - \theta) w_k \end{aligned} \right\}, \quad \text{which implies that} \\
 \left\{ \begin{aligned} \tilde{q}_k &\in C^\infty(\mathbb{R}^N), & \tilde{q}_k &\equiv q_k \text{ in } \mathbb{R}^N - B_1 \\ \tilde{w}_k &\in [C^\infty(\mathbb{R}^N)]^N, & \tilde{w}_k &\equiv w_k \text{ in } \mathbb{R}^N - B_1 \end{aligned} \right\}.
 \end{aligned}
 \tag{A.6}$$

We set also

$$\left\{ \begin{aligned} f_k &= \nabla \tilde{q}_k - \Delta \tilde{w}_k \\ g_k &= \nabla \cdot \tilde{w}_k \end{aligned} \right\}, \quad \text{which implies that} \quad \left\{ \begin{aligned} f_k &\in [D(B_1)]^N \\ g_k &\in D(B_1) \end{aligned} \right\}.
 \tag{A.7}$$

Then it is easy to check that in $\mathbb{R}^N - B_1$,

$$\begin{aligned}
 q_k(x) &= g_k(x) + \frac{1}{S_N} \int_{\mathbb{R}^N} \frac{(x - y) \cdot f_k(y)}{|x - y|^N} dy, \\
 w_k(x) &= e_k + \frac{1}{S_N} \int_{\mathbb{R}^N} \frac{(x - y)}{|x - y|^N} g_k(y) dy + \frac{1}{2(N-2) S_N} \int_{\mathbb{R}^N} \frac{f_k(y)}{|x - y|^{N-2}} dy \\
 &\quad + \frac{1}{2S_N} \int_{\mathbb{R}^N} \frac{(x - y) \cdot f_k(y)}{|x - y|^N} (x - y) dy.
 \end{aligned}
 \tag{A.8}$$

Asymptotically (A.8) becomes

$$\begin{aligned}
 q_k(x) &= \frac{1}{S_N} \frac{x \cdot \int_{B_1} f_k(y) dy}{|x|^N} + O\left(\frac{1}{|x|^N}\right), \\
 w_k(x) &= e_k + \frac{1}{2(N-2)S_N} \frac{\int_{B_1} f_k(y) dy}{|x|^{N-2}} + \frac{x}{2S_N} \left(\frac{x \cdot \int_{B_1} f_k(y) dy}{|x|^N} \right) + O\left(\frac{1}{|x|^{N-1}}\right), \\
 \nabla w_k &= O\left(\frac{1}{|x|^{N-1}}\right). \tag{A.9}
 \end{aligned}$$

Introducing $F_k = - \int_{B_1} f_k(y) dy$ and $r = |x|$ in (A.9) leads to the desired estimates (2.3.25).

It remains to prove equalities (2.3.24). Multiplying equation (2.1.5) by w_i , integrating the result by parts, and using estimates (2.3.25), we obtain

$$F_k \cdot e_i = \int_{\mathbb{R}^N - T} \nabla w_k : \nabla w_i \quad \text{for } i \in \{1, \dots, N\}. \tag{A.10}$$

It is well-known that

$$F_k \cdot e_k = \int_{\mathbb{R}^N - T} \nabla w_k : \nabla w_k = \inf_{w \in E_k} \int_{\mathbb{R}^N - T} \nabla w : \nabla w \tag{A.11}$$

with

$$E_k = \{w \in [H_{loc}^1(\mathbb{R}^N)]^N / w = 0 \text{ in } T, \nabla \cdot w = 0 \text{ in } \mathbb{R}^N - T \text{ and } w = e_k \text{ at infinity}\}.$$

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