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Homogenization and Bloch wave method for fluid tube bundle interaction

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Abstract

The aim of this paper is to investigate the problem of the vibrations of large arrays of elastic rods immersed in a perfect incompressible fluid. The case of an infinite spatially periodic bundle is firstly considered leading to use the Bloch wave method in order to describe the resonance spectrum of the coupled system. When the bundle is contained in a bounded domain, the homogenization technique combined with the Bloch wave method allows to obtain the eigenspectrum which is formed of two eigenfrequencies (of infinite multiplicity), and of a continuous spectrum.

1. Introduction

The study of vibrations of tube bundles immersed in a liquid is of great practical importance in engineering design concerning particularly heat exchangers, condensers and nuclear reactor cores. For this reason, numerous papers and books have been devoted to this subject. Among others, let us mention the works by Blevins [8], Chen [10], Paidoussis [32], Gorman [22], Pettigrew [33] and Gibert [21].

These studies are based on the important notion of added mass matrix describing the reciprocal influences of the different elements of the bundle via the surrounding fluid. The computation of this matrix is often complicated because the number of tubes is generally high in practice, of the order of several thousands as it is the case for heat exchangers and nuclear reactors. However, this complexity may be overcome by means of the homogenization method which delivers constitutive equations and boundary conditions for an homogenized medium equivalent to the coupled fluid-tube system (see [35,16,15]). Another possibility consists in considering the bundle as infinitely large and spatially periodic, so that the fluid flow potential can be decomposed in terms of Bloch waves (cf. [18,17,11]). Obviously, the effects of the boundary of the bundle are ignored in this latter approach.

An alternative method is to combine the homogenization and the Bloch wave techniques. This process has been recently used by two of the authors in order to investigate the behavior of the spectrum of the added mass matrix as the number of tubes becomes infinite [3,4]. This so-called Bloch wave homogenization method is rather involved from a mathematical point of view. The aim of this paper is to present, in a less formal manner, this method for the time-dynamical behavior of large tube arrays.

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2. Some reminders about the added mass matrix

2.1. Notations and equations

We consider a group of parallel cylinders immersed in a quiescent perfect incompressible liquid. The cylinders are rigid and elastically mounted by a spring system allowing transversal motions. In order to simplify the presentation, two-dimensional problems are considered. The generalization to the three-dimensional case, taking into account the tube bending, may easily be done. One denotes by γ_l the wall of tube ℓ and by Γ the rigid wall of the cavity containing the fluid and the bundle; Ω is the region occupied by the liquid. The cylinder motions are supposed small enough so that the geometrical variation of the domain Ω may be neglected.

Under these hypotheses, the coupled system obeys to the following system of equations

$$\begin{cases} \Delta\phi(x,t) = 0 & \text{for } x \in \Omega\,, \\ \dfrac{\partial \phi}{\partial n} = 0\,, & \text{for } x \in \Gamma\,, \\ \dfrac{\partial \phi}{\partial n}(x,t) = \dfrac{\mathrm{d}\vec{s}_\ell}{\mathrm{d}t}\cdot \vec{n} & \text{for } x \in \gamma_\ell\,, \end{cases} \tag{2.1}$$

in which ϕ is the fluid potential, $\vec{s}_\ell$ is the displacement–vector of the tube γ_ℓ, $\vec{n}$ is the unit-normal on γ_ℓ and Γ oriented outside Ω. The vectors $\vec{s}_\ell$ are assumed to satisfy the dynamics equation

$$m_\ell \frac{\mathrm{d}^2\vec{s}_\ell}{\mathrm{d}t^2} = -\rho \int_{\gamma_\ell} \frac{\partial \phi}{\partial t}(x,t)\vec{n}\,\mathrm{d}\gamma_\ell - k_\ell \vec{s}_\ell(t) + \vec{f}_\ell(t)\,, \quad \ell = 1,2,\dots,N\,, \tag{2.2}$$

where m_ℓ is the tube mass (per unit length), k_ℓ is the stiffness of the spring system supporting γ_ℓ; ρ is the fluid specific density and $\vec{f}_\ell$ is an external force acting on the tube ℓ; N is the number of tubes. The first term in the right-hand side of (2.2) is the hydrodynamical force on γ_ℓ (recall that the pressure is equal to $-\rho\,\partial\phi/\partial t$).

2.2. The added mass matrix

It is usual to eliminate ϕ by setting

$$\phi(x,t) = \sum_{\ell=1}^{N} \sum_{j=1}^{2} \chi_{\ell_j}(x) \frac{\mathrm{d}s_{\ell_j}}{\mathrm{d}t}(t)\,, \tag{2.3}$$

where χ_{ℓ_j} satisfies

$$\begin{cases} \Delta\chi_{\ell_j} = 0 & \text{in } \Omega\,, \\ \dfrac{\partial \chi_{\ell_j}}{\partial n} = 0 & \text{on } \Gamma\,, \\ \dfrac{\partial \chi_{\ell_j}}{\partial n} = n_j(x)\delta_{\ell\ell'} & \text{on each } \gamma_{\ell'},\ \ell' = 1,\dots,N \end{cases} \tag{2.4}$$

where n_j is the jth direction-cosine of the normal $\vec{n}$ and s_{ℓ_j} is the jth component of $\vec{s}_\ell$. The χ_{ℓ_j} are chosen such that $\int_\Omega \chi_{\ell_j}\,\mathrm{d}x = 0$, which determines these functions uniquely.

Inserting (2.3) into (2.2), one obtains

$$M \frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{s}(t) = -K\vec{s}(t) - \rho H \frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{s}(t) + \vec{f}(t)\,, \tag{2.5}$$

where $\vec{s} = \mathrm{col}(\vec{s}_1, \vec{s}_2, \dots, \vec{s}_N)$, $\vec{f} = \mathrm{col}(\vec{f}_1, \vec{f}_2, \dots, \vec{f}_N)$ and $M = \mathrm{diag}(m_\ell)$, $K = \mathrm{diag}(k_\ell)$ (m_ℓ and k_ℓ being repeated twice), and H is the matrix of order $2N$ with entries $\int_{\gamma_\ell} \chi_{\ell' j}(x) n_i(x)\,\mathrm{d}\gamma_\ell$. The matrix ρH, so defined, is the added mass matrix: it is symmetric and positive definite, so its eigenvalues are real positive numbers, see (see [25,1]).

An immediate consequence is that the eigenfrequencies ω of the coupled system

$$(M + \rho H)\vec{s} = \frac{K}{\omega^2}\vec{s} \tag{2.6}$$

are real and positive. Moreover, the following inequality holds

$$0 < \omega < \max_{\ell} \sqrt{\frac{k_\ell}{m_\ell}} . \tag{2.7}$$

Actually, a more precise result holds true. Setting $\omega_{0\ell} = \sqrt{k_\ell/m_\ell}$ the eigenfrequency of tube ℓ when placed in vacuum and labeled in such a way that

$$\omega_{01} \leqslant \omega_{02} \leqslant \cdots \omega_{0N} ,$$

then, there are at least 2ℓ eigenfrequencies ω such that

$$0 < \omega < \omega_{0\ell} . \tag{2.8}$$

The proof of (2.8) may be found in the references [24,19]. Remark that an eigenfrequency ω may well exist such that $\omega = \omega_{0\ell}$ (this obviously requires that $\omega_{0\ell} < \omega_{0N}$); for such an eigenfrequency ω, the corresponding flow potential ϕ must satisfy $\int_{\gamma_\ell} \phi\vec{n}\, d\gamma_\ell = 0$, which means that the hydrodynamical force acting on γ_ℓ is zero.

2.3. Behavior of the frequency spectrum when N increases

It is now assumed that all cylinders are identical (same geometry and same mechanical properties, i.e. $m_\ell \equiv m$, $k_\ell \equiv k$); they have therefore the same eigenfrequency $\omega_0 = \sqrt{k/m}$ in vacuum. The eigenfrequencies ω and the eigenvectors ϕ of the fluid-bundle system satisfy the variational equation

$$a(\phi, v) = \frac{\rho\omega^2}{k - m\omega^2} \sum_{\ell=1}^{N} \overrightarrow{N_\ell \phi} \cdot \overrightarrow{N_\ell v} , \tag{2.9}$$

for any smooth-test function $v(x)$, where

$$a(\phi, v) = \int_\Omega \nabla\phi \cdot \nabla v \, dx , \qquad \overrightarrow{N_\ell v} = \int_{\gamma_\ell} v(x)\vec{n}\, d\gamma_\ell .$$

Setting $\lambda = \rho\omega^2/(k - m\omega^2)$ and $\overrightarrow{Nv} = \mathrm{col}(\overrightarrow{N_\ell v})$, the Rayleigh quotient is defined by

$$q(v) = \frac{a(v, v)}{|\overrightarrow{Nv}|^2} ,$$

where $|\overrightarrow{Nv}|$ is the Euclidian norm of the vector $\overrightarrow{Nv}$ of components $\overrightarrow{N_\ell v}$.

The first eigenvalue λ_1, corresponding to the smallest eigenfrequency ω_1, is given by

$$\lambda_1 = \inf_v \left\{ q(v) \,\middle|\, \int_\Omega v \, dx = 0 \right\} ,$$

and the infimum is attained for some function $\phi_1(x)$.

The second eigenvalue λ_2 is

$$\lambda_2 = \inf_v \left\{ q(v) \,\middle|\, \int_\Omega v \, dx = 0 \quad \text{and} \quad \overrightarrow{Nv} \cdot \overrightarrow{N\phi_1} = 0 \right\} ,$$

and $\lambda_2 = q(\phi_2)$ for another function ϕ_2. More generally, we have

$$\lambda_{n+1} = \inf_v \left\{ q(v) \,\middle|\, \int_\Omega v \, dx = 0 \quad \text{and} \quad \overrightarrow{Nv} \cdot \overrightarrow{N\phi_\ell} = 0 \quad \forall\, \ell = 1, \ldots, n \right\} ,$$

and $\lambda_{n+1} = q(\phi_{n+1})$. The recurrence holds true up to $n = 2N$, since the number of eigenfrequencies ω is exactly $2N$ (see [19]). Furthermore, these eigenvalues can be characterized via the min–max principle

$$\begin{cases} \lambda_n = \max_{v_i; i=1,\dots,n-1} \min_{v} q(v) \quad \text{with } \overrightarrow{Nv} \cdot \overrightarrow{Nv_i} = 0, \int_\Omega v \, dx = 0 , \\ \lambda_n = \min_{V_n; \dim V_n = n} \max_{v \in V_n} q(v) . \end{cases} \tag{2.10}$$

Now, our goal is to investigate the behavior of ω_1 and ω_{2N} (corresponding respectively, to λ_1 and λ_{2N}) as N increases. For this purpose, the tube bundle is assumed to be an assembling of identical cells C_ℓ, each one containing one cylinder. If $\Omega^{(m)}$ is a domain made up of m identical cells, then (see Fig. 1)

$$\Omega^{(m)} = \Omega^{(m-1)} \cup C_m .$$

The rigid wall of $\Omega^{(m)}$ is its external boundary. With these definitions, we introduce

$$a_m(v, v) = \int_{\Omega^{(m)}} |\nabla v|^2 \, dx , \qquad q_m(v) = \frac{a_m(v, v)}{\sum_{\ell=1}^{m} |\overrightarrow{N_\ell v}|^2} ,$$

and the eigenvalues $\lambda_n^{(m)}$ associated with each $\Omega^{(m)}$.

A consequence of relation (2.10) is

$$\lambda_{2m}^{(m)} \geq \lambda_{2(m-1)}^{(m-1)} , \tag{2.11}$$

which immediately implies that the largest eigenfrequency increases with the number of tubes (see [16, Proposition 5, p. 107]).

We now turn to the case of the first eigenfrequency. Let ϕ_1 be the eigenfunction associated with the first eigenvalue $\lambda_1^{(m)}$ of $\Omega^{(m)}$

$$\lambda_1^{(m)} = \inf_v q_m(v) = q_m(\phi_1) = \frac{\sum_{\ell=1}^{m} \int_{C_\ell} |\nabla \phi_1|^2 \, dx}{\sum_{\ell=1}^{m} |\overrightarrow{N_\ell \phi_1}|^2}$$

and since the cells C_ℓ are identical, one has

$$\int_{C_\ell} |\nabla \phi_1|^2 \, dx \geq \lambda_1^{(1)} |\overrightarrow{N_\ell \phi_1}|^2$$

from the definition of $\lambda_1^{(1)}$. Hence

$$\lambda_1^{(m)} \geq \lambda_1^{(1)} \quad \text{for any } m . \tag{2.12}$$

On the other hand, we know that $\lambda_1^{(1)} = \inf q_1(v) = q_1(\phi_1^{(1)})$ for some function $\phi_1^{(1)}(x)$ (which satisfies Neumann boundary condition on the cell boundary)

$$\lambda_1^{(1)} = \frac{\int_{C_\ell} |\nabla \phi_1^{(1)}|^2 \, dx}{|\overrightarrow{N_\ell \phi_1^{(1)}}|^2} \quad \text{on each cell } C_\ell ,$$

and we can write

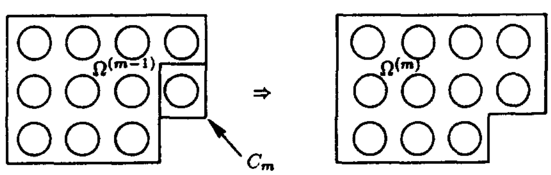

Fig. 1. Fluid domain $\Omega^{(m)}$.

$$\lambda_1^{(1)} = \frac{\sum_{\ell=1}^{m} \int_{C_\ell} |\nabla \phi_1^{(1)}|^2 \, dx}{\sum_{\ell=1}^{m} |\overrightarrow{N_\ell \phi_1^{(1)}}|^2} . \tag{2.13}$$

Let us define a function $\tilde{\phi}_1$ in $\Omega^{(m)}$ which is equal to $\phi_1^{(1)}$ in each C_ℓ. By construction $\tilde{\phi}_1$ may be discontinuous at the interfaces of the cells. So we change the definition of $\tilde{\phi}_1$ in order to obtain a continuous function. For this purpose, it is necessary to assume that the cells are symmetric with respect to their principal axes (cubic symmetry). By a standard reflection procedure with respect to the cell boundaries, we can extend $\phi_1^{(1)}$ from one cell to its neighbors and obtain a continuous function on the entire domain. Its gradient is also continuous since $\partial \phi_1^{(1)} / \partial n = 0$ on the cell boundaries. Thus, (2.13) yields

$$\lambda_1^{(1)} = q_m(\tilde{\phi}_1) .$$

But $\lambda_1^{(m)} = \inf_v q_m(v) \leq q_1(\tilde{\phi}_1) = \lambda_1^{(1)}$, and using (2.12)

$$\lambda_1^{(m)} = \lambda_1^{(1)} \quad \text{for any integer } m . \tag{2.14}$$

Therefore, in the case of symmetric cells, the smallest eigenfrequencies $\omega_1^{(m)}$ of the different domains $\Omega^{(m)}$ are equal, while the greatest ones increase with m. This behavior of the limiting spectrum has been observed by numerical computations (see [25,24]). For instance, for bundles containing, respectively, 16 and 49 tubes we have

$$N = 16, \quad \omega_1 = 40.7 \text{ Hz}, \quad \omega_{32} = 51.8 \text{ Hz} ;$$
$$N = 49, \quad \omega_1 = 40.7 \text{ Hz}, \quad \omega_{98} = 52.4 \text{ Hz} .$$

For largest values of N, some numerical troubles occur because the eigenfrequencies ω become very close one from each other.

Some remarks

When the fluid is compressible, the added mass matrix is time – dependent and the term $H \, d^2\vec{s}/dt^2$ in (2.5) must be replaced by the time-convolution term $H * d^2\vec{s}/dt^2$. Moreover, there exists an infinite set of eigenfrequencies ω's which accumulate at infinity and there are at least $2N$ ω's which are strictly smaller than $\omega_0 = (k/m)^{1/2}$ (see [16,36]).

In [38], the small geometrical variations of the domain Ω caused by the cylinder motions, are taken into account and lead to additional damping and stiffness terms in Eq. (2.5).

The case of a viscous fluid is investigated in great detail in [12,39,13,14].

3. Infinite tube bundle and the Bloch wave method

3.1. The Bloch waves

It has been shown in the previous section that the highest eigenfrequency ω_{2N} increases with the number N of elements. Because the eigenfrequencies are lower than $\omega_0 = (k/m)^{1/2}$, they tend to cluster in a fixed interval. Thus, the effect of large N is to spread out the resonance frequencies over a certain interval located between 0 and ω_0, and this interval is obviously inside the band corresponding to an infinite N.

The case of an infinite bundle can easily be investigated by means of the Bloch wave method. Although this technique is well-known in solid state physics [9,40], it is only recently that it has been introduced in the context of fluid–solid vibrations [16,18,17,11,20]. For the reader's convenience, we briefly remind in the sequel.

The cylinders are supposed to be located at the nodes of a regular square network with period ε (see Fig. 2). Each tube is labeled by a double integer index $\vec{\ell} = (\ell_1, \ell_2)$. Our purpose is to find the eigenmotions of

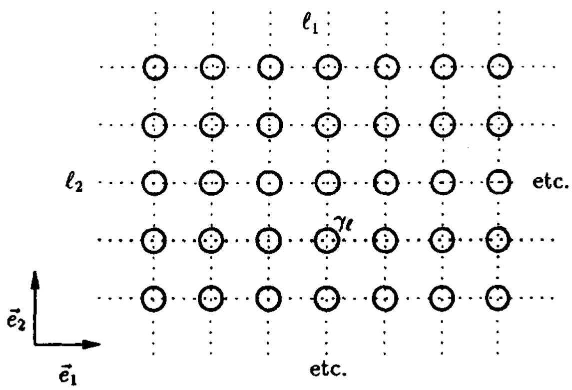

Fig. 2. Infinite array of tubes.

$$\begin{cases} \Delta\phi = 0\,, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\mathrm{d}\vec{s}_\ell}{\mathrm{d}t}\cdot\vec{n} & \text{on each } \gamma_\ell\,, \\ \left(m\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}+k\right)\vec{s}_\ell = -\rho\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{\gamma_\ell}\phi(x,t)\vec{n}\,\mathrm{d}\gamma_\ell\,. \end{cases} \tag{3.1}$$

The Brillouin's book [9] suggests to search the eigendisplacements of frequency ω for which the $\vec{s}_\ell$ are of the form

$$\vec{s}_\ell(t) = \mathrm{e}^{\mathrm{i}(\vec{\ell}\cdot\vec{\theta}+\omega t)}\vec{\xi}(\theta)\,, \tag{3.2}$$

where $\vec{\theta} = (\theta_1, \theta_2)$, in which θ_i is a parameter varying from zero to 2π, $\vec{\xi}(\theta)$ is a vector which does not depend on ℓ, and ω is obviously a function of $\vec{\theta}$. Remark that this idea was also evoked by Paidoussis and his co-workers [29]. An eigenmotion defined by (3.2) is quasi-periodic: this means that for any $\vec{\theta}$ and any $\eta > 0$ there exists a double integer $\vec{p} = (p_1, p_2)$ such that $|\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{\theta}} - 1| < \eta$; clearly, the tubes vibrate with quasi-identical vibration cells containing p_1 rows of p_2 tubes. This phenomenon was also effectively observed by numerical computations for a group of 10×10 cylinders [26].

Let $\phi(x; \vec{\theta})$ be the flow potential associated to (3.2); it necessarily satisfies

$$\phi(x + \varepsilon\vec{e}_i; \vec{\theta}) = \mathrm{e}^{\mathrm{i}\theta_i}\phi(x; \vec{\theta})\,, \quad i = 1, 2 \tag{3.3}$$

in which $(\vec{e}_i)_{i=1,2}$ are orthonormal vectors parallel to the network lines. It is then said that ϕ is *$\vec{\theta}$-periodic*. It can be checked that ϕ solves

$$\begin{cases} \Delta\phi(x; \vec{\theta}) = 0\,, \\ \dfrac{\partial}{\partial n}\phi(x; \vec{\theta}) = \mathrm{i}\omega\vec{\xi}(\vec{\theta})\cdot\vec{n}\,\mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}} & \text{on each } \gamma_\ell\,, \\ (k - m\omega^2)\vec{\xi}(\vec{\theta}) = -\mathrm{i}\omega\rho\displaystyle\int_{\gamma_\ell}\phi(x; \vec{\theta})\vec{n}\,\mathrm{d}\gamma_\ell\,, \end{cases} \tag{3.4}$$

in each cell C_ℓ containing the tube ℓ. Because of (3.3), it is sufficient to solve, in an arbitrary cell, say C_0 (corresponding to $\vec{\ell} = 0$), the eigenproblem

$$\begin{cases} \Delta\phi(x; \vec{\theta}) = 0 & \text{in } C^*\,, \\ \dfrac{\partial}{\partial n}\phi(x; \vec{\theta}) = \beta(\vec{\theta})\vec{n}\cdot\displaystyle\int_\gamma \phi(x; \vec{\theta})\vec{n}\,\mathrm{d}\gamma & \text{on } \gamma\,, \\ \phi \text{ is } \vec{\theta}\text{-periodic}\,, \end{cases} \tag{3.5}$$

where C^* is that part of the cell occupied by the fluid and γ is in fact γ_0. The eigenvalue $\beta(\vec{\theta})$ is related to the eigenfrequency by

$$\beta(\vec{\theta}) = \frac{\rho\omega^2}{k = m\omega^2} \,. \tag{3.6}$$

The corresponding ω's are the Bloch wave eigenfrequencies. Remark that the additional condition $\int_{C^*} \phi(x; \vec{\theta})\, dx = 0$ is necessary only for $\vec{\theta} = 0$. If ϕ is assumed to be generated by a vector $\vec{z} = (z_1, z_2)$ such that $\partial\phi / \partial n = \vec{z} \cdot \vec{n}$ on γ, then

$$\phi(x; \vec{\theta}) = \sum_{i=1}^{2} \chi_i(x; \vec{\theta}) z_i \,,$$

where $\chi_i(x; \vec{\theta})$ is the $\vec{\theta}$-periodic harmonic function satisfying

$$\frac{\partial \chi_i}{\partial n}(x; \vec{\theta}) = n_i(x) \quad \text{on } \gamma$$

(n_i is the ith direction-cosine of the normal $\vec{n}$). Replacing ϕ by its expansion into (3.5), one gets

$$\vec{z} = \beta(\vec{\theta}) B \vec{z} \,,$$

where B is the 2×2 matrix whose entries are

$$b_{ij} = \int_\gamma \chi_j(x; \vec{\theta}) n_i(x)\, d\gamma \,.$$

The matrix B is obviously complex-valued if $\vec{\theta} \neq 0$, and is self-adjoint positive definite. It implies that (3.5) has only two eigenvalues $(\beta_j(\vec{\theta}))_{j=1,2}$ which are real and positive. The associated eigenfunctions $\phi_j(x; \vec{\theta})$ are the so-called Bloch waves while the vectors

$$\vec{v}_j(\vec{\theta}) = \int_\gamma \phi_j(x; \vec{\theta}) \vec{n}\, d\gamma \tag{3.7}$$

are known as the Bloch wave vectors. The $\vec{\xi}(\vec{\theta})$ corresponding to (3.2) are obviously

$$\vec{\xi}_j(\vec{\theta}) = \frac{-i\rho\omega}{k - m\omega^2} \vec{v}_j(\vec{\theta}) \,.$$

The eigenvalues $\beta_j(\vec{\theta})$(and the potentials ϕ_j) are continuous functions of $\vec{\theta}$, except for $\vec{\theta} = 0$ (for which they have a well-determined value); this is due to the fact that the condition $\int_\gamma \phi_i\, dx = 0$ (is necessary when $\vec{\theta} = 0$ see [3,15]).

3.2. *The generalized added mass matrix*

Let $\vec{s}$ be the vector (of infinite length) with components $\vec{s}_\ell$

$$\vec{s} = \text{col}(\vec{s}_\ell) \,.$$

Each component $\vec{s}_\ell$ may be written as

$$\vec{s}_\ell = \int \vec{s}_\ell(\vec{\theta})\, d\theta$$

(where $\int(\cdot)\, d\theta$ means $\int_0^{2\pi} \int_0^{2\pi} (\cdot)\, d\theta_1\, d\theta_2$) with

$$\begin{cases} \vec{s}_\ell(\vec{\theta}) = \sum_{j=1}^{2} a_j(\vec{\theta})\, e^{i\vec{\theta}\cdot\vec{\ell}}\, \vec{v}_j(\vec{\theta}) \,, \\ a_j(\vec{\theta}) = \dfrac{1}{(2\pi)^2} \sum_{\vec{q}} \vec{s}_q \cdot \bar{\vec{v}}_j(\vec{\theta})\, e^{-i\vec{\theta}\cdot\vec{q}} \end{cases} \tag{3.8}$$

where $\vec{q} = (q_1, q_2)$ is a double integer index and the summation on $\vec{q}$ means $\Sigma_{q_1=-\infty}^{\infty} \Sigma_{2=-\infty}^{\infty}$; the upper bar denotes the complex conjugate. In (3.8), the vectors $\vec{v}_j(\vec{\theta})$ are orthonormalized by a convenient choice of the eigenvectors $\vec{z}$ of the matrix B. The quantities $a_j(\vec{\theta})$ are the generalized Fourier coefficients of the infinite vector $\vec{s}$.

The added mass is the operator H which associates to any vector $\vec{s} = \mathrm{col}(\vec{s}_\ell)$ the infinite vector $\vec{h}$ of components $\vec{h}_\ell = \int_{\gamma_\ell} \psi(x)\vec{n}\, \mathrm{d}\gamma_\ell$, where $\psi(x)$ is the harmonic function satisfying $\partial\psi/\partial n(x) = \vec{s}_\ell \cdot \vec{n}(x)$ on each γ_ℓ. The operator H is obviously of infinite order and is symmetric positive definite (see [15,20]). The infinite vector $\vec{h} \equiv H\vec{s}$ has the following decomposition

$$\vec{h}_\ell = \sum_{j=1}^{2} \int d_j(\vec{\theta})\, \mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}}\, \vec{v}_j(\vec{\theta})\, \mathrm{d}\theta$$

with

$$d_j(\vec{\theta}) = \frac{1}{(2\pi)^2} \sum_{\vec{q}} \vec{h}_q \cdot \vec{v}_j(\vec{\theta})\, \mathrm{e}^{-\mathrm{i}\vec{\theta}\cdot\vec{q}} .$$

One needs to relate $d_j(\vec{\theta})$ with the Fourier coefficient $a_j(\vec{\theta})$ of $\vec{s}$. For this purpose, one associates with each Bloch vector $\vec{v}_j(\vec{\theta})$ the $\vec{\theta}$-periodic function $\varphi_j(x; \vec{\theta})$ which is the solution of

$$\begin{cases} \Delta\varphi_j(x;\vec{\theta}) = 0 & \text{in } C^* , \\ \dfrac{\partial\varphi_j}{\partial n}(x;\vec{\theta}) = \vec{n}\cdot\vec{v}_j(x;\vec{\theta}) & \text{on } \gamma , \end{cases}$$

and from the definition of $\vec{v}_j(\vec{\theta})$ and $\phi_j(x;\vec{\theta})$ (see Eq. (3.5)), we have

$$\varphi_j(x;\vec{\theta}) = \frac{\phi_j(x;\vec{\theta})}{\beta_j(\vec{\theta})} . \tag{3.9}$$

On the other hand, $\vec{s}$ is written as

$$\vec{s} = \sum_{j=1}^{2} \int \vec{s}_j(\vec{\theta})\, \mathrm{d}\theta$$

whose components on each γ_ℓ are

$$(\vec{s}_j)_\ell = a_j(\vec{\theta})\, \mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}}\, \vec{v}_j(\vec{\theta}) .$$

Now, we associate with such $\vec{s}_j(\vec{\theta})$, the vector $H\vec{s}_j(\vec{\theta})$ by means of the harmonic $\vec{\theta}$-periodic function $\psi(x;\vec{\theta})$ which satisfies on each γ_ℓ

$$\frac{\partial\psi}{\partial n}(x;\vec{\theta}) = (\vec{s}_j(\vec{\theta}))_\ell \cdot \vec{n} = a_j(\vec{\theta})\, \mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}}\, \vec{v}_j(\vec{\theta}) \cdot \vec{n} ,$$

and then

$$(H\vec{s}_j(\theta))_\ell = \int_{\gamma_\ell} \psi_j(x;\vec{\theta})\vec{n}\, \mathrm{d}\gamma_\ell .$$

It is clear, from (3.9), that

$$\psi_j(x;\vec{\theta}) = a_j(\vec{\theta})\, \mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}}\, \varphi_j(x;\vec{\theta}) = \frac{a_j(\vec{\theta})}{\beta_j(\vec{\theta})}\, \mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}}\, \phi_j(x;\vec{\theta})$$

in each cell C_ℓ. Consequently

$$(H\vec{s}_j(\vec{\theta}))_\ell = \frac{a_j(\vec{\theta})}{\beta_j(\vec{\theta})}\, \mathrm{e}^{\mathrm{i}\vec{\theta}\cdot\vec{\ell}}\, \vec{v}_j(\vec{\theta}) .$$

Thus, the spectral decomposition of H on the Bloch vectors $\vec{v}_j(\vec{\theta})$

$$\begin{cases} (H\vec{s})_\ell = \sum_{j=1}^{2} \int \frac{a_j(\vec{\theta})}{\beta_j(\vec{\theta})} e^{i\vec{\theta}\cdot\vec{\ell}} \vec{v}_j(\vec{\theta})\, d\theta , \quad \text{with} \\ a_j(\vec{\theta}) = \frac{1}{(2\pi)^2} \sum_{\vec{q}} \vec{s}_q \cdot \vec{v}_j(\vec{\theta})\, e^{-i\vec{\theta}\cdot\vec{q}} . \end{cases} \tag{3.10}$$

The relation (3.10) means that $H\vec{s}$ is decomposed on all the vibration eigenmodes of the tube array, characterized by the vectors $\vec{v}_j(\vec{\theta})$ (we note a certain analogy with the spectral decomposition of symmetric matrices).

Such an operator H is continuous in the following sense. If $\|\vec{s}\|^2 = \Sigma_\ell |\vec{s}_\ell|^2$ is finite ($\|\vec{s}\|^2$ represents, up to a multiplicative factor, the total kinetic energy of the tube bundle), there exists a positive constant c such that $\|\vec{s}\| \leq c\|\vec{s}\|$ (the smallest constant c satisfying this inequality is the norm of H). In fact, formulae (3.8) are established for $\vec{s}$ with a finite norm (see [15]).

The spectrum of H is real and included in the interval $[\lambda_{min}, \lambda_{max}]$ where

$$\begin{cases} \lambda_{min} = \min_j \min_{\vec{\theta}} \frac{1}{\beta_j(\vec{\theta})} , \\ \lambda_{max} = \max_j \max_{\vec{\theta}} \frac{1}{\beta_j(\vec{\theta})} . \end{cases} \tag{3.11}$$

Numerical calculations show that the values of $1/\beta_j(0)$ are inside this interval (see [1]). Remark that, from the general theory of selfadjoint operators, the norm of H is λ_{max}. It results from the continuity of $\beta_j(\vec{\theta})$ that the resonance spectrum of the coupled fluid-cylinder system is spread out in the interval $[\omega_{min}, \omega_{max}]$, where

$$\omega_{min} = \sqrt{\frac{k}{m + \rho\lambda_{max}}} , \qquad \omega_{max} = \sqrt{\frac{k}{m + \rho\lambda_{min}}} . \tag{3.12}$$

In fact, each eigenvalue $\beta_j(\vec{\theta})$ is associated to an eigenfrequency

$$\omega_j(\vec{\theta}) = \sqrt{\frac{k}{m + \rho/\beta_j(\vec{\theta})}} , \tag{3.13}$$

so that the frequency spectrum is continuous. The quantity $\rho/\beta_j(\vec{\theta})$ may be interpreted as the *modal added mass*. Numerical computations of the $\beta_j(\vec{\theta})$ have been published and can be found in [18,17,1].

3.3. Dynamical equations

Let $\vec{f}(t) = \text{col}(\vec{f}_\ell(t))$ be an external force applied to the infinite rod array. It induces a motion of the structure defined by the dynamics equation

$$(m + \rho H) \frac{d^2}{dt^2} \vec{s}(t) + k\vec{s}(t) = \vec{f}(t) . \tag{3.14}$$

The force $\vec{f}(t)$ is then written in terms of Bloch wave vectors

$$\vec{f}_\ell = \sum_{j=1}^{2} \int \varphi_j(\vec{\theta}; t)\, e^{i\vec{\theta}\vec{\ell}} \vec{v}_j(\vec{\theta})\, d\theta \tag{3.15}$$

with

$$\varphi_j(\vec{\theta}; t) = \frac{1}{(2\pi)^2} \sum_{\vec{q}} \vec{f}_q(t) \cdot \vec{v}_j(\vec{\theta})\, e^{-i\vec{q}\cdot\vec{\theta}} .$$

If $(a_j(\vec{\theta}; t))$ is the family of Fourier coefficients of $\vec{s}(t)$, then inserting (3.15) into (3.14) and using the spectral representation (3.10) of H yields after identification

$$\left(m + \frac{\rho}{\beta_j(\vec{\theta})}\right) \frac{\mathrm{d}^2}{\mathrm{d}t^2} a_j(\vec{\theta}; t) + k a_j(\vec{\theta}, t) = \varphi_j(\vec{\theta}; t), \quad j = 1, 2, \tag{3.16}$$

whose solution gives the time-behavior of the bundle. For zero initial conditions, we have

$$a_j(\vec{\theta}; t) = \frac{\omega_j(\vec{\theta})}{k} \int_0^t \sin[\omega_j(\vec{\theta})(t - \tau)]\vartheta_j(\vec{\theta}; \tau)\, \mathrm{d}\tau, \tag{3.17}$$

where $\omega_j(\vec{\theta})$ is the eigenfrequency corresponding to $\vec{\theta}$. Clearly, a resonance occurs when $\varphi_j(\vec{\theta}; t)$ is a sinusoidal function of time with pulsation equal to $\omega_j(\vec{\theta})$ for a set of values of $\vec{\theta}$ having a non-zero measure, in which case $\vec{s}(t) = \mathrm{O}(t)$ as $t \to +\infty$ with an oscillating behavior.

REMARK 3.1. Formulae (3.15) to (3.17) are valid if the $\vec{f}_\ell$'s satisfy the condition

$$\|\vec{f}\|^2 \equiv \sum_\ell |\vec{f}_\ell|^2 < \infty .$$

However, in the particular case where $\vec{f}_\ell = \vec{f}^0$, $\vec{f}$ is then a constant vector of infinite norm, and the corresponding response is $\vec{s}_\ell = \vec{s}^0$ for any ℓ (i.e. $\|\vec{s}^0\| = \infty$). The associated potential ϕ is periodic and

$$\phi(x, t) = \sum_{j=1}^{2} \chi_j(x) \frac{\mathrm{d}}{\mathrm{d}t} s_j^0(t)$$

where χ_j satisfies

$$\begin{cases} \Delta\chi_j = 0 & \text{in the fluid,} \\ \dfrac{\partial \chi_j}{\partial n}(x) = n_j(x) & \text{on each } \gamma_\ell, \\ \chi_j & \text{is periodic,} \end{cases}$$

and the dynamics equation is written as

$$(m + \rho H^0) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \vec{s}^0 + k\vec{s}^0 = \vec{f}^0,$$

where the matrix H^0 is constructed with the different terms $\int_{\gamma_0} \chi_j n_i \,\mathrm{d}\gamma_0$.

Now, suppose that $\vec{f}_\ell = \vec{f}^0_\ell + \vec{f}^1_\ell$ where $\vec{f}^0_\ell \equiv \vec{f}^0$ is constant and $\vec{f}^1_\ell$ is such that $\Sigma_\ell |\vec{f}^1_\ell|^2$ is finite. The response $\vec{s}$ is decomposed as $\vec{s} = \vec{s}^0 + \vec{s}^1$, where $\vec{s}^0$ and $\vec{s}^1$ correspond, respectively, to $\vec{f}^0$ and $\vec{f}^1$. The vector $\vec{s}^0$ is obtained via the matrix H^0 defined above while $\vec{s}^1$ is given via formulae (3.15) to (3.17). In other words, $\vec{s}(t)$ takes the form

$$\vec{s}(t) = \vec{s}^0(t) + \sum_{j=1,2} \int \vec{s}_j(\vec{\theta}, t)\, \mathrm{d}\theta .$$

Remark however that the Bloch decompositions of $\vec{f}^1$ and $\vec{s}^1$ contain contributions at the Bloch frequency $\vec{\theta} = \vec{0}$ which are related by the same relationship as that for $\vec{f}^0$ and $\vec{s}^0$.

3.4. Bloch waves along a plane boundary

An interesting case is when the infinite tube array is limited by a plane Γ parallel to the cylinder axes; the bundle is then contained inside a half space (see Fig. 3). The Bloch wave method has been adapted to this case in [4]. In this situation, one must consider cells containing a row of cylinders in the x_2-direction (normal to Γ), and the eigenmotions are searched in the form

$$\vec{s}_{\ell,m}(t) = \mathrm{e}^{\mathrm{i}(\ell\theta + \omega t)}\, \vec{\xi}_m(\theta), \tag{3.18}$$

in which ℓ is the row index and m is the tube index in row ℓ; θ is a one-dimensional parameter in $[0, 2\pi]$;

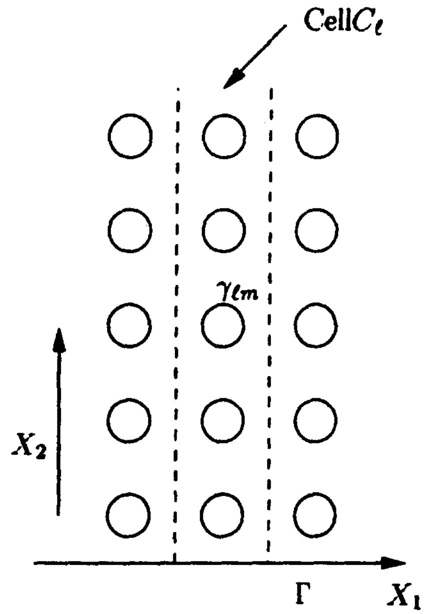

Fig. 3. Semi-infinite array of tubes.

$\vec{\xi}_m(\theta)$ is a two-dimensional vector depending only on θ (one considers in fact Bloch waves only in the x_1-direction).

We are interested only in bundles with finite kinetic energy; so we impose that $\Sigma_\ell \Sigma_m |\vec{s}_{\ell,m}|^2$ is finite, implying that $|\vec{s}_{\ell,m}| \to 0$ as $m \to \infty$ for each row ℓ.

Clearly, the corresponding flow potential is θ-periodic in x_1, i.e.

$$\phi(x_1 + \varepsilon, x_2; \theta) = e^{i\theta} \phi(x_1, x_2; \theta), \tag{3.19}$$

and from (3.18), it satisfies

$$\begin{cases} \Delta\phi(x;\theta) = 0 & \text{in the fluid domain,} \\ \dfrac{\partial \phi}{\partial n}(x;\theta) = i\omega \vec{\xi}_m(\theta)\cdot \vec{n} & \text{on each } \gamma_{\ell,m}, \\ (k - m\omega^2)\vec{\xi}_m(\theta) = -i\omega\rho \displaystyle\int_{\gamma_{\ell,m}} \phi(x;\theta)\vec{n}\, d\gamma_{\ell,m}, & \\ \phi(x;\theta) = 0 & \text{when } x_2 = 0, \\ \dfrac{\partial \phi}{\partial x_2}(x;\theta) \to 0 & \text{as } x_2 \to +\infty; \end{cases} \tag{3.20}$$

this last condition is a consequence of the finite kinetic energy assumption. In other words, we are interested by the eigenmotions concentrating along Γ (the tubes far from Γ do not vibrate).

There is a subtle point here. The eigenvalue problem (3.20) is posed in an infinite domain, and therefore, as is well known, the corresponding operator lacks compactness. This implies that its spectrum is made of, at most, a countable number of eigenvalues of finite multiplicity, and also of the so-called essential spectrum (for a precise definition of the essential spectrum we refer e.g. to [42,15]). Loosely speaking, an element of the essential spectrum is either an eigenvalue of infinite multiplicity, or it does not have a corresponding eigenvector but an infinite sequence of approximated eigenvectors (a so-called Weyl sequence). Such approximated eigenvectors correspond to vibrations of the bundle far away from the boundary (see [4]).

Eqs. (3.20) are written for $\ell = 0$ (for instance) and then, setting $\vec{z}_m = i\omega\vec{\xi}_m$

$$\phi(x;\theta) = \sum_{j=1}^{2} \chi_{mj}(x,\theta) z_{mj}(\theta) \tag{3.21}$$

where z_{mj} is the jth component of $\vec{z}_m$ and χ_{mj} is the solution of

$$\begin{cases} \Delta\chi_{mj}(x,\theta) = 0 & \text{in } C^*, \\ \dfrac{\partial\chi_{mj}}{\partial n}(x,\theta) = n_j(x)\delta_{mm'} & \text{on each } \gamma_{m'}, \\ \chi_{mj}(x,\theta) = 0 & \text{when } x_2 = 0, \\ \dfrac{\partial\chi_{mj}}{\partial x_2}(x,\theta) \to 0 & \text{as } x_2 \to +\infty, \\ \chi_{mj} & \text{is } \theta\text{-periodic in } x_1 \end{cases} \tag{3.22}$$

where $\delta_{mm'}$ is the Kronecker symbol and C^* is that part of C_0 occupied by the liquid; γ_m is the wall of the mth tube in the row C_0. For $\theta = 0$, it is necessary to have the additional condition

$$\int_{C^*} \chi_{mj}\,\mathrm{d}x = 0 .$$

In practical applications, it is enough to suppose that $\partial/\partial x_2 \chi_{mj} = 0$ for a sufficiently large value of x_2, for instance $x_2 = m_0\varepsilon$, m_0 being a positive integer (one only considers rows with m_0 cylinders).

Assuming that the functions $\chi_{mj}(x,\theta)$ have been computed, using (3.21), the relation (3.20) leads to the eigenvalue problem

$$z_{mi}(\theta) = \beta(\theta) \sum_{m',j} \left(\int_{\gamma_m} \chi_{m'j}(x,\theta) n_i(x)\,\mathrm{d}\gamma_m \right) z_{m'j}(\theta) , \tag{3.23}$$

$i = 1$ and 2, $m = 1, 2,$ etc. where $\beta(\theta) = \rho\omega^2/(k - m\omega^2)$.

The infinite matrix (or operator) $B(\theta)$ formed with the different integrals occurring in (3.23) is self-adjoint and positive definite. Considered as an operator in ℓ^2, the space of square summable sequences (or finite kinetic energy displacements), the infinite matrix $B(\theta)$ has positive eigenvalues $\beta(\theta)$. The spectrum of $B(\theta)$ is globally continuous with respect to θ, excepted at $\theta = 0$ [3].

Let $\beta_{mj}(\theta)$, $1 \leq m < +\infty$, $j = 1, 2$, be the eigenvalues and $\phi_{mj}(x,\theta)$ the corresponding eigenpotentials. We define the x_1-Bloch wave vectors by

$$\vec{v}_{mj}(\theta) = \mathrm{col}\left(\int_{\gamma_{m'}} \phi_{mj}(x,\theta)\vec{n}\,\mathrm{d}\gamma_{m'} \right) \tag{3.24}$$

and the eigenvectors may be chosen such that

$$\vec{v}_{mj}(\theta) \cdot \vec{v}_{m'j'}(\theta) = \delta_{mm',jj'} = \begin{cases} 1 & \text{if } m = m',\ j = j', \\ 0 & \text{otherwise}. \end{cases}$$

For any $\vec{s} = \mathrm{col}(\vec{s}_\ell)$, where $\vec{s}_\ell$ is the displacement-vector of the set of tubes of the row ℓ, it is decomposed as follows:

$$\begin{cases} \vec{s} = \displaystyle\sum_{j=1}^{2} \int_0^{2\pi} \vec{s}^{\,j}(\theta)\,\mathrm{d}\theta , \quad \vec{s}^{\,j}(\theta) = \mathrm{col}(\vec{s}^{\,j}_\ell(\theta)) \quad \text{where} \\ \vec{s}^{\,j}_\ell(\theta) = \displaystyle\sum_m a_j^m(\theta)\,\mathrm{e}^{\mathrm{i}\theta\ell}\vec{v}_{mj}(\theta) , \\ a_j^m(\theta) = \dfrac{1}{2\pi} \displaystyle\sum_{q=-\infty}^{\infty} \vec{s}_q \cdot \vec{v}_{mj}(\theta)\,\mathrm{e}^{-\mathrm{i}q\theta} , \end{cases} \tag{3.25}$$

$\vec{s}_q$ being the displacement-vector of the set of tubes of the row q.

It is then possible to express the added mass matrix H of the half infinite bundle by

$$(H\vec{s})_\ell = \sum_{j=1}^{2} \sum_m \int_0^{2\pi} \frac{a_j^m(\theta)}{\beta_{mj}(\theta)}\,\mathrm{e}^{\mathrm{i}\theta\ell}\vec{v}_{mj}(\theta)\,\mathrm{d}\theta \tag{3.26}$$

where ℓ denotes the row index.

Suppose now that the half tube array is subjected to external forces $\vec{f}(t)$ concentrated near Γ. These forces $\vec{f}(t)$

are therefore decomposed in Bloch waves, which leads to the following second order system of differential equations

$$\left[\left(m+\frac{\rho}{\beta_{mj}(\theta)}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}+k\right]a_j^m(\theta;t)=f_j^m(\theta;t) \tag{3.27}$$

where a_j^m and f_j^m are the Fourier coefficients of $\vec{s}$ and $\vec{f}$, respectively.

REMARK 3.2. In the case of an infinite bundle, the band of resonance frequencies may be easily obtained by only computing the functions $\chi_j(x,\vec{\theta})$ on the elementary cell (which is a small domain) and constructing the corresponding matrices $B(\vec{\theta})$ (which are of order two) whose eigenvalues and eigenvectors can be calculated in a trivial manner. Observe, from the previous section, that the spectrum of the half infinite array (limited by a plane) is bounded by the limits of the spectrum corresponding to the entire bundle. However, there may be new eigenvalues inside these limits. For a discussion on this point, see Subsection 5.4.

4. Classical homogenization

4.1. Preliminaries

In the previous section, it was supposed that the infinite tube array occupies the entire space or a half space. Here, we assume that the bundle has a large number of elements but its size is finite. The distance ε between adjacent cylinder centers is consequently small and our aim is to derive a simplified model allowing one to carry out numerical calculations with reasonable cost. This can be done thanks to the homogenization theory which amounts to replacing the fluid-tube system by an homogeneous medium for which homogenized equations have to be found. Homogenization has been intensively studied by Babuška, Bensoussan et al. [6], Sánchez-Palencia et al. [43,42], and many other authors. Applications of this method to fluid–structure interaction have been performed by the authors of this paper during these last few years [35,16,15,3].

4.2. The homogenized equations

These equations are obtained by means of the standard multi-scale asymptotic expansions. The tube bundle is still assumed to be a regular rectangular network of cylinders with spatial period ε (the period is the same in x_1 and x_2 directions, but one could also consider different periods). Because ε is small compared to the size of the bundle, the flow potential is expected to have small fluctuations around a mean value ϕ_0, of the order of ε with a spatial period equal to ε. Then, we a priori set

$$\phi=\phi_0(x,t)+\varepsilon\phi_1(x,y,t)+\varepsilon^2\phi_2(x,y,t)+\cdots, \tag{4.1}$$

where $y=x/\varepsilon$ and the ϕ_k are periodic, of period 1, with respect to y, $\phi_k(x,y+\vec{e}_i,t)=\phi_k(x,y,t)$, for $i=1,2$ and any x, y, t. Now, x is the macroscopic variable (roughly x indicates the position of a tube inside the bundle) and y is the microscopic variable which describes the microscopic geometry near each cylinder. The principal role of the variable y is to model the local periodic variations of the flow generated by the presence of the rods.

In writing (4.1), we a priori suppose that the displacements of any two adjacent cylinders differ only slightly from each other, and thus we can replace the set of tube motions $\vec{s}_\ell$ by a continuous function $\vec{s}(x,t)$ which slowly varies with respect to x. More precisely, the family $(\vec{s}_\ell)$ is extended to a function $\vec{s}(x,t)$ such that each $\vec{s}_\ell$ is given by $\vec{s}_\ell\equiv\vec{s}(x_\ell,t)$ where x_ℓ is the center of γ_ℓ.

It is convenient to introduce for $\vec{s}$ an asymptotic expansion in ε

$$\vec{s}=\vec{s}_0(x,t)+\varepsilon\vec{s}_1(x,t)+\varepsilon^2\vec{s}_2(x,t)+\cdots \tag{4.2}$$

in which the $\vec{s}_k$ are smooth functions of x. Note that y is not present in (4.2) because the cylinder displacements are rigid. Thus, the first terms ϕ_0 and $\vec{s}_0$ will describe the bulk behavior of the fluid and of the mechanical structure.

When applied to each $\phi_k(x, y, t)$, with $y = x/\varepsilon$, the derivative with respect to x_j has to be replaced by $\partial/\partial x_j + \varepsilon^{-1}\,\partial/\partial y_j$ from the rule of derivation for composed functions. Hence

$$\Delta = \Delta_x + 2\varepsilon^{-1}\Delta_{xy} + \varepsilon^{-2}\Delta_y$$

where

$$\Delta_x = \sum_i \frac{\partial^2}{\partial x_i^2}, \qquad \Delta_{xy} = \sum_i \frac{\partial^2}{\partial x_i\,\partial y_i}, \qquad \Delta_y = \sum_i \frac{\partial^2}{\partial y_i^2},$$

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial n_x} + \varepsilon^{-1}\frac{\partial}{\partial n_y} \quad \text{with} \quad \frac{\partial}{\partial n_x} = \sum_i n_i \frac{\partial}{\partial x_i}$$

and an obvious similar definition for $\partial/\partial n_y$; n_i is the ith direction-cosine of the normal $\vec{n}$.

Introducing these definitions into $\Delta\phi = 0$, using (4.1), (4.2) and after identifying the different terms in ε^{-2}, ε^{-1}, etc. one obtains

$$\begin{cases} \Delta_y \phi_0 = 0\,, \\ \Delta_y \phi_1 = -2\Delta_{xy}\phi_0\,, \\ \Delta_y \phi_2 = -\Delta_x \phi_0 - 2\Delta_{xy}\phi_1\,, \quad \text{etc.}, \end{cases} \tag{4.3}$$

Since ϕ_0 does not depend on y the first equation above is obviously satisfied.

Now, some care must be considered for using the boundary condition for ϕ on each tube and, for that, it is necessary to take into account the gradient of $\vec{s}$ on each cell. Because $\vec{s}(x_\ell, t)$ represents the displacement of the tube placed at x_ℓ, the boundary condition on each γ_ℓ can be written as

$$\frac{\partial \phi}{\partial n}(x, t) = \frac{\partial}{\partial t}\vec{s}(x_\ell, t)\cdot \vec{n}(x) \quad \text{for } x \in \gamma_\ell\,,$$

and $\vec{s}(x_\ell, t)$ may be expressed by

$$\vec{s}(x_\ell, t) = \vec{s}(x, t) - \nabla\vec{s}(x, t)\cdot(x - x_\ell) + O(\varepsilon^2)\,.$$

Then, setting $x - x_\ell = \varepsilon y$ and introducing (4.2) in the Taylor expansion of $\vec{s}$, the boundary condition on γ_ℓ gives successively via the identification of the different powers of ε

$$\begin{cases} \dfrac{\partial \phi_0}{\partial n_y} = 0\,, \\ \dfrac{\partial \phi_1}{\partial n_y} = -\dfrac{\partial}{\partial n_x}\phi_0 + \dfrac{\partial \vec{s}_0}{\partial t}\cdot \vec{n}\,, \\ \dfrac{\partial \phi_2}{\partial n_y} = -\dfrac{\partial \phi_1}{\partial n_x} + \dfrac{\partial \vec{s}_1}{\partial t}\cdot \vec{n} - \dfrac{\partial}{\partial t}(\nabla\vec{s}_0(x, t)y)\cdot \vec{n}\,, \quad \text{etc.}\ldots \end{cases} \tag{4.4}$$

The relations (4.3) furnish the partial differential equations, with respect to y, which must be satisfied by ϕ_0, ϕ_1, ϕ_2, etc. on the part Y^* of the enlarged cell Y (in the ratio $1/\epsilon$) occupied by the fluid. The relations (4.4) give the corresponding boundary conditions on the enlarged boundary γ of the tube in the cell Y. The functions ϕ_1, ϕ_2 must satisfy a periodicity condition in y. Grouping two by two Eqs. (4.3) and (4.4), it follows that ϕ_1 is of the form

$$\phi_1(x, y, t) = \sum_{i=1}^{2} \chi_i(y)\left(\frac{\partial s_{0,i}}{\partial t}(x, t) - \frac{\partial \phi_0}{\partial x_i}(x, t)\right), \tag{4.5}$$

in which $\chi_i(y)$ is the solution of

$$\begin{cases} \Delta_y \chi_i(y) = 0 & \text{in } Y^*\,, \\ \dfrac{\partial}{\partial n}\chi_i(y) = n_i(y) & \text{on } \gamma\,, \\ \chi_i(y) & \text{is periodic of period 1 in both directions } y_1 \text{ and } y_2. \end{cases} \tag{4.6}$$

Introducing the expansion (4.5) into the third equations (4.3) and (4.4), we obtain a partial differential equation (with respect to the variable y) for ϕ_2 on Y^*, with a boundary condition on γ. The compatibility condition for this differential condition with its boundary condition leads to the relation

$$-|Y^*|\Delta_x\phi_0 + \sum_{i,j}\int_{Y^*}\frac{\partial\chi_i}{\partial y_j}(y)\,\mathrm{d}y\,\frac{\partial^2\phi_0}{\partial x_i\,\partial x_j} = \sum_{i,j}\int_{Y^*}\frac{\partial\chi_i}{\partial y_j}(y)\,\mathrm{d}y\,\frac{\partial^2 s_{0i}}{\partial t\,\partial x_j}$$

($|Y^*|$ = area of Y^*) or

$$A\phi_0(x,t) = \sum_{i,j} b_{ij}\frac{\partial^2 s_{0i}}{\partial t\,\partial x_j}(x,t) + |\bar{\gamma}|\,\mathrm{div}_x\frac{\partial\vec{s}_0}{\partial t}(x,t)\,, \tag{4.7}$$

where

$$A \equiv \sum_{i,j} a_{ij}\frac{\partial^2}{\partial x_i\,\partial x_j}\,, \qquad a_{ij} = -|Y^*|\delta_{ij} + b_{ij}\,,$$

$$b_{ij} = \int_{Y^*}\frac{\partial\chi_i}{\partial y_j}\,\mathrm{d}y\,, \qquad |\bar{\gamma}| = \text{area of the interior of } \gamma.$$

The reader may find the details of these calculations in [16,15]. The matrix B of coefficients b_{ij} is symmetric and positive definite. Moreover, the central symmetry of the square-shaped elementary cell implies that $b_{12} = b_{21} = 0$ and $b_{11} = b_{22} = b > 0$. On the other hand, $a_{11} = a_{22} = -\alpha < 0$, so that the homogenized equation (4.7) may be rewritten as

$$-\alpha\Delta_x\phi_0(x,t) = (b + |\bar{\gamma}|)\,\mathrm{div}_x\frac{\partial\vec{s}_0}{\partial t}(x,t)\,. \tag{4.8}$$

Now, let us turn back to the dynamics equation

$$\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k\right)\vec{s}_\ell = -\rho\int_{\gamma_\ell}\frac{\partial\phi}{\partial t}\vec{n}\,\mathrm{d}\gamma_\ell + \vec{f}_\ell\,,$$

for each cylinder γ_ℓ; $\vec{f}_\ell$ is an external force acting on γ_ℓ. It is convenient to suppose that $m = m_0\varepsilon^2$, $k = k_0\varepsilon^2$, $\vec{f}_\ell = \varepsilon^2\vec{f}_0(x,t)$ (this point will be discussed in the following section).

The asymptotic expansions of ϕ and $\vec{s}$ and the expression of ϕ_1, after being inserted in the dynamics equation, lead to the homogenized equation (see [16]).

$$\left[(m_0 + \rho B)\frac{\partial^2}{\partial t^2} + k_0\right]\vec{s}_0(x,t) = \rho(B + |\bar{\gamma}|)\nabla_x\frac{\partial\phi_0}{\partial t}(x,t) + \vec{f}_0(x,t) \tag{4.9}$$

where B is defined above ($B = bI$ in case of central symmetry).

Thus, Eqs. (4.7), (4.9), whose coefficients do not depend on x, describe the global behavior of the structure. They obviously hold on the homogenized domain Ω_{hom} which is occupied by both the fluid and the tubes.

4.3. Boundary condition for ϕ_0

The flow potential satisfies $\partial\phi/\partial n = 0$ on Γ which is the wall of the cavity containing both the fluid and the mechanical structure (it is reminded that Γ is rigid) and this boundary condition is true at the microscopic scale. But the first term ϕ_0 does not satisfy a boundary condition of the same type. Suppose indeed that the tubes move towards Γ, the mass conservation law leads one to think that this motion generates a flow in the opposite direction, in other words, that there exists a linear relation between $\partial\phi_0/\partial n$ and $\partial\vec{s}_0/\partial t$ at the macroscopic scale on the boundary Γ.

In truth, the boundary condition is implicitly contained in the variational formulation of our problems. Homogenizing the variational equation automatically gives the homogenized boundary condition. The details, too long to be presented here, may be found in [16].

The result is then

$$\frac{\partial \phi_0}{\partial n_A} = (B + |\bar{\gamma}|)\frac{\partial \vec{s}_0}{\partial t} \cdot \vec{n} \quad \text{on } \Gamma , \tag{4.10}$$

where $\partial / \partial n_A = \Sigma_{i,j}\, a_{ij} n_j\, \partial / \partial x_i$ (conormal derivative), and in the case where we have central symmetry

$$\alpha \frac{\partial \phi_0}{\partial n} = -(b + |\bar{\gamma}|)\frac{\partial \vec{s}_0}{\partial t} \cdot \vec{n} \quad \text{on } \Gamma . \tag{4.11}$$

Let us mention that the homogenized equation (4.7) and its corresponding boundary condition (4.10) have also been derived in [3] by using the two-scale convergence method recently introduced by Nguetseng [31] and Allaire [2]. The main interest in the approach of [3] is that is gives a rigorous proof of convergence for the homogenization process, as well as a detailed study of the spectrum of the homogenized equation (see Subsection 4.5 below). The reader can easily check that (4.10) is compatible with (4.7) by integrating this last equation on the homogenized domain Ω_{hom} and using the Green identity. So, Eqs. (4.7), (4.9) and (4.10) form a well-posed problem when initial values for $\vec{s}_0$ and $\partial \vec{s}_0 / \partial t$ are prescribed (see [16]). It is noted that there is no boundary condition for $\vec{s}_0$.

REMARK 4.1. In certain situations, the cylinders are linked by a system of springs (spacers) that leads to a dynamics homogenized equation containing a differential operator with respect to x (generally of second order) applied to $\vec{s}_0$, requiring then a boundary condition for this variable. This interesting case is discussed in [35,16,7].

4.4. The homogenized added mass operator

We consider only the case of cells with central symmetry, and let us turn back to (4.8) and (4.11)

$$\begin{cases} -\alpha\, \Delta\phi_0 = b_\gamma \operatorname{div} \dfrac{\partial \vec{s}_0}{\partial t} & \text{in } \Omega_{\text{hom}} , \\ \alpha \dfrac{\partial \phi_0}{\partial n} = -b_\gamma \dfrac{\partial \vec{s}_0}{\partial t} \cdot \vec{n} & \text{on } \Gamma , \end{cases}$$

where the subscript x is omitted and

$$b_\gamma = b + |\bar{\gamma}| .$$

Remark that these equations determine ϕ_0 up to an additive arbitrary constant, so that one imposes the condition $\int_{\Omega_{\text{hom}}} \phi_0 \, dx = 0$. The potential ϕ_0 may be expanded in terms of $\partial \vec{s}_0 / \partial t$ via the Green function $g(x; \xi)$ of the operator $-\alpha\Delta$ with Neumann boundary condition. This Green function is expressed with the eigen-elements of the Laplacian operator

$$g(x; \xi) = \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{1}{\mu_k} w_k(x) w_k(\xi)$$

where $\mu_k > 0$, $\|w_k\|_{L^2(\Omega_{\text{hom}})} = 1$, and

$$\begin{cases} -\Delta w_k = \mu_k w_k & \text{in } \Omega_{\text{hom}} , \\ \dfrac{\partial w_k}{\partial n} = 0 & \text{on } \Gamma . \end{cases}$$

The Laplacian has also the eigenvalue $\mu_0 = 0$ with the eigenfunction $w_0(x) \equiv \text{cst.}$, but this eigen-element is not present in the above expression of $g(x; \xi)$. Since $\int_{\Omega_{\text{hom}}} w_k(x)\, dx = 0$ for $k > 0$, it results that

$$\int_{\Omega_{\text{hom}}} g(x; \xi)\, d\xi = 0 .$$

These things being remembered, we then have

$$\phi_0(x) = b_\gamma \int_{\Omega_{\text{hom}}} g(x;\xi)\, \text{div}_\xi \frac{\partial \vec{s}_0}{\partial t}(\xi, t)\, d\xi - b_\gamma \int_\Gamma g(x;\xi) \frac{\partial \vec{s}_0}{\partial t}(\xi, t)\cdot \vec{n}(\xi)\, d\Gamma_\xi . \tag{4.12}$$

Setting

$$-\mathcal{H}_0 \vec{s}_0(x,t) \equiv b_\gamma^2 \nabla_x \left[\int_{\Omega_{\text{hom}}} g(x;\xi)\, \text{div}_\xi \vec{s}_0(\xi, t)\, d\xi - \int_\Gamma g(x;\xi) \vec{s}_0 \cdot \vec{n}(\xi)\, d\Gamma_\xi \right]$$

one gets from (4.9)

$$[m_0 + \rho(b + \mathcal{H}_0)] \frac{\partial^2 \vec{s}_0}{\partial t^2} + k_0 \vec{s}_0 = \vec{f}_0 . \tag{4.13}$$

$\rho\mathcal{H} \equiv \rho(b + \mathcal{H}_0)$ is the *homogenized added mass operator*. The operator $\rho\mathcal{H}_0$ models the interaction of the different tubes via the fluid while ρb is the added mass of each cylinder in its own cell at x. The operator $\mathcal{H}$, so defined, is symmetric and positive definite, so that its spectrum is real and positive, from which it immediately results that the resonance ω-spectrum of the homogenized system is located between zero and $\omega_0 = (k/m)^{1/2}$.

4.5. The eigenfrequencies

This section is devoted to the computation of the eigenfrequencies of the homogenized system which has been first fully performed in [3] and [16]. From now on the cells are assumed to have a central symmetry (the unsymmetric case is more complicated, see [3]). We rewrite the coupled system (4.8), (4.9), (4.11), in terms of the eigenfrequency ω

$$\begin{cases} -\alpha\, \Delta\phi_0 = \mathrm{i}\omega b_\gamma \,\text{div}\, \vec{s}_0 & \text{in } \Omega_{\text{hom}}, \\ \alpha \dfrac{\partial \phi_0}{\partial n} = -\mathrm{i}\omega b_\gamma \vec{s}_0 \cdot \vec{n} & \text{on } \Gamma, \\ (k_0 - m_0'\omega^2)\vec{s}_0 = \mathrm{i}\omega\rho b_\gamma \nabla\phi_0 & \text{in } \Omega_{\text{hom}}, \end{cases} \tag{4.14}$$

where $m_0' = m_0 + \rho b_\gamma$.

Two cases have to be considered. If $k_0 - m_0'\omega^2 = 0$ with $\vec{s}_0 \neq 0$, then $\nabla\phi_0 \equiv 0$ and ϕ_0 is constant over Ω_{hom} which leads to $\text{div}\, \vec{s}_0 = 0$. The boundary condition implies $\vec{s}_0 \cdot \vec{n} \equiv 0$ on Γ. Then,

$$\omega_1 = \sqrt{\frac{k_0}{m_0 + \rho b_\gamma}} \tag{4.15}$$

is an eigenfrequency of infinite multiplicity: its eigenspace is made of all the divergence-free displacements $\vec{s}$ satisfying $\vec{s} \cdot \vec{n} = 0$ on Γ.

Suppose now $k_0 - m_0'\omega^2 \neq 0$. Plugging the third equation of (4.14) into the first one yields

$$-\alpha(k_0 - m_0'\omega^2)\, \Delta\phi_0 = -\omega^2 \rho b_\gamma^2\, \Delta\phi_0 .$$

If $\Delta\phi_0 = 0$, which is equivalent to $\text{div}\, \vec{s}_0 = 0$ and corresponds to an eigenvector of the previous eigenspace, we are back to the eigenfrequency ω_1. Therefore, a new eigenfrequency is obtained if we assume that $\Delta\phi_0 \neq 0$, or equivalently $\text{div}\, \vec{s}_0 \neq 0$. Then, necessarily

$$k_0 - \left(m_0' + \frac{\rho b_\gamma^2}{\alpha}\right)\omega^2 = 0 , \tag{4.16}$$

whose unique root is

$$\omega_2 = \sqrt{\frac{k_0}{m_0' + \dfrac{\rho b_\gamma^2}{\alpha}}} . \tag{4.17}$$

If (4.16) holds, $\vec{s}_0$ must be proportional to $\nabla\phi_0$, while $\vec{s}_0 \cdot \vec{n}$ may take any values on Γ. Therefore, ω_2 is also an eigenfrequency with infinite multiplicity, and its eigenspace is made of all $\vec{s}$ which are the gradient of some potential and have arbitrary values on Γ (for details, see [3]).

Thus, the spectrum of resonance of the homogenized tube bundle is formed of two points ω_1 and ω_2 with infinite multiplicity. The homogenization process has the effect of concentrating the spectrum in two points whereas the fluid–structure interaction spread out the eigenfrequencies over an interval when the number of cylinders is high, as is confirmed by numerical computation. This anomaly is due to our assumption that two neighbor rods have similar displacements. However, there may well be eigenmodes with different motions for two neighboring tubes. To avoid this difficulty, a method of homogenization by packets has been suggested in [35], and rigorously applied in [3], which considers unit cells containing several tubes instead of just one. However, in certain applications, like the effects of a seism on a nuclear reactor core, the standard homogenization may be sufficient, as it has been confirmed by comparison with experiments, with a relatively good agreement (see Hammami [23]), because the inertial force acting on the tubes is constant through the domain Ω_{hom}. Another experimental confirmation of homogenization may be found in [37] concerning the acoustical eigenfrequencies of heat exchangers.

REMARK 4.2. In certain situations where spacers are present between adjacent rods, the resonance spectrum may be infinitely discrete or continuous depending on the nature of the spacers (see [16,36,7]).

4.6. Homogenization by packets of tubes

Numerical calculations and experiments show that the cylinder bundles can vibrate with certain identical patterns in which the motions of two adjacent rods may be different at any instant. In order to take into account this possible situation, the elementary cell must contain several tubes ([35,3]) and it is necessary to have a displacement function $\vec{s}^{\ell}(x,t)$ for each tube of the cell (of size ε' small compared to the diameter of the bundle; ε' is a multiple of ε); x is the position of the cell. This obviously supposes that the external forces, acting on the rods, present a certain periodicity characterized by ε'. The second term of the asymptotic expansion of ϕ must then be written as

$$\phi_1(x,y,t) = \sum_{\ell} \sum_{j=1,2} \chi_j^{\ell}(y,t)\left(\frac{\partial s_{0,j}^{\ell}}{\partial t}(x,t) - \frac{\partial \phi_0}{\partial x_j}(x,t)\right), \tag{4.18}$$

where $\vec{s}_0^{\,\ell}$ is the first term of the asymptotic expansion of the displacement $\vec{s}^{\,\ell}$ of the tube γ^{ℓ} of the packet (i.e. the cell of size ε'); the summation on ℓ is done on this packet. Each function $\chi_j^{\ell}(y)$ is the solution of

$$\begin{cases} \Delta_y \chi_j^{\ell}(y) = 0 & \text{in } Y^*, \\ \dfrac{\partial \chi_j^{\ell}}{\partial n}(y) = n_j(y)\delta_{\ell\ell'} & \text{on each } \gamma^{\ell'} \text{ of the packet}, \\ \chi_j^{\ell}(y) & \text{is 1-periodic in } y \text{ with } \displaystyle\int_{Y^*} \chi_j^{\ell}(y)\,\mathrm{d}y = 0\,. \end{cases}$$

Using (4.18), the homogenized equation for the flow potential becomes, after some tedious calculations

$$A\phi_0 \equiv \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 \phi_0}{\partial x_i\, \partial x_j}(x,t) = \sum_{\ell,\ell'} \mathrm{div}_x (B^{\ell\ell'} + |\bar{\gamma}^{\ell'}|\delta_{\ell\ell'}) \frac{\partial \vec{s}_0^{\,\ell'}}{\partial t} \tag{4.19}$$

in which $B^{\ell\ell'}$ is the matrix with entries

$$b_{ij}^{\ell\ell'} = \int_{\gamma^{\ell}} \chi_j^{\ell'}(y) n_i(y)\,\mathrm{d}\gamma^{\ell}\,, \qquad a_{ij} = \sum_{\ell,\ell'} b_{ij}^{\ell\ell'} - |Y^*|\delta_{ij}\,,$$

$|\bar{\gamma}^{\ell}|$ = interior area of γ^{ℓ}, $A = \Sigma_{ij}\, a_{ij}\, \partial^2/(\partial x_i\, \partial x_j)$.

The associated boundary condition has then the form

$$\frac{\partial \phi_0}{\partial n_A} = \sum_{\ell,\ell'} (B^{\ell\ell'} + |\overline{\gamma}^{\ell'}|\delta_{\ell\ell'}) \frac{\partial \vec{s}_0^{\ell'}}{\partial t} \cdot \vec{n} \quad \text{on } \Gamma . \tag{4.20}$$

We note, in the right-hand sides of Eqs. (4.19) and (4.20), the interaction of the cylinders of the cells via the matrices $B^{\ell\ell'}$. For the dynamics equation, the first term of the asymptotic expansion of $\vec{s}^{\ell}$ satisfies

$$\left(m_0^{\ell} \frac{\partial^2}{\partial t^2} + k_0^{\ell}\right)\vec{s}_0^{\ell} = \rho\left[|\overline{\gamma}^{\ell}| \nabla_x \frac{\partial \phi_0}{\partial t} - \frac{\partial}{\partial t} \int_{\gamma^{\ell}} \phi_1(x, y, t)\vec{n}(y)\, \mathrm{d}\gamma^{\ell} \right] + \vec{f}_0^{\ell} .$$

Again, using (4.18) yields the homogenized dynamics equation

$$\frac{\partial^2}{\partial t^2}\left[m_0^{\ell}\vec{s}_0^{\ell} + \rho \sum_{\ell'} B^{\ell\ell'}\vec{s}_0^{\ell'} \right] + k_0^{\ell}\vec{s}_0^{\ell} = \rho(B^{\ell} + |\overline{\gamma}^{\ell}|)\nabla_x \frac{\partial \phi_0}{\partial t} + \vec{f}_0^{\ell} , \tag{4.21}$$

in which $B^{\ell} = \Sigma_{\ell'} B^{\ell\ell'}$. The matrix B^{ℓ} is symmetric and positive definite.

It is noted, in this approach, that the tubes inside a same cell can be different (different mass, stiffness and geometry) and placed in any position inside the cell, but all the packets must be identical.

Suppose now the tubes of each cell are identical. Replacing $\partial/\partial t$ by $\mathrm{i}\omega$ and eliminating ϕ_0, the following eigenfrequency problem is obtained (we do not present the details)

$$[k_0 - \omega^2(m_0 + \rho B_0 + \rho \mathscr{H}_0)]\vec{s}_0 = 0$$

where $\vec{s} = \mathrm{col}(\vec{s}^1, \vec{s}^2, \ldots, \vec{s}^n)$, n is the number of tubes in a cell. ρB_0 is the added mass matrix of the tube in a same packet, $\rho\mathscr{H}_0$ is an integro-differential operator (with respect to the macroscopic variable x) describing the interaction of the different packets. It can be shown that B_0 and $\mathscr{H}_0$ are symmetric and positive definite. This yields the well-known fact that the eigenfrequencies ω are located between zero and $\omega_0 = (k/m)^{1/2}$.

If one assembles the tubes by groups of p rows of q elements and p, q are varied, it can be expected, from this manner, to obtain all the eigenfrequencies but the amount of computation becomes high for large p and q, so that the advantages of the homogenization technique disappears. A solution to overcome this difficulty consists in combining homogenization and Bloch wave techniques. This will be the object of the next section.

4.7. A particular case

Suppose the bundle is formed by tubes of circular cross-sections. The homogenized equations are

$$\begin{cases} -\alpha\, \Delta\phi = b_\gamma \operatorname{div} \dfrac{\partial \vec{s}}{\partial t} , \\ \left[(m_0 + \rho b) \dfrac{\partial^2}{\partial t^2} + k_0 \right]\vec{s} = b_\gamma \nabla \dfrac{\partial \phi}{\partial t} + \vec{f}_0 , \end{cases} \tag{4.22}$$

with adequate boundary conditions on Γ.

In certain applications, the tubes are grouped in identical assemblies Ω_ℓ (of square cross – sections), and Ω is the union of such Ω_ℓ. The tubes of each assembly are linked by means of rigid grids, and consequently their motions are identical at any time. Such a situation occurs in the dynamical analysis of cores of pressurized water nuclear reactors (PWR): each Ω_ℓ is an assembly of fuel pencils.

This implies that ϕ is harmonic inside each assembly and consequently the first equation of (4.22) can be written on the common interface $\Sigma_{k\ell}$ of Ω_ℓ and Ω_k

$$-\alpha\, \Delta\phi = b_\gamma \left(\frac{\mathrm{d}\vec{s}_\ell}{\mathrm{d}t} \cdot \vec{n}_\ell + \frac{\mathrm{d}\vec{s}_k}{\mathrm{d}t} \cdot \vec{n}_k \right) \delta_{\Sigma_{k\ell}} \tag{4.23}$$

where $\delta_{\Sigma_{k\ell}}$ is the Dirac distribution on $\Sigma_{k\ell}$, $\vec{s}_\ell$ is the displacement of Ω_ℓ, $\vec{n}_\ell$ is the outward unit-normal on $\partial\Omega_\ell$. The dynamics equation of each assembly is, $|\Omega_\ell|$ denoting the area of Ω_ℓ;

$$|\Omega_\ell|\left[(m_0+\rho b)\frac{d^2}{dt^2}+k_0\right]\vec{s}_\ell=\rho b_\gamma\frac{d}{dt}\int_{\Omega_\ell}\nabla\phi\, dx+\int_{\Omega_\ell}\vec{f}_0\, dx\,. \tag{4.24}$$

The potential ϕ may be expressed as follows, N being the number of assemblies,

$$\phi(x,t)=\sum_{\ell=1}^{N}\sum_{j=1}^{2}\chi_{\ell j}(x)\frac{ds_{\ell j}}{dt}(t)\,,$$

where $\chi_{\ell j}$ satisfies

$$-\alpha\,\Delta\chi_{\ell j}=b_\gamma(\delta_{\Sigma^{(+)}_{\ell j}}-\delta_{\Sigma^{(-)}_{\ell j}})\,,$$

$\Sigma^{(+)}_{\ell j}$ and $\Sigma^{(-)}_{\ell j}$ are, respectively, the upstream and downstream faces of Ω_ℓ in the x_j-direction (δ is the Dirac distribution associated with the corresponding face); $\chi_{\ell j}$ must also satisfy certain boundary conditions on Γ which are left to the reader's sagacity [34].

Then, using the expansion of ϕ, we get a relation of the form

$$\left[(m_0+\rho b)\frac{d^2}{dt^2}+k_0\right]\vec{s}(t)=-\rho H\frac{d^2\vec{s}}{dt^2}+\vec{F}$$

where $\vec{s}=\mathrm{col}(\vec{s}_1,\vec{s}_2,\ldots,\vec{s}_N)$ and ρH is the (symmetric and positive definite) added mass matrix of all the assemblies.

5. Combination of standard homogenization and Bloch wave method

5.1. Preliminaries

Let us consider a regular tube network of step ε with ε small compared to the size of the entire domain. When the small parameter ε goes to zero, it is interesting to investigate the behavior of the added mass matrix $H\equiv H_\varepsilon$ (which obviously depends on ε) and its associated eigenfrequencies ω.

Let β^{-1} be the eigenvalues of the matrix H_ε. Their eigenpotentials ϕ satisfy the variational equation

$$a_\epsilon(\phi,v)=\beta\sum_\ell\int_{\gamma_\ell}\phi\vec{n}\, d\gamma_\ell\cdot\int_{\gamma_\ell}v\vec{n}\, d\gamma_\ell\equiv\beta\overrightarrow{N\phi}\cdot\overrightarrow{Nv}$$

for any test-function $v(x)$ such that $\int_{\Omega_\varepsilon}v\, dx=0$, is the fluid domain (it depends on ε), and

$$a_\varepsilon(\phi,v)\equiv\int_{\Omega_\varepsilon}\nabla\phi\cdot\nabla v\, dx\,.$$

The eigenfrequency ω of the fluid–structure system is related to β by

$$\beta=\frac{\rho\omega^2}{k-m\omega^2}\,.$$

Denoting by β_1^{-1} the highest eigenvalue of H_ε, there exist positive constants C and C' such that as $\varepsilon\to0$ (see Appendix A)

$$C\varepsilon^{-2}\geq\beta_1\geq C'\varepsilon^{-2}\,. \tag{5.1}$$

It results, since H_ε is a symmetric matrix, that its Euclidian norm is

$$\|H_\varepsilon\|=\beta_1^{-1}=O(\varepsilon^2)\,. \tag{5.2}$$

Suppose that m and k are left fixed, hence independent of ε. Clearly, since $\beta_1\to\infty$ as $\varepsilon\to0$, the eigenfrequencies tend to ω_0 which is the resonance frequency of the tubes in vacuum. Thus, as the tubes become finer and finer (and closer and closer), the added mass effects vanish because the fluid pressure force on each cylinder is then insignificant. Viewed from the fluid, each tube seems to be infinitely rigid and heavy. This contradiction shows that it is necessary to make m and k dependent of ε during the homogenization analysis.

Now suppose that

$$m = m_0\varepsilon^2 , \qquad k = k_0\varepsilon^2 . \tag{5.3}$$

Doing that, the ω's no longer converge to ω_0 and remain smaller than this frequency. As the tubes are finer and finer, thanks to (5.3), their eigenfrequency in vacuum remain equal to ω_0.

The first condition (5.3) is quite evident because the tube mass (per unit length) is of the order of the square of the cylinder radius r, hence of ε. Generally, k represents the stiffness coefficient associated with the first bending eigenmode of the tube, so that k is proportional to EI, where E is the Young modulus and I is the inertia moment with respect to a characteristic diameter of the tube cross-section. Because the tube is hollow with a thickness e (the tube generally contains a liquid or a gas which does not contribute to stiffness), I behaves consequently as $O(r^3e)$. As we impose that the ratio r/ε is constant which implies that $I = O(e_0\varepsilon^4)$ with $e = e_0\varepsilon$. The Young coefficient is often high (the tubes are made of steel) and so we can take $E = E_0\varepsilon^{-2}$, whence the second equality (5.3). These relations are justified by the fact, repeated again, that we are interested by computations done with the present value of ε. On the other hand, the external forces applied to the cylinders are generally proportional to their cross-section area, and we can therefore set

$$\vec{f} = \varepsilon^2\vec{f}_0 .$$

These things being reminded, the scaled added mass $S_{0\varepsilon} = \varepsilon^{-2}H_\varepsilon$ is introduced. This new matrix obviously depends on ε but it remains bounded from below and from above; it is then expected that the scaled added mass converges to a certain linear operator S. The spectral properties of the limit-operator has been studied using functional analysis arguments (see [3,4]); the proofs are done in the particular case where the flow potential is zero on the external boundary but they can be extended, with minor changes, to more realistic situations.

The main novelty of [3] is twofold. First, by using a mixture of two-scale convergence and Bloch waves, this paper furnishes a unified method which combines the advantages of the Bloch wave method (as described in Section 3) and the classical homogenization method (as described in Section 4). It allows to obtain in a single asymptotic analysis a limit spectrum of $S_{0\varepsilon}$ which is made of the previously known Bloch spectrum (characterized in Subsection 3.2) and the homogenized spectrum (characterized in Subsection 4.5). Second, a so-called completeness result is proved which states that the complete limit spectrum of $S_{0\varepsilon}$ is precisely made of these Bloch and homogenized spectra, plus a so-called boundary layer spectrum which corresponds to tubes which vibrate only in the vicinity of the boundary of Ω_{hom}. No other situation may occur: a limit vibration mode has to be either a homogenized mode, or a Bloch wave mode, or a boundary layer mode. Finally, the companion paper completely characterizes the boundary layer spectrum for a rectangular domain. Each part of the boundary is locally approximated as an infinite hyperplane. This allows to use Bloch waves along a plane boundary, as described in Subsection 3.4. The resulting boundary layer spectrum may contain new eigenfrequencies corresponding to tubes vibrating only close to the boundary (their displacements decay exponentially fast away from the boundary).

The reason for combining Bloch waves and homogenization is the following. Recall that the major interest of Bloch waves is that they reduce the original problem to the solutions of a family of similar problems on the elementary cell, which is a much smaller domain (leading to save numerical computational time and programming). As this method does not take into account the influence of the bundle boundary, the idea is to mix the Bloch wave decomposition with standard homogenization techniques in which boundary conditions on the external frontier are included. This will allow to split the force field acting on the rods into two distinct contributions: one leading to a homogenized uniform behavior of the bundle (where boundary effects are taken into account), and the other one yielding a quasi-periodic behavior of the bundle, made of a superposition of Bloch waves (which are indifferent to the boundary condition).

5.2. Homogenization process with $\vec{\theta}$-periodic functions

Because the bundle is large and periodic, it is natural to consider it as an infinite array at the region far from the physical boundary Γ. Consequently, by virtue of Section 3, we may intuitively assume that the flow potential is a superposition of $\vec{\theta}$-periodic functions (it may be justified, see [3]).

Suppose that the external forces $\vec{f}_\ell$ are zero in the cells near the boundary. These forces can be decomposed via the Bloch wave method

$$\vec{f}_\ell(t) = \int \vec{f}_\ell(\vec{\theta}, t)\, d\theta \quad (0 \leq \theta_i \leq 2\pi),$$

where the functions $\vec{f}_\ell(\vec{\theta}, t)$ are defined by relations of the same type as (3.25). Each $\vec{f}_\ell(\vec{\theta}, t)$ is in fact of the form $\vec{\varphi}(\vec{\theta})\, e^{i\vec{\theta}\cdot\vec{\ell}}$, where $\vec{\varphi}(\vec{\theta})$ does not depend on ℓ.

To each field $(\vec{f}_\ell(\vec{\theta}, t))$ corresponds a set of displacements $\vec{s}_\ell(\vec{\theta}, t)$ and a flow potential $\phi(x, \vec{\theta}, t)$ which is $\vec{\theta}$-periodic, whence the idea of computing these quantities by means of homogenization using quasi – periodic functions.

After replacing the field $\vec{s}_\ell(\vec{\theta}, t)$ by a smooth function $\vec{s}(x, \vec{\theta}, t)$, the following asymptotic expansions are introduced

$$\begin{cases} \phi = \phi_0(x, y, \vec{\theta}, t) + \varepsilon\phi_1(x, y, \vec{\theta}, t) + \varepsilon^2\phi_2(x, y, \vec{\theta}, t) + \cdots \\ \vec{s} = \vec{s}_0(x, \vec{\theta}, t) + \varepsilon\vec{s}_1(x, \vec{\theta}, t) + \varepsilon^2\vec{s}_2(x, \vec{\theta}, t) + \cdots \end{cases} \tag{5.4}$$

with $y = x/\varepsilon$ and each ϕ_k is $\vec{\theta}$-periodic in y, i.e.

$$\phi_k(x, y + \vec{e}_j, \vec{\theta}, t) = e^{i\theta_j}\phi_k(x, y, \vec{\theta}, t)$$

for any x, y, t, and $j = 1, 2$.
Plugging (5.4) into the set of equations (2.1), and identifying the different powers of ε, we obtain

$$\begin{cases} \Delta_y\phi_0 = 0 & \text{in } Y^*, \\ \dfrac{\partial\phi_0}{\partial n_x} = 0 & \text{on } \gamma, \end{cases} \tag{5.5}$$

$$\begin{cases} \Delta_y\phi_1 = -2\Delta_{xy}\phi_0 & \text{in } Y^*, \\ \dfrac{\partial\phi_1}{\partial n_y} = \dfrac{\partial\vec{s}_0}{\partial t}\cdot\vec{n} - \dfrac{\partial\phi_0}{\partial n_x} & \text{on } \gamma, \end{cases} \tag{5.6}$$

$$\begin{cases} \Delta_y\phi_2 = -2\Delta_{xy}\phi_1 - \Delta_x\phi_0 & \text{in } Y^*, \\ \dfrac{\partial\phi_2}{\partial n_y} = \dfrac{\partial\vec{s}_1}{\partial t}\cdot\vec{n} - \dfrac{\partial\phi_1}{\partial n_x} - \nabla_x\dfrac{\partial\vec{s}_0}{\partial t}(x, \vec{\theta}, t)\vec{y}\cdot\vec{n} & \text{on } \gamma, \end{cases} \tag{5.7}$$

where Y^* is the part of the unit cell Y occupied by the fluid and γ is the frontier of the cylinder.

The case $\vec{\theta} = 0$ has already been studied in Subsections 4.2 and 4.3. Hence, we consider the case $\vec{\theta} \neq 0$ in some detail in what follows. Eqs. (5.5), together with the $\vec{\theta}$-periodicity condition, show that ϕ_0 is constant with respect to the microscopic variable y, but a constant function cannot be $\vec{\theta}$-periodic when $\vec{\theta} \neq 0$. Consequently, ϕ_0 must be equal to zero.

Eqs. (5.6), with $\phi_0 = 0$, leads then to

$$\phi_1(x, y, \vec{\theta}, t) = \sum_{i=1}^{2} \chi_i(y, \vec{\theta}) \frac{\partial s_{0i}}{\partial t}(x, \vec{\theta}, t), \tag{5.8}$$

where $\chi_i(y)$ is the unique solution of

$$\begin{cases} \Delta_y\chi_i(y, \vec{\theta}) = 0 & \text{in } Y^*, \\ \dfrac{\partial\chi_i}{\partial n}(y, \vec{\theta}) = n_i(y) & \text{on } \gamma, \\ \chi_i \text{ is } \vec{\theta}\text{-periodic}. \end{cases} \tag{5.9}$$

It is easily seen that ϕ_2 is therefore of the following form

$$\phi_2(x, y, \vec{\theta}, t) = \mathcal{H}(y, \vec{\theta})\frac{\partial\vec{s}_0}{\partial t} + \vec{\chi}(y, \vec{\theta})\cdot\frac{\partial\vec{s}_1}{\partial t} \tag{5.10}$$

where $\vec{\chi} = \mathrm{col}(\chi_i)$ and $\mathscr{K}$ is a differential operator defined as follows:

$$\mathscr{K} \frac{\partial \vec{s}_0}{\partial t} = \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(\vec{\psi}_j(y, \vec{\theta}) \cdot \frac{\partial \vec{s}_0}{\partial t}(x, \vec{\theta}, t) \right)$$

in which $\vec{\psi}_j = \mathrm{col}(\psi_{ij})$ with ψ_{ij} solution of

$$\begin{cases} \Delta_y \psi_{ij}(y, \vec{\theta}) = -2 \dfrac{\partial \chi_i}{\partial y_j}(y, \vec{\theta}) & \text{in } Y^*, \\ \dfrac{\partial \psi_{ij}}{\partial n}(y, \vec{\theta}) = -n_j(y) \chi_i(y, \vec{\theta}) & \text{on } \gamma, \\ \psi_{ij} & \text{is } \vec{\theta}\text{-periodic}. \end{cases}$$

As customary, one sets

$$m = m_0 \varepsilon^2, \qquad k = k_0 \varepsilon^2, \qquad \vec{f}_\ell(\vec{\theta}, t) = \varepsilon^2 \vec{f}_0(x, \vec{\theta}, t)$$

and the dynamics equation gives after the standard identification process

$$\left(m_0 \frac{\partial^2}{\partial t^2} + k_0 \right) \vec{s}_0(x, \vec{\theta}, t) = -\rho \frac{\partial}{\partial t} \int_\gamma \phi_1(x, y, \vec{\theta}, t) \vec{n} \, \mathrm{d}\gamma + \vec{f}_0(x, \vec{\theta}, t) \tag{5.11}$$

$$\left(m_0 \frac{\partial^2}{\partial t^2} + k_0 \right) \vec{s}_1(x, \vec{\theta}, t) = -\rho \frac{\partial}{\partial t} \int_\gamma \phi_2(x, y, \vec{\theta}, t) \vec{n} \, \mathrm{d}\gamma \,. \tag{5.12}$$

The relations (5.11) and (5.8) yield the homogenized equation

$$\left[(m_0 + \rho B(\vec{\theta})) \frac{\partial^2}{\partial t^2} + k_0 \right] \vec{s}_0(x, \vec{\theta}, t) = \vec{f}_0(x, \vec{\theta}, t) \tag{5.13}$$

where $B(\vec{\theta})$ is the matrix of order two whose coefficients are

$$\int_\gamma \chi_j(y, \vec{\theta}) n_i(y) \, \mathrm{d}\gamma \,,$$

so that $B(\vec{\theta})$ is self-adjoint and positive definite. Actually, $\rho B(\vec{\theta})$ is the added mass of the cylinder in its own cell. Note that this matrix is the same as $B(\vec{\theta})$ defined in Subsection 3.1.

Note that the interaction of the different tubes does not explicitly appears in Eq. (5.13), but we must have in mind that $\vec{f}_0$ and $\vec{s}_0$ are dependent of $\vec{\theta}$; the coupling occurs when one integrates $\vec{s}_0$ with respect to $\vec{\theta}$. Eqs. (5.10) and (5.12) give

$$\left[(m_0 + \rho B(\vec{\theta})) \frac{\partial^2}{\partial t^2} + k_0 \right] \vec{s}_1(x, \vec{\theta}, t) = -\rho \int_\gamma \left(\mathscr{K}(y, \vec{\theta}) \frac{\partial^2 \vec{s}_0}{\partial t^2} \right) \vec{n} \, \mathrm{d}\gamma \,, \tag{5.14}$$

which is similar to (5.13). One sees that the coupling of the cells is explicitly present in (5.14) via the term $\mathscr{K}\vec{s}_0$ since $\mathscr{K}$ is a differential operator with respect to the macroscopic variable.

5.3. The eigenfrequencies ω

Setting $\vec{f}_0 = 0$ and replacing $\partial/\partial t$ by $\mathrm{i}\omega$ leads to the eigenfrequency problem in ω. We yet know that there exist only two eigenfrequencies ω_1 and ω_2 of infinite multiplicity when $\vec{\theta} = 0$ (see Eqs. (4.15) and (4.17)). The situation is quite different when $\vec{\theta}$ is different from zero. In this case, (5.14) yields

$$k_0 (m_0 + \rho B(\vec{\theta}))^{-1} \vec{s}_0(x, \vec{\theta}) = \omega^2 \vec{s}_0(x, \vec{\theta}) \tag{5.15}$$

and it is clear that ω is a *Bloch wave eigenfrequency* defined in Section 3 and there are two eigenfrequencies for each value of $\vec{\theta} \neq 0$. Since the spectrum of the matrix $B(\vec{\theta})$ depends continuously on $\vec{\theta}$, except at $\vec{\theta} = 0$, we have a continuous spectrum of Bloch wave eigenfrequencies.

Thus, the frequency spectrum of the operator S defined in Subsection 5.1 contains two eigenfrequencies ω_1

and ω_2 associated with $\vec{\theta}=0$ and resulting of the standard homogenization process, and a continuous Bloch wave spectrum corresponding to $\vec{\theta}\neq 0$. A rigorous proof of this result is given in [3].

As a consequence, and from the self-adjointness of the homogenized added mass operator, the response $\vec{s}(x,t)$ corresponding to the external forces $\vec{f}_\ell$ can be written, by analogy with Remark 3.1, as

$$\vec{s}(x,t)=\vec{s}_0(x,0,t)+\int \vec{s}_0(x,\vec{\theta},t)\,\mathrm{d}\theta\ , \tag{5.16}$$

in which $\vec{s}_0(x,0,t)$ is obtained from the classical homogenized equations while $\vec{s}_0(x,\vec{\theta},t)$ is the solution of the second-order differential equation (5.13). It is understood that the integral in (5.16) is taken on the square $[0,2\pi]^2$. The flow potential is given by

$$\phi(x,t)=\phi_0(x,0,t)+\varepsilon\sum_{i=1}^{2}\int \chi_i\Big(\frac{x}{\varepsilon},\vec{\theta}\Big)\frac{\partial s_{0,i}}{\partial t}(x,\vec{\theta},t)\,\mathrm{d}\theta+\mathrm{O}(\varepsilon^2) \tag{5.17}$$

in which $\phi_0(x,0,t)$ results from the classical homogenized equations.

REMARK 5.1. Let us denote by $\vec{\xi}_0(\vec{\theta})$ an eigenvector (of dimension two) of the matrix $k_0^{-1}(m_0+\rho B(\vec{\theta}))$ with the normalization $|\vec{\xi}_0(\vec{\theta})|=1$. As the matrix $B(\vec{\theta})$ does not depend on x, the same is true for $\vec{\xi}_0$. This implies that $\vec{s}_0(x,\vec{\theta})$ satisfying (5.15) may be written as

$$\vec{s}_0=\varphi(x)\vec{\xi}_0(\vec{\theta})\ ,$$

in which φ is an arbitrary function of x. Consequently, the eigenfrequencies have an infinite multiplicity.

5.4. Boundary layer frequency spectrum

It is proved in [4] that the fluid-tube bundle system has also a spectrum whose cylinder eigen-displacements are concentrated near the frontier Γ and it is called *boundary layer spectrum.* It may be excited by external forces acting on the tubes and concentrated near Γ. When Γ is piecewise parallel to the reference axes, this spectrum, denoted by σ_Γ, is explicitly characterized in [4] by means of the Bloch wave method along Γ described in the subsection 3.4 (replace $\mathrm{d}/\mathrm{d}t$ by $\mathrm{i}\omega$ and set $\vec{f}\equiv 0$); we refer to [4] for more details.

As already mentioned in Subsection 3.4, the boundary layer spectrum σ_Γ is made of two parts: a so-called essential spectrum $\sigma_\Gamma^{\mathrm{ess}}$, and a so-called discrete spectrum $\sigma_\Gamma^{\mathrm{disc}}$ made of finite multiplicity eigenvalues (which is at most countable, but may well be finite or empty). The essential spectrum is a classical mathematical notion defined e.g. in [42,15]. Loosely speaking, an element of the essential spectrum is either an eigenvalue of infinite multiplicity, or it is an 'almost' eigenvalue in the following sense: there exists an infinite sequence of 'almost' eigenvectors which satisfy the spectral equation up to a remainder term that goes to zero. In any case, each element of the essential spectrum is characterized by a so-called Weyl sequence of infinitely many approximated eigenvectors (see Theorem 3.27 in chapter 1 of [15], or Proposition 3.2 in chapter 4 of [42]). It is proved in Proposition 2.3.1 of [4] that such approximated eigenvectors correspond to vibrations of the bundle far away from the boundary, and therefore the essential spectrum $\sigma_\Gamma^{\mathrm{ess}}$ is contained in the previously known Bloch spectrum (described in Subsection 3.2). On the other hand, Proposition 2.3.5 in [4] shows that an eigenvector corresponding to an eigenvalue of finite multiplicity in σ_Γ decays to 0 exponentially fast away from Γ. Therefore, the only new contributions to the limit spectrum due to the boundary layer spectrum σ_Γ stem from eigenvectors concentrated on a few cells near the boundary Γ.

It is believed that generically $\sigma_\Gamma^{\mathrm{disc}}$ is not empty, but we shall prove below that in the special case of a symmetric cell $\sigma_\Gamma^{\mathrm{disc}}$ is indeed empty. Therefore, although interesting from a theoretical point of view, the study of the boundary layer spectrum σ_Γ is unnecessary in practice since for a symmetric cell $\sigma_\Gamma=\sigma_\Gamma^{\mathrm{less}}$ which is already characterized by the usual method of Bloch waves. This result does not mean that there does not exist eigenvectors concentrated near the boundary, but simply that their corresponding eigenfrequency belongs to the Bloch spectrum (remark that such an eigenvector is necessarily of infinite multiplicity since $\sigma_\Gamma^{\mathrm{disc}}$ is empty). In other words, taking into account the boundary Γ does not create new eigenfrequencies, although it may have an effect on the spatial distribution of the eigenvectors. Let us remark in passing that the numerical computations in

[26] strongly suggest that there are eigen-displacements concentrated on a band of a few cells thickness parallel to the boundary.

We now prove that $\sigma_\Gamma^{\text{disc}}$ is empty if the unit cell containing the rod is symmetric with respect to its principal axes (cubic symmetry). Assume the converse is true, i.e. there exists an eigenvalue of finite multiplicity ω and a corresponding non-zero eigen-displacement $(\vec{s}_m)_{m\geq 1}$ for each tube γ_m aligned in a semi-infinite band G orthogonal to the boundary Γ (see Fig. 4). To fix ideas, we assume that Γ is the axis $x_2 = 0$.

From Section 3.4 we already know that, by using Bloch waves in the single direction x_1, it is enough to consider a θ-periodic potential ϕ which is the solution of an equation similar to (3.20)

$$\begin{cases} \Delta\phi(x) = 0 & \text{in the fluid domain } x_2 \geq 0, \\ \dfrac{\partial \phi}{\partial n} = \mathrm{i}\omega \vec{s}_m \cdot \vec{n} & \text{on each } \gamma_m,\ m \geq 1, \\ (k - m\omega^2)\vec{s}_m = -\mathrm{i}\omega\rho \displaystyle\int_{\gamma_m} \phi(x)\vec{n}\, \mathrm{d}\gamma_m\,, & \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{when } x_2 = 0, \\ x_1 \to \mathrm{e}^{-\mathrm{i}\theta x_1}\phi(x_1, x_2) & \text{periodic of period 1}, \\ \dfrac{\partial \phi}{\partial x_2}(x) \to 0 & \text{as } x_2 \to +\infty, \end{cases} \tag{5.18}$$

where the last condition is a consequence of the decay at infinity of the eigenmotions $(\vec{s}_m)_{m\geq 1}$. Although the potential ϕ is only defined for $x_2 \geq 0$, we extend it to the whole space by defining

$$\phi(x_1, x_2) = \phi(x_1, -x_2) \quad \text{if } x_2 \leq 0\,.$$

It is readily checked that, by definition, ϕ is continuous through Γ, and that, since it satisfies a Neumann boundary condition on Γ, its gradient is also continuous through Γ.

We now define an image domain by symmetry with respect to the Γ axis. Let $(\gamma_{-m})_{m\geq 1}$ be the image of the tubes $(\gamma_m)_{m\geq 1}$ by this symmetry. Similarly, let $(\vec{s}_{-m})_{m\geq 1}$ be their corresponding displacements defined by

$$\vec{s}_{-m} \cdot \vec{e}_1 = \vec{s}_m \cdot \vec{e}_1 \quad \text{and} \quad \vec{s}_{-m} \cdot \vec{e}_2 = -\vec{s}_m \cdot \vec{e}_2 \quad \text{for } m \geq 1\,.$$

It is not difficult to check that the extended potential is a solution of

$$\begin{cases} \Delta\phi(x) = 0 & \text{in the fluid domain } -\infty < x_2 < +\infty, \\ \dfrac{\partial \phi}{\partial n} = \mathrm{i}\omega \vec{s}_m \cdot \vec{n} & \text{on each } \gamma_m,\ m \geq 1 \text{ and } m \leq -1 \\ (k - m\omega^2)\vec{s}_m = -\mathrm{i}\omega\rho \displaystyle\int_{\gamma_m} \phi(x)\vec{n}\, \mathrm{d}\gamma_m\,, & \\ x_1 \to \mathrm{e}^{\mathrm{i}\theta x_1}\phi(x_1, x_2) & \text{periodic of period 1}, \\ \dfrac{\partial \phi}{\partial x_2}(x;\theta) \to 0 & \text{as } |x_2| \to +\infty, \end{cases} \tag{5.19}$$

But note now that ϕ is a solution of the spectral equation in an infinite band $-\infty < x_2 < +\infty$ we can apply the Bloch wave decomposition in the x_2 direction as is done in Section 3. This yields that ω must be an eigenfrequency for at least one Bloch wave mode. Then, using the technique of Proposition 2.3.1 of [4], it is not difficult to construct from that Bloch wave eigenmode a Weyl sequence of approximated eigenvectors for ω in the semi-infinite band G. Consequently, this proves that ω belongs to the essential spectrum $\sigma_\Gamma^{\text{ess}}$. However, by definition the intersection of $\sigma_\Gamma^{\text{disc}}$ and $\sigma_\Gamma^{\text{ess}}$ is empty. Therefore, $\sigma_\Gamma^{\text{disc}}$ is itself empty.

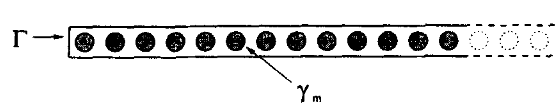

Fig. 4. Semi-infinite band G.

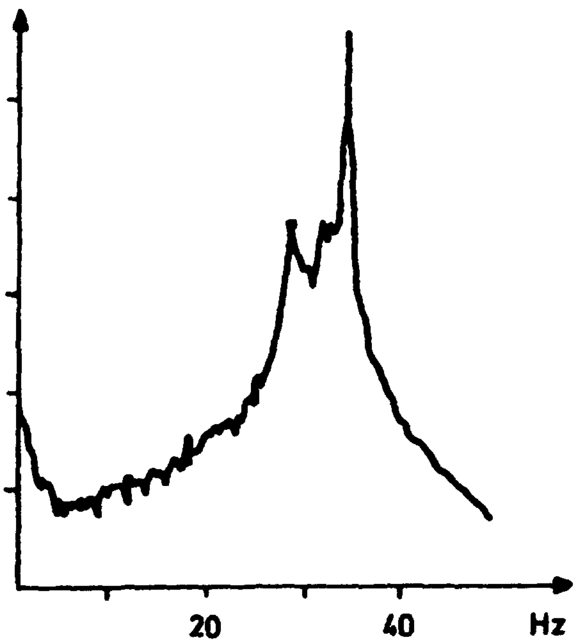

Fig. 5. Frequency response by Rémy and Campistron [41].

REMARK 5.2. The same argument of image domain obtained by symmetry with respect to the boundary Γ can be applied in the case of a Dirichlet boundary condition for the potential ϕ. Recall that $\phi = 0$ on Γ can be interpreted as a boundary condition on the fluid pressure.

5.5. Some additional comments

The display phenomenon of the eigenfrequencies of the tube bundles due to the presence of a fluid is confirmed by the experiments carried out, several years ago at the Electricité De France Research Center, by Rémy and Campistron [41]. These authors have studied the behavior of a group of 49 cylinders (7×7), elastically supported by means of piano strings (with a resonance frequency in vacuum equal to 40 Hz) and placed in a water crossflow. The bundle was excited by the turbulent forces generated by a high pressure drop between upstream and downstream. It is observed that Fig. 5 (taken out from [41]) presents several peaks at 27, 30 and 34 Hz, while the foot of the tube-response 'hill' is relatively wide, corresponding probably to the continuous Bloch wave spectrum.

On the other hand, when the 'exciting' force $\vec{f}$ is spatially constant through the domain Ω_{hom}, $\vec{f}$ has no contribution both on the Bloch wave vectors $\vec{v}_j(\vec{\theta})$ for $\vec{\theta} \neq 0$, corresponding to the interior of Ω_{hom} and the boundary layer. Consequently, classical homogenization can be used, as it is verified by the experiments done at the Atomic Energy Commission (CEA) Center of Saclay, France, see [23].

Before concluding, let us mention a homogenized model proposed by Shimogo and Shinohara [44], and recently considered again by Jacquelin et al. [27,28]. Their method leads to enclose into the homogenized added mass a new partial differential self-adjoint operator of second order. The presence of this differential operator has for effect to display the spectrum (which is an advantage), but it also introduces spurious eigenfrequencies; a part of these ones are filtered by the discretization process and consequently they do not seem to have a great influence on numerical results by comparison with experiments.

6. Conclusion

Several methods describing the global mechanical behavior of large tube banks immersed in a fluid have been presented, allowing to carry out computations with a low cost. In particular, in the case of symmetric rigid rods elastically mounted, it is shown that the limit resonance spectrum is formed of two eigenfrequencies with infinite multiplicity, and a continuous spectrum corresponding to the Bloch waves for an infinite spatially periodic structure. The Bloch wave method gives the lower and upper bounds of the eigenfrequencies in an easy manner, requiring only to solve elliptic partial differential equations on the elementary cell of the bundle.

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Appendix A. Proof of the inequalities (5.1)

The lowest eigenvalue β_1 of the coupled fluid-tube system is given by

$$\beta_1 = \min_v q(v)\,, \tag{A.1}$$

with

$$q(v) = \frac{\int_{\Omega_\varepsilon} |\nabla v|^2 \, dx}{\sum_\ell \left| \int_{\gamma_\ell} v\vec{n} \, d\gamma_\ell \right|^2}\,.$$

Let μ_1 and $v_1(x)$ be the first positive eigenvalue and the associated eigenfunction of the Laplacian on the homogenized domain Ω_{hom}, with Neumann boundary condition on Γ. Clearly, the following bound holds

$$\beta_1 \leq \frac{\int_{\Omega_\varepsilon} |\nabla v_1|^2 \, dx}{\int_{\Omega_\varepsilon} |v_1|^2 \, dx} \, \frac{\int_{\Omega_{\text{hom}}} |v_1|^2 \, dx}{\sum_\ell \left| \int_{\gamma_\ell} v_1 \vec{n} \, d\gamma_\ell \right|^2}\,. \tag{A.2}$$

From a homogenization result by Vanninathan [45],

$$\frac{\int_{\Omega_\varepsilon} |\nabla v_1|^2 \, dx}{\int_{\Omega_\varepsilon} |v_1|^2 \, dx} \rightarrow \frac{\mu_1}{d}\,, \tag{A.3}$$

where d is a positive number (actually $d = \alpha/|Y^*|$). On the other hand, for a smooth function v_1, a Taylor expansion around the center x_ℓ of each boundary γ_ℓ yields

$$\int_{\gamma_\ell} v_1 \vec{n} \, d\gamma_\ell = -\pi r_\varepsilon^2 \nabla v_1(x_\ell) + O(r_\varepsilon^3)\,,$$

where r_ε is the radius of γ_ℓ, and

$$\sum_\ell \left| \int_{\gamma_\ell} v_1 \vec{n} \, d\gamma_\ell \right|^2 = \frac{\pi^2 r_\varepsilon^4}{\varepsilon^2} \int_{\Omega_{\text{hom}}} |\nabla v_1|^2 \, dx + O(\varepsilon^3)\,. \tag{A.4}$$

If $r_\varepsilon = r_0 \varepsilon$, then (A.2), (A.3) and (A.4) imply

$$\beta_1 \leqslant \frac{1}{\pi^2 r_0^4 \varepsilon^2} + O(\varepsilon^2)\,. \tag{A.5}$$

It remains to prove a reverse inequality. Let ϕ_1 be the eigenfunction associated with β_1. By application of a result of Tartar [30], there exists an extension operator P_ε such that $P_\varepsilon\phi_1$ is defined on the homogenized domain $\Omega_{\rm hom}$, $P_\varepsilon\phi_1 \equiv \phi_1$ on Ω_ε and

$$\int_{\Omega_{\rm hom}} |\nabla P_\varepsilon \phi_1|^2 \, dx \leqslant C_0 \int_{\Omega_\varepsilon} |\nabla \phi_1|^2 \, dx\,, \tag{A.6}$$

where C_0 is a positive constant which does not depend on ϕ_1. This yields

$$\beta_1 = \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla \phi_1|^2 \, dx}{\displaystyle\sum_\ell \left| \int_{\gamma_\ell} \phi_1 \vec{n} \, d\gamma_\ell \right|^2} \geqslant \frac{1}{C_0} \frac{\displaystyle\int_{\Omega_{\rm hom}} |\nabla P_\varepsilon \phi_1|^2 \, dx}{\displaystyle\sum_\ell \left| \int_{\gamma} P_\varepsilon \phi_1 \vec{n} \, d\gamma_\ell \right|^2}\,,$$

and using an equality analogous to (A.4) for $\nabla P_\varepsilon \phi_1$, we obtain

$$\beta_1 \geqslant C_0' \varepsilon^{-2}\,, \tag{A.7}$$

where C_0' is another positive constant. Combining (A.5) and (A.7) leads to the desired inequalities.

References

[1] F. Aguirre and C. Conca, Eigenfrequencies of a tube bundle immersed in a fluid, Appl. Math. Optim. 18(1) (1988) 1–38.

[2] G. Allaire G., Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (1992) 1482–1518.

[3] G. Allaire and C. Conca, Bloch wave homogenization for a spectral problem in fluid–solid structures, Arch. Rat. Mech. Anal. 135(3) (1996) 197–257.

[4] G. Allaire and C. Conca, Boundary layers in the homogenization of a spectral problem in fluid–solid structures, SIAM J. Math. Anal. 29(2) (1998) 343, 379.

[5] I. Babuška, Solution of problems with interfaces and singularities, in: Mathematical Aspects of Finite Elements in Partial Differential Equations (Academic Press, New York, 1974) 213–277.

[6] A. Bensoussan, J.L. Lions amd G. Papanicolaou, Asymptotic Analysis for Periodic Structures (North-Holland, Amsterdam, 1978).

[7] S. Berete, J. Planchard and J. Sánchez-Hubert, Spectral properties of an elastic rod-spring system immersed in a fluid, Math. Models Methods Appl. Sci. 2(4) (1992) 441–460.

[8] R.D. Blevins, Flow-Induced Vibrations (Van Nostrand Reinhold, New York, 1977).

[9] L. Brillouin, Propagation of Waves in Periodic Structures (Dover, New York, 1953).

[10] S.S. Chen, Flow-Induced Vibration of Cylindrical Structures (Hemisphere Publishing Corporation, Washington, 1987).

[11] C. Conca, M. Durán and J. Planchard, The Bloch wave method and vibrations of elastic cylinder arrays in a fluid, J. Sound Vib. 170(4) (1994) 473–493.

[12] C. Conca, M. Durán and J. Planchard, A quadratic eigenvalue problem involving Stokes equations, Comput. Methods Appl. Mech. Engrg. 100 (1992) 295–313.

[13] C. Conca, A. Osses and J. Planchard, Asymptotic analysis relating spectral models in fluid-solid vibrations, SIAM J. Numer. Anal. (1997) in press.

[14] C. Conca, A. Osses and J. Planchard, Added mass and damping in fluid-structure interaction, Comput. Methods Appl. Mech. Engrg. 146 (1997) 387–405.

[15] C. Conca, J. Planchard and M. Vanninathan, Fluids and Periodic Structures, Collection RMA 38 (J. Wiley/Masson, New York, Paris, 1996).

[16] C. Conca, J. Planchard, B. Thomas and M. Vanninathan, Problèmes Mathématiques en Couplage Fluide-Structure: Applications aux Faisceaux Tubulaires, Collection des Etudes et Recherches d'Electricité de France (Eyrolles, Paris, 1994).

[17] C. Conca, J. Planchard and M. Vanninathan, Limit behavior of a spectral problem in fluid-solid structures, Asympt. Anal. 6(4) (1993) 365–389.

[18] C. Conca, J. Planchard and M. Vanninathan, Limits of the resonance spectrum of tube arrays immersed in a fluid, J. Fluids Struct. 4 (1990) 541–558.

[19] C. Conca, J. Planchard and M. Vanninathan, Existence and location of eigenvalues for fluid–solid structures, Comput. Methods Appl. Mech. Engrg. 77 (1989) 253–291.

[20] C. Conca and M. Vanninathan, A spectral problem arising in fluid–solid structures, Comput. Methods Appl. Mech. Engrg. 69 (1988) 215–242.

[21] R.J. Gibert, Vibrations des Structures, Interaction avec les Fluides, Sources d'Excitations Aléatoires, Collection des Etudes et Recherches d'Electricité de France (Eyrolles, Paris, 1988).

[22] D.J. Gorman, Experimental study of the flow-induced vibration of multi-span heat exchanger tube bundles in liquid cross-flow, in: Trans. 6th SMIRT Meeting (Paris, 1981).

[23] L. Hammami, Etude de l'interaction fluide-structure par une méthode d'homogénéisation: application à l'analyse sismique des coeurs RNR, Thesis, Université Pierre et Marie Curie, Paris VI, 1990).

[24] M. Ibnou-Zahir, Problèmes de valeurs propres avec des conditions aux limites non-locales, Thesis, Université Pierre et Marie Curie, Paris VI, 1984.

[25] M. Ibnou-Zahir and J. Planchard, Natural frequencies of a tube bundle in an incompressible fluid, Comput. Methods Appl. Mech. Engrg. 41 (1983) 47–68.

[26] M. Ibnou-Zahir and J. Planchard, Fréquences propres et modes propres d'un faisceau de 100 tubes supportés élastiquement et plongés dans un liquide, Electricité de France, Etudes et Recherches, Internal Report HI 4419-07, 1983.

[27] E. Jacquelin, Comportement sous sollicitations dynamiques des coeurs de réacteurs à eau pressurisée, Thesis, Ecole Centrale de Lyon, France, 1994.

[28] E. Jacquelin, J.P. Lainé and L. Jezéquel, Densité d'énergie d'une rangée de poutres immergées, C.R. Acad. Sci. Paris Série IIb, tome 323 (1996) 321–326.

[29] D. Mavriplis, M.P. Paidoussis and S.J. Price, A potential flow theory for the dynamics of cylinder arrays in cross-flow, J. Fluid Mech. 146 (1984) 227–272.

[30] F. Murat and L. Tartar, H-convergence, in: A. Cherkaev and R. Kohn, eds., Topics in the Mathematical Modelling of Composite Materials, pp. 21–44, Progress in Nonlinear Differential Equations and their Applications, 31, Birkhaüser, Boston (1997). (Original French version in Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger, mimeographed notes, 1978).

[31] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. 20 (1989) 608–623.

[32] M.P. Paidoussis, Fluidelastic vibration of cylinder arrays in axial and cross flow: state of the art, J. Sound Vib. 76(3) (1981) 329–360.

[33] M.J. Pettigrew, Flow-induced vibration phenomena in nuclear power station components, Power Indust. Res. (1981) 97–133.

[34] J. Planchard, Modèle de simulation dynamique des assemblages combustible d'un réacteur, Electricité de France, Etudes et Recherches, Internal Report 95NB 00043, 1994.

[35] J. Planchard, Global behavior of large elastic tube bundle immersed in a fluid, Comput. Mech. 2 (1987) 105–118.

[36] J. Planchard, Comportement dynamique des faisceaux de tubes immergés dans un fluide, in: Y. Bamberger, J. Donéa, H. Laval, J. Planchard and R.P. Shaw, eds., Aspects Théoriques et Numériques de la Dynamique des Structures, Collection des Etudes et Recherches d'Electricité de France (Eyrolles, Paris, 1988).

[37] J. Planchard, F.N. Rémy and P. Sonneville, A simplified method for determining acoustic and tube eigenfrequencies in heat exchangers, J. Pressure Vessel Technol. 104 (1982) 175–179.

[38] J. Planchard and B. Thomas, On the added mass, damping and stiffness matrices for an elastic structure placed in a potential cross-flow, in: J.M. Crolet and R. Ohayon eds., Computational Methods for Fluid–Structure Interaction, Pitman Research Notes in Mathematics Series (Longman, 1994).

[39] J. Planchard and B. Thomas, On the dynamical stability of tube bundles in cross-flow, J. Fluids Struct. 7 (1993) 321–339.

[40] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV Analysis of Operators (Academic Press, New York, 1978).

[41] F.N. Rémy and R. Campistron, Etude des vibrations induites par un écoulement d'eau sur un faisceau de tubes, Electricité de France, Etudes et Recherches, Internal Report HT/34-79-05, 1979.

[42] J. Sánchez-Hubert and E. Sánchez-Palencia, Vibration and Coupling of Continuous Systems. Asymptotic Methods (Springer Verlag, Berlin, 1989).

[43] E. Sánchez-Palencia, Non-Homogeneous Media and Vibration Theory, Lecture Notes in Physics 127 (Springer-Verlag, Berlin, 1980).

[44] T. Shimogo and Y. Shinohara, Vibrations of square and hexagonal cylinders in a liquid, J. Pressure Vessel Technol. 103 (1981) 233–239.

[45] M. Vanninathan, Homogenization of eigenvalue problems in perforated domains, Proc. Indian Acad. Sci. Math. 90 (1981) 239–271.