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Bloch-Wave Homogenization for a Spectral Problem in Fluid-Solid Structures

GRÉGOIRE ALLAIRE & CARLOS CONCA

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Abstract

This paper is concerned with the study of the vibrations of a coupled fluid-solid periodic structure. As the period goes to zero, an asymptotic analysis of the spectrum (i.e., the set of eigenfrequencies) is performed with the help of a new method, the so-called Bloch-wave homogenization method (which is a blend of two-scale convergence and Bloch-wave decomposition). The limit spectrum is made of three parts: the macroscopic or homogenized spectrum, the microscopic or Bloch spectrum, and the boundary-layer spectrum. The two first parts are completely characterized: The homogenized and the Bloch spectra are purely essential, and have a band structure. The boundary-layer spectrum is shown to be empty in the special case of periodic boundary condition.

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1. Introduction

1.1. Presentation of the main results

This paper is devoted to the asymptotic analysis of the spectrum of an elliptic operator defined in a periodic bounded domain whose period goes to zero. The motivation of such a problem is the study of the vibration frequencies of a coupled system of solid tubes immersed in a perfect incompressible fluid. A detailed description of this problem is given in the second part of this introduction. For the moment, we content ourselves with giving a brief statement of the problem, and we focus instead on the main tools and results obtained hereafter.

We consider a periodic bounded domain Ω_ε obtained from a fixed domain Ω by removing a collection of identical, periodically distributed holes $(T_p^\varepsilon)_{1 \leqq p \leqq n(\varepsilon)}$. The distance between adjacent holes, as well as their size, are both of the order of ε. Correspondingly, the number of holes $n(\varepsilon)$ is of the order of ε^{-N}, where N is the spatial dimension. The spectral problem we are interested in is to find the real eigenvalues λ_ε and the corresponding normalized eigenvectors u_ε that satisfy

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \lambda_\varepsilon \frac{\partial u_\varepsilon}{\partial n} &= \varepsilon^{-N} \boldsymbol{n} \cdot \int_{\partial T_p^\varepsilon} u_\varepsilon \boldsymbol{n}\, ds && \text{on } \partial T_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon), \\ u_\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\boldsymbol{n}$ denotes the exterior unit normal to Ω_ε. This model is due to J. PLANCHARD [27, 28], and it has already attracted the attention of several authors (see [1, 11, 12]). We call σ_ε the subset of $\mathbb{R}$ made of such eigenvalues λ_ε. A key feature of this model is that σ_ε is made of a finite number (proportional to the number $n(\varepsilon)$ of holes) of eigenvalues which are uniformly bounded away from zero and from infinity (see Proposition 1.2.1 and Lemma 1.2.2 below). As the period ε goes to zero, this spectrum σ_ε converges to a limit set $\sigma_\infty \subset \mathbb{R}$. The asymptotic analysis of this spectral problem is to find a characterization of the limit spectrum σ_∞.

To our knowledge, there are two methods to study the asymptotic behavior of the partial differential equation (1). As it is presented, the most obvious one is the *homogenization method* for periodic structures (see, e.g., [5, 6, 33]). However, it turns out that, in the present situation, it is also the most difficult to work out, and so far, no significant progress has been made in this direction. Consequently, the previous works of C. CONCA, M. VANNINATHAN, and their co-workers (see [1, 10, 12]), have used a second method, the so-called *Bloch-wave method*, also called the non-standard homogenization procedure in [11] (for other applications of this method see, e.g., [31]). The differences in the application of these two methods are easy to understand. In the homogenization process the overall domain Ω is kept fixed while the size ε of the microstructure goes to zero (see Figure 1). Thus, in the limit, the homogenized fluid domain is Ω where the tubes have disappeared, but their influence is still manifest in the effective (or homogenized) coefficients of the limit equation. On the contrary, for the Bloch-wave method, the

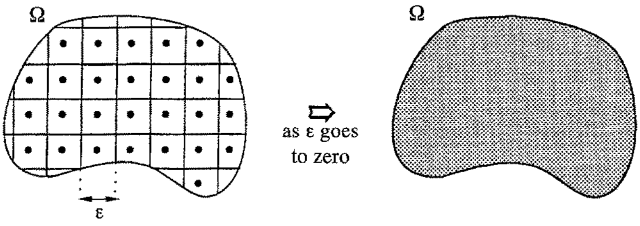

Fig. 1. Homogenization process

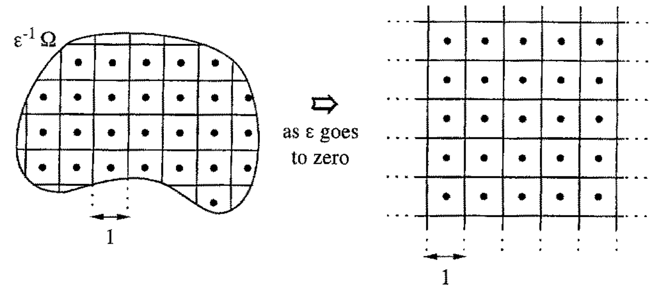

Fig. 2. Bloch-wave limit

ε-network of tubes is first rescaled to size 1. Thus, as ε goes to zero, the tubes remain fixed with a constant unit size, while the boundary of the fluid domain $\varepsilon^{-1}\Omega_\varepsilon$ goes to infinity. In the limit, the fluid domain is the entire space $\mathbb{R}^N$ minus an infinite periodic arrangement of unit tubes (see Figure 2). Then, the limit problem is amenable to the celebrated Bloch-wave decomposition (also known as the Floquet decomposition; see the original work of F. BLOCH [7], or the first mathematical proofs in [14, 24, 39], or the books [6, 29].

The first goal of this paper is to homogenize the spectral problem (1). This task is achieved in Section 2 by using the two-scale convergence method introduced by G. ALLAIRE [2] and G. NGUETSENG [23]. More precisely, we introduce a compact self-adjoint operator $\tilde{S}_\varepsilon$ acting on $L^2(\Omega)^N$, having characteristic values which coincide with the eigenvalues of (1). We prove that this operator $\tilde{S}_\varepsilon$ converges *strongly but not uniformly* to a non-compact self-adjoint operator S in $L^2(\Omega)^N$ (see Theorem 2.1.1). We characterize the spectrum $\sigma(S)$ of the homogenized operator S (see Theorems 2.1.4 and 2.1.5), but the "poor" convergence of the sequence $\tilde{S}_\varepsilon$ implies merely that

$$\sigma(S) \subset \sigma_\infty,$$

where the inclusion is generically strict. As a matter of fact, the spectrum $\sigma(S)$ of the homogenized problem does not coincide with the limit spectrum obtained by the Bloch-wave method in [12]. Therefore, the second goal of this paper is to understand this discrepancy between the "homogenized" and "Bloch" spectra, and to recover *all the limit set* σ_∞.

This goal is achieved in Section 3 by introducing a new method, called the *Bloch-wave homogenization method,* which is a blend of the two-scale convergence and of the Bloch-wave decomposition. This allows us to improve the analysis of Section 2 in the sense that we are able to recover simultaneously the homogenized and the Bloch spectra. Since we believe this new method is of intrinsic interest, we briefly give a flavor of its main underlying idea. The starting point is the definition of the "usual" two-scale convergence where the period of oscillations of the test functions is a multiple K of the unit cube $Y = [0, 1]^N$.

Theorem. *Let u_ε be a bounded sequence in $L^2(\Omega)$. Let K be any positive integer. There exist a subsequence (still denoted by ε) and a two-scale limit $u_0^K(x, y) \in L^2(\Omega \times KY)$ such that $u_\varepsilon(x)$* two-scale converges to *$u_0^K(x, y)$ in the sense that*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|KY|} \int_\Omega \int_{KY} u_0^K(x, y)\phi(x, y) dx\, dy, \tag{2}$$

for any function $\phi(x, y)$ in $\mathscr{D}[\Omega;\ C_\#^\infty(KY)]$.

Note that in the convergence (2) the test function ϕ is KY-periodic in the variable y. Since the choice of the integer K is arbitrary, it remains to find a relation between all the different two-scale limits u_0^K. This is done by introducing the so-called discrete Bloch-wave decomposition (see [1]).

Theorem. *Let u_ε be a sequence in $L^2(\Omega)$ which, for any integer K, two-scale converges to a limit $u_0^K(x, y) \in L^2(\Omega \times KY)$. There exists a countable family of complex-valued functions of $L^2(\Omega; L_\#^2(Y))$, denoted by $(u_{j/K}(x, y))_{0 \leqq j \leqq K-1 < +\infty}$ (where j is a multi-index, the components of which vary between* 0 *and* $K - 1$, *while K run in $\mathbb{N}$), such that, for any K,*

$$u_0^K(x, y) = \sum_{0 \leqq j \leqq K-1} u_{j/K}(x, y)\, e^{2\pi i \theta_j \cdot y}, \quad \textit{where } \theta_j = \frac{j}{K}.$$

Furthermore, the Parseval identity holds:

$$\frac{1}{|KY|} \int_\Omega \int_{KY} |u_0^K(x, y)|^2\, dx\, dy = \sum_{0 \leqq j \leqq K-1} \int_Y |u_{j/K}(x, y)|^2\, dx\, dy.$$

Remark that, as j and K vary over their range, the so-called Bloch frequencies j/K become dense in $[0, 1]^N$, and the above *discrete* Bloch-wave decomposition becomes very "close" to the usual *continuous* one. Of course, two Bloch components $u_{j/K}$ and $u_{j'/K'}$ coincide if their corresponding frequencies j/K and j'/K' are equal. Therefore, the Bloch components $u_{j/K}(x, y)$ actually depend on three variables: the macroscopic variable $x \in \Omega$, the microscopic $y \in [0, 1]^N$, and the Bloch variable $\theta_j = j/K \in [0, 1]^N$. It is easily seen that this approach combines the advantages of both the homogenization and the Bloch-wave method.

More precisely in Section 3, we introduce a compact self-adjoint operator S_ε^K acting on $L^2(\Omega \times KY)^N$, having characteristic values which coincide with the eigenvalues of the spectral problem (1). We prove that it converges *strongly* to a non-compact self-adjoint operator S^K in $L^2(\Omega \times KY)^N$ (see Theorem 3.2.1). As K goes to infinity, the spectrum of the limit operator S^K converges to the union of the previous homogenized spectrum $\sigma(S)$ and of the Bloch spectrum σ_{Bloch} (see (70)). However, the convergence of the sequence S_ε^K is still not uniform and we merely conclude that

$$\lim_{K \to +\infty} \sigma(S_K) = (\sigma(S) \cup \sigma_{\text{Bloch}}) \subset \sigma_\infty.$$

Having reconciled the Bloch-wave and the homogenization methods, we may hope that it is enough to completely characterize the limit spectrum σ_∞ of the spectral problem (1). It turns out that it is not exactly the case because there may exist some sequences of eigenvectors of (1) which concentrate near the boundary $\partial\Omega$ of the domain. These sequences, which behave like boundary layers, are captured neither by the homogenization method (since they converge to zero inside the domain), nor by the Bloch-wave method (since it does not take into account a possible interaction between the boundary and the periodic network of tubes). However, for any other type of sequences of eigenvectors (not concentrating on the boundary), the limits of the corresponding sequences of eigenvalues actually belong to the spectrum of the homogenized problem or to the Bloch spectrum. In other words, the main theorem of this paper is the following *completeness* result (see Theorem 3.2.9).

Theorem. *Denote by σ_{boundary} the set of all limits of sequences of eigenvalues such that the corresponding sequences of eigenvectors concentrate on the boundary $\partial\Omega$. The limit spectrum σ_∞ is precisely given by*

$$\sigma_\infty = \sigma(S) \cup \sigma_{\text{Bloch}} \cup \sigma_{\text{boundary}}.$$

In general, we are unable to characterize σ_{boundary}. We suspect its definition depends on the sequence of periods ε (a fact which is reminiscent of a recent work of F. Santosa & M. Vogelius [34]). However, when the domain is exactly made of an integral number of periodicity cells, we are able to adapt our Bloch-wave homogenization method to take care of the boundary layers. The analysis of σ_{boundary} in this case will be the focus of a future paper [3]. For the moment, we make the obvious remark that, if the domain Ω is a torus (i.e., a compact manifold without boundary), then there is no contribution of boundary layers in the limit spectrum.

Corollary. *Let Ω be a parallelpiped, $]0, L_1[\times]0, L_2[\times \cdots]0, L_N[$, where the $(L_p)_{1 \leq p \leq N}$ are positive integers. Define the sequence of periods $\varepsilon_n = 1/n$. Assume that the unit tube in the periodic cell has cubic symmetry, or replace the Dirichlet boundary condition in the spectral problem* (1) *by a periodicity condition. Then, the limit spectrum reduces to*

$$\sigma_\infty = \sigma(S) \cup \sigma_{\text{Bloch}}.$$

The proof of this completeness theorem is the focus of Section 3.4. It involves a new technical tool: the so-called Bloch and rescaled Bloch measures. Their purpose is to quantify the amount of oscillations for any sequence of eigenvectors of the spectral problem. They act as a filter on the length scale and the directions of propagation of the oscillations. The *Bloch measure* selects only the oscillations on the ε scale and distinguishes their corresponding Bloch frequencies. Although specific to the present context, it can be seen as a type of Wigner, or semi-classical, measure (see [16] and [20]). The *rescaled Bloch measure* is sensitive only to those oscillations which have a length scale larger than ε, and sorts them out according to their directions (in Fourier space). It is again specific to the present situation, but it obviously bears some resemblance to the H-measures of P. GÉRARD [15] and L. TARTAR [37].

Our analysis of the limit spectrum σ_∞ leaves open three important questions. The first is to characterize the "boundary layer" spectrum σ_{boundary}. As already said, this is the focus of our next paper [3] in the case of a domain built with entire periodic cells. However, for a general domain with a smooth boundary (not coinciding with the cell boundaries), we are helpless in the matter. The second question is to find the rate of convergence of σ_ε to σ_∞. The answer is unclear, and as before we suspect it depends on the form of the boundary $\partial\Omega$ and on the sequence of periods ε. Such effects occur in the case of the standard wave equation in a periodic domain, as recognized by F. SANTOSA & M. VOGELIUS [34]. The third open question is to understand the consequences of our spectral analysis in terms of the associated time-dependent problem (a wave-type equation; see (6) below). This is the topic of future research, and we hope it could shed some light on questions of geometrical optics. Let us also indicate that we have recently applied our Bloch-wave homogenization method to study the asymptotic behavior of the spectrum of the wave equation in a periodic domain [4].

We conclude this subsection by warning the reader that the first part of this introduction has deliberately been kept to a minimal size. Nevertheless, a detailed presentation and discussion of our results is available by simply reading in a first pass the Subsections 1.2, 2.1, 3.1, and 3.2 (the remaining subsections are devoted to the proofs). Specifically, Subsection 1.2 furnishes a complete description of the spectral problem (1). Subsection 2.1 gives the main results of the classical homogenization process for (1). Then, Subsection 3.1 introduces the Bloch waves, and Subsection 3.2 contains the final results of our Bloch-wave homogenization method, along with many additional comments (see Remarks 3.2.10 to 3.2.15).

1.2. Physical background and mathematical setting of the problem

We begin with the definition of the geometry of the fluid domain. As usual in periodic homogenization, we first define a unit cell, which, upon rescaling to size ε (a small positive parameter), becomes the period of a periodic domain. Let $Y = (0;1)^N$ be the unit cube and T be a smooth, simply connected, closed set, with a non-empty interior, *strictly included* in Y (i.e., such that T does not touch the boundaries of the cell Y). We call the set $Y^* = Y \setminus T$ the fluid cell, and the set T the reference hole (or rod).

Introducing a smooth, bounded, open set Ω in $\mathbb{R}^N$ and a sequence of positive parameters ε going to zero, for each value of ε we define a fluid domain Ω_ε obtained by removing from the reference domain Ω a collection of perforations εT distributed in a periodic manner with period εY. More precisely, we denote by (T_p^ε) the family of all translates of the hole εT by vectors εp (where p is a multi-index in $\mathbb{Z}^N$), and by (Y_p^ε) the corresponding family of cells. Analogously, (Γ_p^ε) denotes the corresponding family of boundaries $\varepsilon \partial T$. We consider only those cells which are strictly included in the domain Ω so that no hole meets the boundary $\partial\Omega$, and even more importantly, so that each hole lies at a distance larger than the order of ε from $\partial\Omega$. It is easily seen that the total number of holes $n(\varepsilon)$ is asymptotically equal to $\varepsilon^{-N}\,|\Omega|$. Then, the fluid domain is defined by

$$\Omega_\varepsilon = \Omega_\varepsilon \setminus \bigcup_{p=1}^{n(\varepsilon)} T_p^\varepsilon. \tag{3}$$

As we shall soon explain, the underlying physical problem of fluid-solid interactions is purely two-dimensional. Nevertheless, since from a mathematical point of view there is no conceptual difficulty in higher dimensions, we shall state all our results in any space dimension $N \geqq 2$. We consider the simplest model of vibrations of a tube-bundle immersed in a perfect incompressible fluid inside a cavity with a constant planar section Ω (this model is mainly due to J. PLANCHARD [27, 28]; more complex models are available in, e.g., [11]). A tube bundle is made of parallel tubes, long enough for three-dimensional effects to be ignored, which can move transversally under the action of the fluid pressure and of repelling forces created by some binding device. The holes T_p^ε represent the cross sections of the tubes and Ω_ε is the part of Ω occupied by the fluid.

When such a vibrating tube bundle is immersed in the fluid, both the tubes and the fluid vibrate. The result is a non-stationary partial differential equation in the fluid region coupled with a system of ordinary differential equations which represents the oscillations of the tubes. The fluid-solid interactions are taken into account by the coupling between the partial differential equation and the system of ordinary differential equations.

More precisely, in the model we study in this paper the tubes are also assumed to be rigid and only small oscillations of the fluid around the state of rest are allowed. The velocity $\boldsymbol{U}_{\varepsilon 0}$ of the fluid derives then from a potential function $u_{\varepsilon 0} = u_{\varepsilon 0}(x,t)$. Since the fluid is incompressible, its motion is governed by the Laplace equation

$$\Delta u_{\varepsilon 0} = 0 \quad \text{in } \Omega_\varepsilon \times \mathbb{R}. \tag{4}$$

The fluid is not allowed to escape the cavity and so $u_{\varepsilon 0}$ satisfies

$$\frac{\partial u_{\varepsilon 0}}{\partial n} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}. \tag{5}$$

On each Γ_p^ε, the normal velocity of the fluid should coincide with the normal component of the tube's velocity. Thus we have

$$\frac{\partial u_{\varepsilon 0}}{\partial n} = \frac{d\boldsymbol{r}_{0p}}{dt}\cdot\boldsymbol{n} \quad \text{on } \Gamma_p^\varepsilon \times \mathbb{R}, \quad p = 1, \ldots, n(\varepsilon),$$

where $\boldsymbol{n}$ is the outward unit normal on the boundary of Ω_ε and $\boldsymbol{r}_{0p}(t)$ is the transverse displacement vector at the instant t of the p-th tube which, due to the assumption of rigidity, depends only on t. It is furthermore assumed that there is no interaction between the tubes and that the motion of tube p is that of a simple harmonic oscillator with a forcing term modeled by its interaction with the fluid. Since the fluid is assumed to be perfect, this term depends only on the pressure $p_{\varepsilon 0}$ of the fluid. More exactly, $\boldsymbol{r}_{0p}$ satisfies the second-order ordinary differential equation

$$m\frac{d^2\boldsymbol{r}_{0p}}{dt^2} + k\boldsymbol{r}_{0p} = \int_{\Gamma_p^\varepsilon} p_{\varepsilon 0}(x,t)\boldsymbol{n}\, ds \quad \text{in } \mathbb{R}, \tag{6}$$

where m and k are two positive constants corresponding to the mass per unit length and the stiffness of the tubes, and $p_{\varepsilon 0}(x,t)$ is the pressure of the fluid at the point $x \in \Omega$ and at time t. Next, let us consider the Euler equation in order to link $p_{\varepsilon 0}$ and $u_{\varepsilon 0}$:

$$\frac{\partial \boldsymbol{U}_{\varepsilon 0}}{\partial t} + (\boldsymbol{U}_{\varepsilon 0}\cdot\nabla)\boldsymbol{U}_{\varepsilon 0} + \frac{1}{\rho}\nabla p_{\varepsilon 0} = 0,$$

where $\rho > 0$ is the density of the fluid. Given that only small oscillations are being considered, linearizing the Euler equation, we obtain the well-known Bernoulli relationship

$$p_{\varepsilon 0} = -\rho\frac{\partial u_{\varepsilon 0}}{\partial t} + c(t),$$

where $c(t)$ is an arbitrary constant (in space, not in time). As usual in vibration models, we seek sinusoidal solutions of the form

$$u_{\varepsilon 0}(x,t) = u_\varepsilon e^{i\omega_\varepsilon t}, \quad \boldsymbol{r}_{0p}(t) = \boldsymbol{r}_p e^{i\omega_\varepsilon t},$$

where ω_ε is the unknown (resonant) vibration frequency of the coupled system and i is the usual square root of -1. We can solve explicitly the ordinary differential equation for $\boldsymbol{r}_{0p}$ and get

$$\boldsymbol{r}_{0p}(t) = -\frac{i\rho\omega_\varepsilon e^{i\omega_\varepsilon t}}{k - m\omega_\varepsilon^2}\int_{\Gamma_p} u_\varepsilon \boldsymbol{n}\, ds.$$

It is now possible to eliminate the unknowns $\boldsymbol{r}_{0p}$. To simplify the notation, we define a rescaled frequency

$$\lambda_\varepsilon = \frac{k - m\omega_\varepsilon^2}{\varepsilon^N \rho\omega_\varepsilon^2}.$$

To avoid the problem of having the potential u_ε defined up to a constant, we replace the physical Neumann boundary condition (5) on $\partial\Omega$ by the more convenient Dirichlet one. As we shall see below (cf. Remark 2.1.7), this change has basically no influence on the results, and greatly simplifies the exposition. Then, $(\lambda_\varepsilon, u_\varepsilon)$ has to be a solution of the following spectral problem in Ω_ε

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \lambda_\varepsilon \frac{\partial u_\varepsilon}{\partial n} &= \frac{1}{|Y_p^\varepsilon|} \boldsymbol{n} \cdot \int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n}\, ds && \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon), \\ u_\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{7}$$

where $\boldsymbol{n}$ is the exterior unit normal on the boundary of Ω_ε.

To obtain the solutions of (7), we introduce a finite-dimensional operator S_ε, mapping the space $\mathbb{R}^{Nn(\varepsilon)}$ into itself, whose characteristic values coincide with the eigenvalues λ_ε of (7). A vector $\boldsymbol{s}$ in $\mathbb{R}^{Nn(\varepsilon)}$ is decomposed into components $(\boldsymbol{s}_p)_{1 \leqq p \leqq n(\varepsilon)}$, each $\boldsymbol{s}_p \in \mathbb{R}^N$ being associated with the corresponding hole T_p^ε. This operator S_ε is then defined by

$$S_\varepsilon \boldsymbol{s} = \left(\frac{1}{|Y_p^\varepsilon|} \int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n}\, ds \right)_{1 \leqq p \leqq n(\varepsilon)} \qquad \text{for any } \boldsymbol{s} \in \mathbb{R}^{Nn(\varepsilon)}, \tag{8}$$

where u_ε is now the unique solution in $H^1(\Omega_\varepsilon)$ of the boundary-value problem

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} &= \boldsymbol{s}_p \cdot \boldsymbol{n} && \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon), \\ u_\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{9}$$

Remark that this choice of the operator S_ε is somewhat arbitrary, but this nonuniqueness is not important at this point.

According to J. Planchard [27], C. Conca & M. Vanninathan [12], the operator S_ε is self-adjoint and positive-definite, and its characteristic values coincide with the spectrum of (7). More precisely, they proved

Proposition 1.2.1. *There exist $Nn(\varepsilon)$ positive reals $0 < \lambda_\varepsilon(1) \leqq \ldots \leqq \lambda_\varepsilon(Nn(\varepsilon))$ (not necessarily distinct) and $Nn(\varepsilon)$ non-zero functions $u_\varepsilon(1), \ldots, u_\varepsilon(Nn(\varepsilon))$ in $H^1(\Omega_\varepsilon)$ such that*

1. *for each $j = 1, \ldots, Nn(\varepsilon)$, $[\lambda_\varepsilon(j), u_\varepsilon(j)]$ is a solution of (7),*
2. *the pairs $[\lambda_\varepsilon(j), u_\varepsilon(j)]_{1 \leqq j \leqq Nn(\varepsilon)}$ describe all the solutions of (7).*

An important feature of the spectrum of S_ε is that it is bounded away from zero and infinity, *uniformly in* ε. This behavior is deduced from the following lemma, the proof of which can be found in Section 2.2 (a similar result can also be found in [12]).

Lemma 1.2.2. *There exist two positive constants c and C with $0 < c \leqq C < +\infty$, which do not depend on ε, such that*

$$c \leqq \lambda_\varepsilon(1) \leqq \cdots \leqq \lambda_\varepsilon(Nn(\varepsilon)) \leqq C.$$

The determination of the spectral eigenvalues λ_ε is of great importance in many applications (see [27] and references therein). However, for small ε the number of eigenvalues is very large and their precise values are irrelevant. Rather, we are interested in the qualitative asymptotic behavior of the spectrum. Thus, the goal of this paper is to study the limit of the spectrum $[\lambda_\varepsilon(j), u_\varepsilon(j)]_{1 \leqq j \leqq Nn(\varepsilon)}$ of (7) as the period ε goes to zero. In practice we achieve this goal by studying the convergence of the sequence of operators S_ε. Since these operators are not defined on the same fixed space, but rather on a sequence of spaces $\mathbb{R}^{Nn(\varepsilon)}$, depending on ε, we shall introduce an extension $\tilde{S}_\varepsilon$ of each S_ε defined on a fixed Hilbert space. This will allow us to use the classical theory of perturbations of linear operators and the powerful tools on spectral convergence developed by F. RELLICH (see his original work [30] or modern textbooks such as [18] or [31]).

However, the question of choosing the right extension of S_ε (i.e., of choosing in which fixed space $\mathbb{R}^{Nn(\varepsilon)}$ should be embedded) is a subtle one and is addressed in great detail in Section 3.1. At this point, let us simply warn the reader that different extensions yield various limit operators, possibly with different limit spectra. This seemingly paradoxical result is due to the fact that, although S_ε is a nice, finite-rank operator, it never converges, in any sense, to a compact limit operator. This loss of compactness is the main feature of this problem, which makes it both difficult and interesting.

2. Classical homogenization: a macroscopic limit operator

2.1. Main results

In this section, we extend the operators S_ε to the Hilbert space $L^2(\Omega)^N$, and we study the convergence of these extensions, denoted by $\tilde{S}_\varepsilon$, in the fixed space $L^2(\Omega)^N$. In some sense, it is the most *natural* choice of extension, since, as ε goes to zero, a vector $s_\varepsilon \in \mathbb{R}^{Nn(\varepsilon)}$, interpreted as a function constant in each cell Y_p^ε, "converges" to a function $s(x)$ defined in Ω. This choice also has the advantage that the convergence behavior of $\tilde{S}_\varepsilon$ is obtained through a periodic homogenization problem, amenable to classical techniques such as two-scale convergence.

To begin with, we introduce two continuous linear operators: P_ε which maps $L^2(\Omega)^N$ onto $\mathbb{R}^{Nn(\varepsilon)}$, and E_ε which maps $\mathbb{R}^{Nn(\varepsilon)}$ into $L^2(\Omega)^N$. More precisely, they are defined by

$$\begin{aligned} &P_\varepsilon : L^2(\Omega)^N \to \mathbb{R}^{Nn(\varepsilon)}, \\ &s(x) \to \left(\frac{1}{|Y_p^\varepsilon|} \int_{Y_p^\varepsilon} s(x)\, dx \right)_{1 \leqq p \leqq n(\varepsilon)}, \end{aligned} \tag{10}$$

$$
\begin{aligned}
&E_\varepsilon : \mathbb{R}^{Nn(\varepsilon)} \to L^2(\Omega)^N, \\
&(s_p)_{1 \leq p \leq n(\varepsilon)} \to s(x) = \sum_{p=1}^{n(\varepsilon)} s_p \chi_{Y_p^\varepsilon}(x)
\end{aligned}
\tag{11}
$$

where $\chi_{Y_p^\varepsilon}(x)$ is the characteristic function of the set Y_p^ε. It is easily seen that the composition $P_\varepsilon E_\varepsilon$ is nothing but the identity in $\mathbb{R}^{Nn(\varepsilon)}$, while $E_\varepsilon P_\varepsilon$ is the standard projection operator from $L^2(\Omega)^N$ onto its subspace of piecewise constant functions on each cell Y_p^ε.

We are now in a position to define the extension $\tilde{S}_\varepsilon$ by means of the following composition rule

$$
\begin{aligned}
&\tilde{S}_\varepsilon : L^2(\Omega)^N \to L^2(\Omega)^N \\
&\qquad s(x) \to E_\varepsilon S_\varepsilon P_\varepsilon s(x).
\end{aligned}
\tag{12}
$$

One can easily check that $E_\varepsilon^* = \varepsilon^N P_\varepsilon$ and $P_\varepsilon^* = \varepsilon^{-N} E_\varepsilon$ (where the symbol $*$ denotes the adjoint operator). Since S_ε is self-adjoint and has finite rank, its extension $\tilde{S}_\varepsilon$ is obviously self-adjoint and has finite rank too (thus, it is compact). Since P_ε is surjective, the spectrum $\sigma(\tilde{S}_\varepsilon)$ of $\tilde{S}_\varepsilon$ is nothing but that of S_ε plus the eigenvalue 0, which has an infinite multiplicity. As a consequence, in this framework the asymptotic analysis of (7) is reduced to the study of the convergence of the sequence $\tilde{S}_\varepsilon$ in the space of linear continuous operators on $L^2(\Omega)^N$. Our main result is a strong (but not uniform) convergence of this sequence to a non-compact limit operator S.

Theorem 2.1.1. *The sequence of operators $\tilde{S}_\varepsilon$ converges strongly to a limit S on $L^2(\Omega)^N$, i.e., for any function $s \in L^2(\Omega)^N$,*

$$\tilde{S}_\varepsilon s \to Ss \text{ in } L^2(\Omega)^N \text{ strongly,}$$

and the limit operator S is defined by

$$Ss = (A - I)\nabla u - (A - \theta I)s, \tag{13}$$

where I denotes the identity matrix, $\theta = |Y^*|$ denotes the volume fraction of fluid in the unit cell, u is the unique solution in $H_0^1(\Omega)$ of the boundary-value problem

$$
\begin{aligned}
-\operatorname{div}(A\nabla u) &= \operatorname{div}((I - A)s) \quad && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{14}
$$

and the matrix A is defined by

$$A_{ij} = \int_{Y^*} (\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j)\, dy, \tag{15}$$

where, for $1 \leq i \leq N$, w_i is the unique solution of the so-called cell problem

$$
\begin{aligned}
&-\operatorname{div}_y(\nabla_y w_i + e_i) = 0 \quad && \text{in } Y^*, \\
&(\nabla_y w_i + e_i) \cdot n = 0 && \text{on } \partial T, \\
&y \to w_i(y) \text{ is } Y\text{-periodic.}
\end{aligned}
\tag{16}
$$

The proof of Theorem 2.1.1 is given in Section 2.2. The essential ingredient for proving the strong convergence of $\tilde{S}_\varepsilon$ is a careful analysis of the homogenization of the boundary-value problem (9) for the potential u_ε. Our main tool is the two-scale convergence method recently introduced by G. ALLAIRE [2] and G. NGUETSENG [23].

Remark 2.1.2. In view of definition (13) of the limit operator S, it seems likely that it is not compact. This is indeed the case by virtue of Theorem 2.1.4 below. Since the operators $\tilde{S}_\varepsilon$ are compact, this theorem indicates the optimality of the strong convergence, in $L^2(\Omega)^N$, of this sequence. One cannot hope to prove a sharper result, namely, the uniform convergence of $\tilde{S}_\varepsilon$, because this convergence would imply the compactness of the limit S.

In order to describe the spectrum of S, we recall a well-known lemma on the properties of the homogenized matrix A.

Lemma 2.1.3. *Let A be the homogenized matrix defined by* (15). *Then, both A and $(\theta I - A)$ are symmetric and positive-definite. Furthermore, if the fluid cell Y^* has cubic symmetry, then the homogenized matrix A is proportional to the identity.*

Theorem 2.1.4. *The limit operator S, defined on $L^2(\Omega)^N$ by* (13), *is positive-definite, self-adjoint, and not compact. Furthermore, its spectrum $\sigma(S)$ coincides with its essential spectrum. Since the matrix*

$$B(\lambda) = A - (I - A)(A + (\lambda - \theta)I)^{-1}(I - A)$$

is diagonizable by virtue of Lemma 2.1.3, the spectrum of S is exactly the set of all values of λ such that $B(\lambda)$ has either an infinite or a zero eigenvalue, or has simultaneously positive and negative eigenvalues. Thus, the spectrum of S is a finite union of intervals of $\mathbb{R}$.

We recall that the essential spectrum of a self-adjoint operator is the subset of its spectrum whose elements are not isolated eigenvalues of finite multiplicity. Thus, S cannot be compact since the essential spectrum of a compact operator can contain only the single point zero. In the case where the homogenized matrix A is a multiple of the identity, we can improve Theorem 2.1.4.

Theorem 2.1.5. *Assume that the fluid cell Y^* has cubic symmetry; then, by virtue of Lemma 2.1.3, the homogenized matrix A is equal to αI, with $0 < \alpha \leqq \theta$. In this isotropic case, the spectrum $\sigma(S)$ of S consists of two eigenvalues, $\lambda_1 \leqq \lambda_2$, of infinite multiplicity:*

$$\lambda_1 = \theta - \alpha, \quad \lambda_2 = \theta + \frac{1 - 2\alpha}{\alpha}$$

whose eigenspaces are respectively

$$E_1 = \{s \in L^2(\Omega)^N \text{ such that } \operatorname{div} s = 0 \text{ in } \Omega\},$$

$$E_2 = \{s \in L^2(\Omega)^N \text{ such that } s = \nabla q \text{ with } q \in H_0^1(\Omega)\}.$$

The proofs of Theorems 2.1.4, 2.1.5 are given in Section 2.3.

Remark 2.1.6. It is easily seen that the direct sum of the eigenspaces E_1 and E_2 is precisely $L^2(\Omega)^N$, as it should be. From a physical point of view, E_1 is the set of all macroscopically incompressible displacements of the tubes (corresponding to a zero fluid velocity), while E_2 contains the gradient-type displacements which yield a non-zero fluid velocity. In the anisotropic case (in which A is not proportional to the identity), such an orthogonal decomposition of $L^2(\Omega)^N$, with respect to S, is not easily available. Furthermore, the spectrum of S exhibits a band structure as can be readily checked from its definition.

Remark 2.1.7. The spectrum $\sigma(S)$ is defined by an algebraic criterion involving the eigenvalues of the matrix $B(\lambda)$, regardless of the precise type of boundary condition satisfied by the potential u. In particular, if we change the definition of the operator S by replacing the Dirichlet boundary condition on $\partial\Omega$ by a Neumann condition in (14), this does not affect its spectrum $\sigma(S)$ (considered as a subset of the real line). The same holds true for other types of "usual" boundary conditions like mixed or Fourier conditions. Of course, a value λ in the spectrum may or may not be an eigenvalue, depending on the precise type of boundary conditions. Moreover, the definition of the corresponding eigenspaces depend also on the boundary condition in (14). However, as far as the characterization of $\sigma(S)$ is concerned, the entire analysis performed in this paper for a Dirichlet boundary condition carries over mutatis mutandis for the more physical case of a Neumann boundary condition.

It remains to see in which sense $\sigma(\tilde{S}_\varepsilon)$ converges to $\sigma(S)$. If the convergence of the sequence $\tilde{S}_\varepsilon$ were uniform, and the limit operator S compact (which is not the case by virtue of Theorem 2.1.4), then it would be a classical matter to prove pointwise convergence of the eigenvalues (see, e.g., [25, 26, 38]). Unfortunately, the situation here is non-standard in the sense that neither $\tilde{S}_\varepsilon$ nor its resolvent converge uniformly to their limits. Therefore, following the ideas of [31, 32], we prove the convergence of $\sigma(\tilde{S}_\varepsilon)$ to $\sigma(S)$ in a much weaker sense, i.e., by means of the so-called spectral families. We first recall the classical definition of a spectral family for a self-adjoint operator on a Hilbert space (for details see, e.g., [18, 31]).

Proposition 2.1.8. *Let S be a self-adjoint operator on a Hilbert space H. Its spectral family is the unique function $\mathscr{E}(\lambda)$, defined on $\mathbb{R}$ with values in the space of orthogonal projections on H, satisfying the properties*
1. $S = \int_{-\infty}^{+\infty} \lambda\, d\mathscr{E}(\lambda)$ *in the sense of Stieltjes integrals.*
2. $\mathscr{E}(\lambda)$ *is non-decreasing, i.e.,* $\mathscr{E}(\lambda)\mathscr{E}(\mu) = \mathscr{E}(\min(\lambda, \mu))$.

3. *for any $u \in H$, $\mathscr{E}(\lambda)u$ converges strongly to 0 in H when λ goes to $-\infty$ and to u in H when λ goes to $+\infty$.*
4. *for any $u \in H$, $\lambda \to \mathscr{E}(\lambda)u$ is continuous on the right in the strong topology of H.*

An important property of the spectral family $\mathscr{E}(\lambda)$ is that the spectrum $\sigma(S)$ coincides with the set of real values λ such that $\mathscr{E}(\lambda)$ is not constant in a neighborhood of λ. Furthermore, the points of discontinuity of $\mathscr{E}(\lambda)$ correspond to eigenvalues (of finite or infinite multiplicity), while the points where $\mathscr{E}(\lambda)$ is continuous and not constant belong to the continuous spectrum of S.

It is very easy to compute the spectral family of $\tilde{S}_\varepsilon$, but we are able to compute that of S only in the isotropic case (i.e., $A = \alpha I$). For $1 \leqq j \leqq Nn(\varepsilon)$, with $s_j^\varepsilon(x)$ denoting the normalized eigenvector associated with the eigenvalue λ_j^ε of $\tilde{S}_\varepsilon$, its spectral family $\mathscr{E}_\varepsilon(\lambda)$ is simply defined by

$$\mathscr{E}_\varepsilon(\lambda)s(x) = \sum_{\{j \mid \lambda_j^\varepsilon \leqq \lambda\}} \left(\int_\Omega s_j^\varepsilon(x) \cdot s(x)\, dx \right) s_j^\varepsilon(x).$$

In the isotropic case, a well-known result states that the direct sum of the eigenspaces E_1 and E_2 is exactly $L^2(\Omega)^N$. In other words, for any $s \in L^2(\Omega)^N$, there exists a unique decomposition

$$s = t + \nabla q \quad \text{with } \operatorname{div} t = 0 \text{ and } q \in H_0^1(\Omega).$$

Then, the spectral family $\mathscr{E}(\lambda)$ of S is defined by

$$\mathscr{E}(\lambda)s = \begin{cases} 0 & \text{if } \lambda < \lambda_1, \\ t & \text{if } \lambda_1 \leqq \lambda < \lambda_2, \\ s & \text{if } \lambda_2 \leqq \lambda. \end{cases} \tag{17}$$

As a direct consequence of Rellich's theorem, which implies the convergence of spectral families from the strong convergence of a sequence of self-adjoint operators (see the original work of RELLICH [30], or more recent textbooks such as [18, 31]), we deduce from Theorem 2.1.1

Theorem 2.1.9. *Let $\tilde{S}_\varepsilon$ and S be the operators on $L^2(\Omega)^N$ defined by* (12) *and* (13) *respectively. If λ is not an eigenvalue of S, then the spectral family $\mathscr{E}_\varepsilon(\lambda)$ of $\tilde{S}_\varepsilon$ converges strongly to that $\mathscr{E}(\lambda)$ of S as ε goes to zero, in the sense that for any $s \in L^2(\Omega)^N$,*

$$\mathscr{E}_\varepsilon(\lambda)s \to \mathscr{E}(\lambda)s \quad \text{in } L^2(\Omega)^N \text{ strongly.}$$

Remark 2.1.10. This convergence of $\mathscr{E}_\varepsilon(\lambda)$ holds for all values of λ not eigenvalues of S. In the isotropic case, this condition rules out only the two values λ_1 and λ_2. However, in the anisotropic case we do not know the point spectrum of S (see Section 2.3 for more details). Of course, we know that the spectrum $\sigma(S)$ of S coincides with its essential spectrum, but this yields no information on the decomposition of $\sigma(S)$ in its point and continuous parts. Thus, it could happen, unfortunately, that the above convergence is useless on a large part of $\sigma(S)$.

Convergence of spectral families is a sort of *weak convergence* for the spectrum, not so much from a physical point of view (for a more complete discussion of this type of convergence and very interesting applications, we refer to [31]). As mentioned before, we are primarily interested in the pointwise convergence of the spectrum. From Theorem 2.1.1, we can obtain such a result which, strictly speaking, is weaker than the Rellich Theorem 2.1.9 (for its proof see Section 2.3).

Proposition 2.1.11. *The strong convergence of $\tilde{S}_\varepsilon$ to S implies that*
1. *For any $\lambda \in \sigma(S)$, there exists a sequence $\lambda_\varepsilon \in \sigma(\tilde{S}_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$.*
2. *There may exist non-zero sequences $\lambda_\varepsilon \in \sigma(\tilde{S}_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$, where the limit λ does not belong to $\sigma(S)$. In this case, any associated sequence of normalized eigenvectors converges weakly to zero in $L^2(\Omega)^N$.*

In view of this result, it seems likely that $\sigma(S)$ does not contain all possible limits of sequences in $\sigma(\tilde{S}_\varepsilon)$. This means that S is perhaps not the "best" limit operator available. To obtain a different limit operator, the only possibility is to change the Hilbert space in which we perform the asymptotic analysis. Indeed, we are at liberty to extend the original operator S_ε, merely defined in $\mathbb{R}^{Nn(\varepsilon)}$, to a space other than $L^2(\Omega)^N$. To recover the "largest" possible limit spectrum, we have to carefully choose this new extension, following the crucial hint, given by Proposition 2.1.11, that sequences of eigenvalues which "escape" in the limit from $\sigma(S)$ are associated with eigenvectors converging weakly to zero in $L^2(\Omega)^N$. The usual belief that weak convergence corresponds to oscillations (in the absence of concentration effects) indicates that the new Hilbert space, in which $\mathbb{R}^{Nn(\varepsilon)}$ is embedded, must capture these oscillations. In Section 3 we propose such a new extension by using a combination of two-scale convergence and Bloch-wave theory.

Remark 2.1.12. Let us briefly discuss the previous work of C. Conca & M. Vanninathan [12] on the same problem. The key in comparing our results to theirs is to recognize that the spectral problem (7) is invariant upon rescaling by a factor ε. In other words, by the change of variables

$$y = \frac{x}{\varepsilon},$$

the problem of finding all (λ_η, u_η) solutions of

$$\begin{aligned} -\Delta u_\eta &= 0 && \text{in } \Omega_\eta^*, \\ \lambda_\eta \frac{\partial u_\eta}{\partial n} &= \boldsymbol{n} \cdot \int_{\Gamma_p} u_\eta \boldsymbol{n}\, ds && \text{on } \Gamma_p \text{ for } 1 \leqq p \leqq n(\eta), \\ u_\eta &= 0 && \text{on } \partial\Omega_\eta \end{aligned} \tag{18}$$

is equivalent to our spectral problem (7) with the definitions $n(\eta) = n(\varepsilon)$, $\Omega_\eta = \varepsilon^{-1}\Omega$, $\Omega_\eta^* = \varepsilon^{-1}\Omega_\varepsilon$, $\Gamma_p = \varepsilon^{-1}\Gamma_p^\varepsilon$, $Y_p = \varepsilon^{-1}Y_p^\varepsilon$ and the identities

$$u_\eta(y) = u_\varepsilon(x), \quad \lambda_\eta = \lambda_\varepsilon.$$

The above spectral problem (18) is exactly that studied in [12] by means of the Bloch-wave theory. It turns out that the resulting limit problem, defined in an unbounded domain, has nothing to do with the results obtained in this section by classical homogenization. Thus, another motivation of Section 3 is to understand this difference. As we see, we are able to recover the results of [12] by using our Bloch-wave homogenization method developed in Section 3 (see Remark 3.2.13).

2.2. *Convergence of the classical homogenization process*

This section is devoted to the proof of Theorem 2.1.1 concerning the strong convergence of the sequence of operators $\tilde{S}_\varepsilon$ defined by

$$\begin{aligned} &\tilde{S}_\varepsilon : L^2(\Omega)^N \to L^2(\Omega)^N, \\ &\boldsymbol{s}(x) \to \tilde{S}_\varepsilon \boldsymbol{s}(x) = \sum_{p=1}^{n(\varepsilon)} \left(\frac{1}{|Y_p^\varepsilon|} \int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n}\, ds \right) \chi_{Y_p^\varepsilon}(x), \end{aligned} \tag{19}$$

where u_ε is the unique solution in $H^1(\Omega_\varepsilon)$ of

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} &= (P_\varepsilon \boldsymbol{s}) \cdot \boldsymbol{n} && \text{on } \Gamma_p^\varepsilon, \text{ for } 1 \leqq p \leqq n(\varepsilon), \\ u_\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{20}$$

the right-hand side $P_\varepsilon \boldsymbol{s}$ being defined by

$$P_\varepsilon \boldsymbol{s} = \frac{1}{|Y_p^\varepsilon|} \int_{Y_p^\varepsilon} \boldsymbol{s}(x)\, dx \quad \text{on each } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon).$$

Since the definition (19) of $\tilde{S}_\varepsilon$ involves the solution u_ε of the boundary-value problem (20), our first task is to homogenize this problem as ε goes to zero. To do so, we use the two-scale convergence method (recently introduced in [2] and [23]), which is well-suited to the present periodic setting. It turns out that the determination of the limit of $\tilde{S}_\varepsilon$ (i.e., the weak convergence of $\tilde{S}_\varepsilon$ to its limit S) requires only a corrector result for u_ε, or equivalently the Γ-convergence of the energy associated with (20). Such results can also be obtained via standard homogenization procedures such as two-scale asymptotic expansions (see, e.g., [5, 6, 33]) rigorously justified by G- or H-convergence (see, e.g., [13, 22, 35, 36]). However, the strong convergence of the sequence $\tilde{S}_\varepsilon$ seems to rely crucially on the two-scale convergence method (see the proof of Proposition 2.2.6). We begin by recalling the necessary results on two-scale convergence (we follow the notation of [2]).

Propositon 2.2.1. 1. *Let u_ε be a bounded sequence in $L^2(\Omega)$. There exist a subsequence (still denoted by ε) and a two-scale limit $u_0(x, y) \in L^2(\Omega \times Y)$ such that $u_\varepsilon(x)$*

two-scale converges to $u_0(x, y)$ in the sense that

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|Y|} \int_\Omega \int_Y u_0(x, y)\phi(x, y)\, dx\, dy$$

for any function $\phi(x, y)$ in $\mathscr{D}[\Omega; C^\infty_\#(Y)]$.

2. *Let u_ε be a bounded sequence in $H^1(\Omega)$. Then, up to a subsequence, u_ε two-scale converges to a limit $u(x) \in H^1(\Omega)$, and ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$, where the function $u_1(x, y)$ belongs to $L^2[\Omega; H^1_\#(Y)/\mathbb{R}]$.*

3. *Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$. Assume further that*

$$\lim_{\varepsilon \to 0} \| u_\varepsilon \|^2_{L^2(\Omega)} = \frac{1}{|Y|} \| u_0 \|^2_{L^2(\Omega \times Y)}.$$

(*Then u_ε is said to two-scale converge* strongly *to its limit u_0.*) *For any sequence v_ε which two-scale converges simply to a limit $v_0(x, y) \in L^2(\Omega \times Y)$,*

$$u_\varepsilon(x)v_\varepsilon(x) \rightharpoonup \frac{1}{|Y|} \int_Y u_0(x, y)v_0(x, y)\, dy \quad \text{in } L^1(\Omega) \text{ weakly.}$$

Since we are studying an homogenization problem in a perforated domain Ω_ε, we use a well-known technical lemma [9] for extending the solution u_ε of (20) to the whole limit domain Ω. This allows us to study the convergence of the sequence u_ε in the fixed space $H^1(\Omega)$. We remark in passing that two-scale convergence can handle homogenization problems in perforated domains without using any extension operator (see Section 2 in [2]), but the following extension lemma simplifies the presentation of the results.

Lemma 2.2.2. *Let Ω_ε be a perforated domain defined by* (3). *There exists a bounded extension operator X_ε acting from $H^1(\Omega_\varepsilon)$ into $H^1(\Omega)$ and a positive constant C (independent of ε) such that*

$$X_\varepsilon v = v \quad \text{in } \Omega_\varepsilon, \; \| X_\varepsilon v \|_{H^1(\Omega)} \leqq C \| v \|_{H^1(\Omega)}$$

for any $v \in H^1(\Omega_\varepsilon)$.

To simplify the notation further, we denote by u_ε both the solution of (20) in $H^1(\Omega_\varepsilon)$ and its bounded extension $X_\varepsilon u_\varepsilon$ in $H^1(\Omega)$. The next step is to obtain a priori estimates for the solution u_ε (which, by the way, provides a proof of Lemma 1.2.2).

Lemma 2.2.3. *Let u_ε be the unique solution of* (20) *in $H^1(\Omega_\varepsilon)$ (extended to the whole domain Ω). There exist two positive constants c and C independent of ε and $\boldsymbol{s}$ with $0 < c \leqq C$ such that*

$$c \| P_\varepsilon \boldsymbol{s} \|_{L^2(\Omega)} \leqq \| u_\varepsilon \|_{H^1(\Omega)} \leqq C \| \boldsymbol{s} \|_{L^2(\Omega)}.$$

Proof. Multiplying (20) by u_ε and integrating by parts yields

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \sum_{p=1}^{n(\varepsilon)} \left(\frac{1}{|Y_p^\varepsilon|} \int_{Y_p^\varepsilon} \boldsymbol{s}(x) \, dx \right) \cdot \left(\int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n} \, ds \right). \tag{21}$$

Since the holes are isolated, we have

$$\left| \int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n} \, ds \right| = \left| \int_{T_p^\varepsilon} \nabla u_\varepsilon \, dx \right| \leqq \varepsilon^{N/2} \, \| \nabla u_\varepsilon \|_{L^2(T_p^\varepsilon)}.$$

On the other hand, the Cauchy-Schwarz inequality gives

$$\left| \frac{1}{|Y_p^\varepsilon|} \int_{Y_p^\varepsilon} \boldsymbol{s} \, dx \right| \leqq \varepsilon^{-N/2} \, \| \boldsymbol{s} \|_{L^2(Y_p^\varepsilon)}.$$

Thus, the right-hand side of (21) is bounded by

$$\sum_{p=1}^{n(\varepsilon)} \| \nabla u_\varepsilon \|_{L^2(Y_p^\varepsilon)} \, \| \boldsymbol{s} \|_{L^2(Y_p^\varepsilon)} \leqq \| \nabla u_\varepsilon \|_{L^2(\Omega)} \, \| \boldsymbol{s} \|_{L^2(\Omega)}.$$

This gives an upper bound for the norm of u_ε. To get a lower bound, we multiply (20) by a test function v_ε defined in each cell Y_p^ε, of center $\boldsymbol{x}_p$, by

$$v_\varepsilon = -(P_\varepsilon \boldsymbol{s} \cdot (\boldsymbol{x} - \boldsymbol{x}_p)) \, \theta \left(\frac{x}{\varepsilon} \right),$$

where $\theta(y)$ is a Y-periodic function which is identically equal to 1 on the hole T and vanishes on a neighborhood of ∂Y. Integrating by parts yields

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, dx = \sum_{p=1}^{n(\varepsilon)} (P_\varepsilon \boldsymbol{s}) \cdot \left(\int_{\Gamma_p^\varepsilon} v_\varepsilon \boldsymbol{n} \, ds \right). \tag{22}$$

The right-hand side of (22) is easily seen to be equal to $|T| \varepsilon^N \sum_{p=1}^{n(\varepsilon)} |P_\varepsilon \boldsymbol{s}|^2$, while its left-hand side can be estimated by remarking that v_ε is uniformly bounded in $H^1(\Omega)$, giving the lower bound for the norm of u_ε.

We are now in a position to give the homogenized system for (20).

Proposition 2.2.4. *The sequence u_ε of solutions of* (20) (*extended to the whole domain* Ω) *converges weakly in $H_0^1(\Omega)$ to the unique solution u of the homogenized problem*

$$\begin{aligned} -\operatorname{div}(A \nabla u) &= \operatorname{div}((I - A)\boldsymbol{s}) \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{23}$$

where the matrix A is defined by $A_{ij} = \int_{Y^*} (\nabla_y w_i + \boldsymbol{e}_i) \cdot (\nabla_y w_j + \boldsymbol{e}_j) \, dy$, and where $(w_i)_{1 \leqq i \leqq N}$ is the family of solutions of the cell problems (16). *Furthermore,* ∇u_ε

two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$, *with*

$$u_1(x, y) = \sum_{i=1}^{N} w_i(y)\left(\frac{\partial u}{\partial x_i}(x) - s_i(x)\right),$$

and the energies converge:

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx = \int_\Omega \int_{Y^*} |\nabla_x u + \nabla_y u_1(x, y)|^2\, dx\, dy. \tag{24}$$

Proof. We know from Lemma 2.2.3 that the sequence u_ε is bounded in $H_0^1(\Omega)$. By Proposition 2.2.1, up to a subsequence, u_ε two-scale converges to a limit $u(x)$ (which, of course, coincides with the usual weak $H_0^1(\Omega)$ limit), and ∇u_ε two-scale converges to $\nabla u(x) + \nabla_y u_1(x, y)$ where $u_1 \in L^2[\Omega; H_\#^1(Y)]$. Then, we multiply (20) by a test function $\phi(x) + \varepsilon\phi_1(x, x/\varepsilon)$, with $\phi \in \mathscr{D}(\Omega)$ and $\phi_1 \in \mathscr{D}[\Omega; C_\#^\infty(Y)]$. Integrating by parts, we obtain

$$\begin{aligned}
&\int_\Omega \chi\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \cdot \left(\nabla\phi(x) + \nabla_y\phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\nabla_x\phi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx \\
&\quad = \sum_{p=1}^{n(\varepsilon)} \left(\frac{1}{|Y_p^\varepsilon|}\int_{Y_p^\varepsilon} \boldsymbol{s}(x)\, dx\right) \cdot \left(\int_{\Gamma_p^\varepsilon} \left(\phi(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)\right)\boldsymbol{n}\, ds\right),
\end{aligned} \tag{25}$$

where $\chi(y)$ is the characteristic function of Y^*. Green's formula in each hole T_p^ε gives

$$\int_{\Gamma_p^\varepsilon} \left(\phi(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)\right)\boldsymbol{n}\, ds = -\int_{T_p^\varepsilon} \left(\nabla\phi(x) + \nabla_y\phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\nabla_x\phi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx. \tag{26}$$

Therefore, if we use the definitions (10) and (11) of P_ε and E_ε, the right-hand side of (25) becomes

$$\int_\Omega \left(\chi\left(\frac{x}{\varepsilon}\right) - 1\right) E_\varepsilon P_\varepsilon \boldsymbol{s} \cdot \left(\nabla\phi(x) + \nabla_y\phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\nabla_x\phi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx. \tag{27}$$

Since the size of the cells goes to zero as ε does, it is easy to prove (we leave this to the reader) that

$$E_\varepsilon P_\varepsilon \boldsymbol{s} \to \boldsymbol{s} \text{ in } L^2(\Omega)^N \text{ strongly} \quad \text{for each } \boldsymbol{s} \in L^2(\Omega)^N. \tag{28}$$

Thus, we can easily pass to the limit in (27). On the other hand, passing to the two-scale limit (by Proposition 2.2.1) in the left-hand side of (25), we obtain

$$\int_\Omega \int_{Y^*} (\nabla u(x) + \nabla_y u_1(x, y)) \cdot (\nabla\phi(x) + \nabla_y\phi_1(x, y))\, dx\, dy =$$

$$= - \int_\Omega \int_T s \cdot (\nabla \phi(x) + \nabla_y \phi_1(x, y))\, dx\, dy$$

$$= |T| \int_\Omega \phi(x) \operatorname{div}_x s\, dx + \int_\Omega \int_{\partial T} s(x) \cdot n_y \phi_1(x, y)\, dx\, ds. \tag{29}$$

Equation (29) is nothing else but a variational formulation for (u, u_1) in the space $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y^*)/\mathbb{R}]$. By application of the Lax-Milgram lemma, it is easily seen that (29) admits a unique solution. Thus the entire sequence u_ε converges to its limit u. From (29) we derive the associated system of equations (the so-called two-scale homogenized problem)

$$\begin{aligned}
&- \Delta_y u_1(x, y) = 0 && \text{in } \Omega \times Y^*, \\
&- \operatorname{div}_x \left(\textstyle\int_{Y^*} (\nabla u(x) + \nabla_y u_1(x, y))\, dy \right) = |T| \operatorname{div}_x s && \text{in } \Omega, \\
&u(x) = 0 && \text{on } \partial\Omega. \\
&(\nabla u(x) + \nabla_y u_1(x, y) - s(x)) \cdot n_y = 0 && \text{on } \partial T \times \Omega, \\
&y \to u_1(x, y) \text{ is } Y\text{-periodic.}
\end{aligned} \tag{30}$$

By linearity, the solution u_1 in (30) can be computed in terms of $\nabla u(x)$ and of the solutions $(w_i)_{1 \leq i \leq N}$ of the cell problems (16):

$$u_1(x, y) = \sum_{i=1}^N w_i(y) \left(\frac{\partial u}{\partial x_i}(x) - s_i(x) \right). \tag{31}$$

Finally, eliminating the y variable in (30) yields the homogenized equation (23) (for details, see Section 2 in [2]) by remarking that

$$\int_{Y^*} (\nabla u(x) + \nabla_y u_1(x, y))\, dy = \theta \nabla u(x) + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}(x) - s_i(x) \right) \int_{Y^*} \nabla w_i(y)\, dy,$$

$$\int_{Y^*} \nabla w_i(y)\, dy = (A - \theta I)\, e_i, \tag{32}$$

where the matrix A has been defined in (15). To prove the remaining statement (24), we multiply (20) by u_ε, and integrate the product by parts:

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx = \sum_{p=1}^{n(\varepsilon)} \left(\frac{1}{|Y_p^\varepsilon|} \int_{Y_p^\varepsilon} s(x)\, dx \right) \cdot \left(\int_{\Gamma_p^\varepsilon} u_\varepsilon n\, ds \right)$$

$$= \int_\Omega \left(\chi\left(\frac{x}{\varepsilon}\right) - 1 \right) E_\varepsilon P_\varepsilon s \cdot \nabla u_\varepsilon\, dx. \tag{33}$$

To pass to the limit in (33), we recall that $E_\varepsilon P_\varepsilon \boldsymbol{s}$ converges strongly to $\boldsymbol{s}$ in $L^2(\Omega)^N$ (see (28)). Thus the limit of (33) is the same as that of

$$\int_\Omega \left(\chi\left(\frac{x}{\varepsilon}\right) - 1\right) \boldsymbol{s} \cdot \nabla u_\varepsilon \, dx. \tag{34}$$

By definition of two-scale convergence, the limit of (34) is simply

$$-\int_\Omega \int_T \boldsymbol{s} \cdot (\nabla u(x) + \nabla_y u_1(x, y)) \, dx \, dy = \int_\Omega \int_{Y^*} |\nabla_x u + \nabla_y u_1(x, y)|^2 \, dx \, dy,$$

thanks to an easy integration by parts in (30).

With the help of the above homogenization result, we can compute the limit operator S.

Proposition 2.2.5. *The sequence $\tilde{S}_\varepsilon$ converges weakly in $L^2(\Omega)^N$ to its limit S defined by*

$$S\boldsymbol{s} = (A - I)\nabla u - (A - \theta I)\boldsymbol{s}, \tag{35}$$

where $\theta = |Y^*|$, and u is the unique solution in $H^1(\Omega)$ of the homogenized problem (23).

Proof. We study the convergence of $\tilde{S}_\varepsilon$ by inspecting the limit of $\int_\Omega \tilde{S}_\varepsilon \boldsymbol{s} \cdot \boldsymbol{s} dx$. From the energy relation (33) we deduce that

$$\int_\Omega \tilde{S}_\varepsilon \boldsymbol{s} \cdot \boldsymbol{s} \, dx = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx. \tag{36}$$

We have just computed the limit of the right-hand side of (36) (see (24)) which defines a continuous, self-adjoint, linear operator S acting on $L^2(\Omega)^N$ by

$$\int_\Omega S\boldsymbol{s} \cdot \boldsymbol{s} \, dx = \int_\Omega \int_{Y^*} |\nabla u + \nabla_y u_1(x, y)|^2 \, dx \, dy, \tag{37}$$

where $(u(x), u_1(x, y))$ is the solution of the two-scale homogenized problem (30). Replacing u_1 by its value (31) and using the relationship (32) yield

$$S\boldsymbol{s} = (A - I)\nabla u - (A - \theta I)\boldsymbol{s}.$$

To complete the proof of Theorem 2.1.1, it remains to establish.

Proposition 2.2.6. *The sequence $\tilde{S}_\varepsilon$ converges strongly in $L^2(\Omega)^N$ to its limit S.*

Proof. Let $\boldsymbol{t}_\varepsilon$ be any sequence which converges weakly to a function $\boldsymbol{t}$ in $L^2(\Omega)^N$. We need to show that

$$\lim_{\varepsilon \to 0} \int_\Omega \tilde{S}_\varepsilon \boldsymbol{s} \cdot \boldsymbol{t}_\varepsilon \, dx = \int_\Omega S\boldsymbol{s} \cdot \boldsymbol{t} \, dx.$$

By introducing a function v_ε, defined as the unique solution in $H^1(\Omega_\varepsilon)$ of

$$
\begin{aligned}
-\Delta v_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\
\frac{\partial v_\varepsilon}{\partial n} &= (P_\varepsilon \boldsymbol{t}_\varepsilon)\cdot \boldsymbol{n} && \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon), \\
v_\varepsilon &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{38}
$$

we easily obtain

$$
\int_\Omega \tilde{S}_\varepsilon \boldsymbol{s}\cdot \boldsymbol{t}_\varepsilon \, dx = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, dx. \tag{39}
$$

To pass to the limit in (39) we again use two-scale convergence. To this end, we need first to homogenize equation (38) for v_ε. Note that, thanks to the a priori estimate in Lemma 2.2.3, v_ε is also uniformly bounded in $H_0^1(\Omega)$. Thus, the homogenization of (38) follows the same procedure as that in Proposition 2.2.4. The only difference comes from the argument used to pass to the limit in the right-hand side of the variational formulation (25):

$$
\begin{aligned}
&\sum_{p=1}^{n(\varepsilon)} \left(\frac{1}{|Y_p^\varepsilon|} \int_{Y_p^\varepsilon} \boldsymbol{t}_\varepsilon(x)\, dx\right)\cdot\left(\int_{\Gamma_p^\varepsilon} \left(\phi(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)\right) \boldsymbol{n}\, ds\right) \\
&= \int_\Omega \left(\chi\left(\frac{x}{\varepsilon}\right) - 1\right) E_\varepsilon P_\varepsilon \boldsymbol{t}_\varepsilon \cdot \left(\nabla\phi(x) + \nabla_y \phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x \phi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx \\
&= \int_\Omega \boldsymbol{t}_\varepsilon \cdot E_\varepsilon P_\varepsilon \left(\left(\chi\left(\frac{x}{\varepsilon}\right) - 1\right)\left(\nabla\phi(x) + \nabla_y \phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x \phi_1\left(x, \frac{x}{\varepsilon}\right)\right)\right) dx.
\end{aligned}
\tag{40}
$$

It is not difficult to check that, for any smooth function $\psi(x, y) \in \mathscr{D}(\Omega; C_\#^\infty(Y))$, $E_\varepsilon P_\varepsilon \psi(x, x/\varepsilon)$ converges strongly to $\int_Y \psi(x, y)\, dy$ in $L^2(\Omega)$. With this property, it is now straightforward to pass to the limit in (40). Denoting by $v(x)$ the weak limit of v_ε, and by $v_1(x, y)$ the function such that ∇v_ε two-scale converges to $\nabla_x v + \nabla_y v_1$, we find that the sequence v_ε converges in the space $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y^*)/\mathbb{R}]$ to the unique solution (v, v_1) of the same homogenized problem (30) where the right-hand side $\boldsymbol{s}$ is simply changed for $\boldsymbol{t}$.

Then, to pass to the limit in (39) we use the so-called *strong* two-scale convergence of ∇u_ε (see part 3 of Proposition 2.2.1), which is a consequence of the energy convergence (24). Finally, we obtain

$$
\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, dx = \int_\Omega \int_{Y^*} (\nabla_x u + \nabla_y u_1)\cdot(\nabla_x v + \nabla_y v_1)\, dx\, dy,
$$

which gives the desired result up to an easy integration by parts in the homogenized system (30).

Remark 2.2.7. In the definition of the original operator S_ε, we asked that each tube displacement $\boldsymbol{s}_p$ be constant on its section T_p. From a physical point of view, this ensures that the tube displacements are rigid. To define a proper extension $\tilde{S}_\varepsilon = E_\varepsilon S_\varepsilon P_\varepsilon$ of S_ε, it is absolutely necessary to define a projection operator P_ε which maps $L^2(\Omega)^N$ onto piecewise constant functions on each tube boundary, in order to preserve this requirement of the model. Previous attempts to homogenize system (20) have failed because the violation of this property leads to a different homogenized problem and an unphysical limit operator. For example, our results do not agree with those in [28] because uncorrect asymptotic expansions were used that lead to the implicit boundary condition

$$\frac{\partial u_\varepsilon}{\partial n} = \boldsymbol{s}(x)\cdot\boldsymbol{n} \quad \text{on } \Gamma_p^\varepsilon$$

which contradicts the model's requirement of rigid displacements.

2.3. *Spectrum of the limit operator*

This section is devoted to the study of the spectrum $\sigma(S)$ of the limit operator S, i.e., to the proofs of Theorems 2.1.4, 2.1.5, and Proposition 2.1.11. Recall that the limit operator S is defined by

$$S\boldsymbol{s} = (A - I)\nabla u - (A - \theta I)\boldsymbol{s}, \tag{41}$$

where u is the unique solution in $H^1(\Omega)$ of the homogenized problem

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= \operatorname{div}((I - A)\boldsymbol{s}) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{42}$$

and A is the constant homogenized matrix defined by (15). Since A is symmetric, and thus diagonizable, we may assume with no loss of generality, that A is diagonal in the canonical basis $(\boldsymbol{e}_i)_{1 \leqq i \leqq N}$, with eigenvalues $0 < \alpha_1 \leqq \cdots \leqq \alpha_N \leqq \theta$.

A generic eigenvalue λ and eigenvector $\boldsymbol{s}$ of S satisfy

$$S\boldsymbol{s} = \lambda\boldsymbol{s},$$

which, combined with (41), implies that the corresponding potential u is given by

$$\nabla u = (A - I)^{-1}(A + (\lambda - \theta)I)\boldsymbol{s}. \tag{43}$$

Introducing a matrix $B(\lambda)$ defined by

$$B(\lambda) = A - (I - A)(A + (\lambda - \theta)I)^{-1}(I - A),$$

we find that the potential u must be a solution in $H_0^1(\Omega)$ of

$$\begin{aligned} -\operatorname{div} B(\lambda)\nabla u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{44}$$

Note that (44) is either elliptic or hyperbolic (but always with a Dirichlet boundary condition for *all* variables) and that $B(\lambda)$ may have eigenvalues equal to infinity. Investigating the possible solutions of (44) should give us a characterization of the eigenvalues of S. However, since S is not compact, its spectrum is not merely made of eigenvalues, but can also contain an essential part. To characterize the essential spectrum of S we use Weyl's criterion (see, e.g., [31]).

Lemma 2.3.1. *A real λ belongs to the essential spectrum of S if and only if there exists a sequence $s_n \in L^2(\Omega)^N$ such that*

$$\begin{aligned} s_n &\rightharpoonup 0 \quad \text{in } L^2(\Omega)^N \text{ weakly, with } \|s_n\|_{L^2(\Omega)^N} = 1, \\ Ss_n - \lambda s_n &\to 0 \quad \text{in } L^2(\Omega)^N \text{ strongly.} \end{aligned}$$

Proposition 2.3.2. *Let $\lambda = \theta - \alpha_i$ for some $i \in \{1, \ldots, N\}$. In this case $B(\lambda)$ has an eigenvalue equal to infinity. Then λ is an eigenvalue of infinite multiplicity of S.*

Proof. By multiplying equation (43) by the eigenvector $\boldsymbol{e}_i$ and integrating against any test function on Ω, we obtain

$$\frac{\partial u}{\partial x_i}(x) = 0 \quad \text{in } \Omega.$$

Thanks to the Dirichlet boundary condition on $\partial\Omega$, this implies that u is identically zero in Ω. Thus, (41) and (42) yield

$$\begin{aligned} (A - \alpha_i I)\boldsymbol{s} &= 0 \quad \text{in } \Omega, \\ \operatorname{div}(I - A)\boldsymbol{s} &= 0 \quad \text{in } \Omega, \end{aligned}$$

which gives the definition of the associated eigenspace in $L^2(\Omega)^N$, which is infinite-dimensional, as can easily be checked by taking

$$\boldsymbol{s}(x) = \phi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)\boldsymbol{e}_i,$$

where the scalar function ϕ does not depend on x_i.

Proposition 2.3.3. *Let $\lambda = \theta + (1 - 2\alpha_i)/\alpha_i$ for some $i \in \{1, \ldots, N\}$. In this case $B(\lambda)$ has a zero eigenvalue. Then λ belongs to the essential spectrum of S.*

Proof. In the isotropic case ($\alpha_i = \alpha$ for all $i \in \{1, \ldots, N\}$), one can easily prove that λ is an eigenvalue of infinite multiplicity whose eigenspace is $\{\nabla q$ such that $q \in H_0^1(\Omega)\}$. In the anisotropic case, it is generally not true that λ is an eigenvalue. For example, in two dimensions with $\alpha_1 < \alpha_2$, if we assume with no loss of generality, that $\lambda = \theta + (1 - 2\alpha_1)/\alpha_1$, then the associated potential u satisfies $\partial^2 u/\partial x_2^2 = 0$, which, together with the Dirichlet boundary condition, implies that u is identically zero in Ω. Then, by (43), s must also be equal to zero, which means that λ is not an eigenvalue.

Thus, we apply Weyl's criterion to show that λ belongs to $\sigma(S)$. Let $\phi(y)$ be a Y-periodic function defined by

$$\phi(y) = \begin{cases} y_i & \text{if } 0 < y_i < \frac{1}{2}, \\ 1 - y_i & \text{if } \frac{1}{2} \leqq y_i < 1. \end{cases}$$

Let us check Weyl's criterion for the sequence s_n defined by

$$s_n(x) = \nabla(n^{-1}\phi(nx)).$$

The function s_n takes alternatively the values e_i and $-e_i$ with periodicity n^{-1}, where e_i is the eigenvector associated with the eigenvalue α_i. It is easily seen that s_n converges weakly to zero in $L^2(\Omega)^N$, while its norm is uniformly bounded away from zero. To show that $Ss_n - \lambda s_n$ converges strongly to zero, we must first compute the associated potential u_n, the solution in $H_0^1(\Omega)$ of

$$-\operatorname{div} A\nabla u_n = \operatorname{div}(I - A)\, s_n. \tag{45}$$

Studying the asymptotic behavior of the solution of u_n (45) is again a problem of homogenization. Then, it is a classical matter to prove (for example, by using two-scale convergence) that

$$u_n(x) = -\frac{1 - \alpha_i}{\alpha_i} n^{-1}\phi(nx) + r_n(x),$$

where r_n is a remainder term which converges strongly to zero in $H^1(\Omega)$. Thus,

$$Ss_n - \lambda s_n = (A - I)\nabla r_n,$$

which proves the desired result.

Proposition 2.3.4. *Let λ be such that the matrix $B(\lambda)$ is either positive-definite, or negative-definite. Then, λ does not belong to $\sigma(S)$.*

Proof. It is easily seen from (44) that the potential u must be zero. Then, from (43) we deduce that s is also zero. Thus λ is not an eigenvalue of S. To show that λ does not even belong to the essential spectrum of S, we try to get a contradiction from Weyl's criterion. Assume that there exists a normalized sequence $s_n \in L^2(\Omega)^N$ such that

$$s_n \rightharpoonup 0 \text{ weakly}, \quad r_n = Ss_n - \lambda s_n \to 0 \text{ strongly}.$$

From the homogenized equation we get

$$-\operatorname{div} B(\lambda)\nabla u_n = -\operatorname{div}(I - A)(A + (\lambda - \theta)I)^{-1} r_n.$$

From standard regularity results for this elliptic equation, we infer that u_n is compact in $H^1(\Omega)$, which is in contradiction with the fact that s_n cannot converge strongly, since

$$s_n = (A + (\lambda - \theta)I)^{-1}((A - I)\nabla u_n - r_n).$$

Proposition 2.3.5. *Let λ be such that $B(\lambda)$ has neither a zero nor an infinite eigenvalues, but has simultaneously positive and negative eigenvalues. Then, λ belongs to the essential spectrum of S.*

Proof. In this case, the spectral equation (44) is hyperbolic: we do not know if it has a solution in general. (For some special choices of Ω and B, one can show that there exist an infinite number of solutions.) Thus, we do not try to prove that λ is an eigenvalue; rather, we again apply Weyl's criterion to a suitably chosen sequence $s_n \in L^2(\Omega)^N$. We assume that the matrix $B(\lambda)$ has two non-zero eigenvalues β_1^2 and $-\beta_N^2$ such that β_1 and β_N are positive reals. Let us consider in the rectangular cell $Z = (0, \beta_1) \times (0, 1) \times \cdots \times (0, 1) \times (0, \beta_N)$ the hyperbolic equation

$$\begin{aligned} -\operatorname{div}_y(B(\lambda)\nabla_y w) = 0 \quad \text{in } Z, \\ y \to w(y) \text{ is } Z\text{-periodic.} \end{aligned} \tag{46}$$

As is well known, (46) has infinitely many solutions. From among them, we choose

$$w(y) = \phi_+(\beta_1^{-1} y_1 + \beta_N^{-1} y_N) + \phi_-(\beta_1^{-1} y_1 - \beta_N^{-1} y_N)$$

where ϕ_+ and ϕ_- are two 1-periodic functions of a single real variable. Then, from (43) we define s_n by

$$s_n(x) = (A + (\lambda - \theta)I)^{-1}(A - I)\nabla(n^{-1}w(nx)).$$

It is not difficult to check that s_n converges weakly to zero in $L^2(\Omega)^N$, while its norm is uniformly bounded away from zero. As in Proposition 2.3.3, we must compute the associated potential u_n to show that $Ss_n - \lambda s_n$ converges strongly to zero. A standard homogenization result implies that

$$u_n(x) = n^{-1}w(nx) + r_n(x),$$

where r_n is a remainder term which converges strongly to zero in $H^1(\Omega)$. Thus, we have

$$Ss_n - \lambda s_n = (A - I)\nabla r_n,$$

which proves the desired result.

To conclude this section, it remains to prove Proposition 2.1.11 on the pointwise convergence of the spectra of a sequence of strongly convergent operators.

Proof of Proposition 2.1.11. Let $\lambda \in \sigma(S)$, and assume that λ is not the limit of any sequences of eigenvalues of S_ε. This means that there exists a positive constant $\delta > 0$ such that

$$|\lambda_\varepsilon - \lambda| \geqq \delta$$

for sufficiently small ε and for any eigenvalue $\lambda_\varepsilon \in \sigma(S_\varepsilon)$. Obviously this implies that

$$\| \tilde{S}_\varepsilon s - \lambda s \|_{L^2(\Omega)^N} \geqq \delta \tag{47}$$

for any function $s(x) \in L^2(\Omega)^N$. Since the convergence of S_ε to S is strong, one can pass to the limit in (47) and obtain

$$\| Ss - \lambda s \|_{L^2(\Omega)^N} \geqq \delta,$$

for any function $s(x)$, which contradicts the fact that λ belongs to the spectrum of S. Thus, λ is attained as a limit of a sequence $\lambda_\varepsilon \in \sigma(S_\varepsilon)$.

To complete the proof of Proposition 2.1.11, it remains to show that if a sequence of eigenvalues λ_ε converges to a limit λ outside $\sigma(S)$, then any associated sequence of eigenvectors $s_\varepsilon(x)$ converges to zero weakly in $L^2(\Omega)^N$. The spectral equation is

$$\tilde{S}_\varepsilon s_\varepsilon = \lambda_\varepsilon s_\varepsilon. \tag{48}$$

Thanks to the strong convergence of $\tilde{S}_\varepsilon$, we can pass to the limit (up to a subsequence) in (48), and denoting by s the weak limit of the sequence s_ε we obtain

$$Ss = \lambda s.$$

Since λ does not belong to $\sigma(S)$, this necessarily implies that the limit s is equal to zero.

3. Bloch-wave homogenization: a coupled macro-microscopic limit operator

3.1. Motivation and discrete Bloch waves

As we already discussed at the end of Section 2.1, the convergence analysis of the operator $\tilde{S}_\varepsilon$ in $L^2(\Omega)^N$ is not entirely satisfactory. Indeed we proved the convergence of the spectral family of $\tilde{S}_\varepsilon$ to that of the limit operator S, which only gives a very weak convergence of the corresponding spectra. We recall that any element of the spectrum of S is attained by a sequence of eigenvalues of $\tilde{S}_\varepsilon$ as ε goes to zero. However, it may happen that some sequence of eigenvalues of $\tilde{S}_\varepsilon$ converges to a limit which does not belong to the spectrum of S. In this case, by virtue of Proposition 2.1.11, the corresponding sequence of normalized eigenvectors converges weakly to zero in $L^2(\Omega)^N$.

This indicates that the above lack of continuity of the spectrum of $\tilde{S}_\varepsilon$ as ε goes to zero is due to the particular choice of $\tilde{S}_\varepsilon$ as an extension of the finite-dimensional operator S_ε. Recall that S_ε was originally defined on $\mathbb{R}^{Nn(\varepsilon)}$, which was embedded in $L^2(\Omega)^N$ to define its extension $\tilde{S}_\varepsilon$ on a fixed Hilbert space independent of ε. Although "natural", this choice of $L^2(\Omega)^N$ is somewhat arbitrary, and is actually the cause of our troubles. In other words, we can select a different extension of S_ε and a different reference space in which $\mathbb{R}^{Nn(\varepsilon)}$ is embedded that yield a different limit operator having a much larger spectrum. Of course, the question is how to find such an extension. In our quest we should be guided by the fact that the eigenvalues of $\tilde{S}_\varepsilon$ which "escape" from the spectrum of S correspond to eigenvectors which converge weakly to zero in $L^2(\Omega)^N$. Thus, our new choice of extension should "capture" the oscillations contained in these weak convergences. In particular, eigenvectors of $\tilde{S}_\varepsilon$ have a tendency to oscillate on the scale ε of the periodic arrangement of the tubes. A natural candidate for the reference space is that of periodically oscillating functions $\varphi(x, x/\varepsilon)$, where $\varphi(x, y)$ is periodic in y for all x in Ω.

From a physical point of view it is also clear that the macroscopic limit operator S cannot contain all the limit eigenfrequencies of S_ε. As mentioned in Remark 2.1.6, the spectrum of S includes only those frequencies corresponding to a macroscopic displacement of the tubes. More precisely, the smoothness of the associated eigenvectors (which belong to $L^2(\Omega)^N$) implies that neighboring tubes have similar displacements. In particular, this does not take into account the physically reasonable situation where two adjacent tubes vibrate with opposite phases (their displacements take opposite values). This phenomenon suggests that we consider a period containing a bundle of tubes having uncorrelated vibrations rather than a single tube as we did in Section 2.

Having this in mind, we now regard the fluid domain Ω_ε, defined in Section 1.2, as a periodic domain with a new period εK corresponding to a new reference cell $KY = (0; K)^N$. (Here K is a given positive integer.) In the reference cell KY there are K^N tubes $(T_j)_{0 \leq j \leq K-1}$ indexed by a multi-integer $j = (j_1, \ldots, j_N)$, where each component belongs to $\{0, 1, \ldots, K-1\}$. To each tube T_j in the periodic reference cell KY, we associate the subcell Y_j and the fluid subcell $Y_j^* = Y_j \backslash T_j$ analogous to Y and Y^* respectively (see Figure 3).

The main idea of this new framework is to attach to each tube T_j in the reference cell KY a different displacement function $s_j(x)$. This procedure allows us to extend the original finite-dimensional operator S_ε to an operator S_ε^K acting on $[L^2(\Omega)^N]^{K^N}$ (instead of $L^2(\Omega)^N$ for $\tilde{S}_\varepsilon$). We emphasize that the family $(s_j(x))_{0 \leq j \leq K-1}$ is equivalent to a single oscillating displacement $s(x, y) \in L^2(\Omega \times KY)^N$ which is

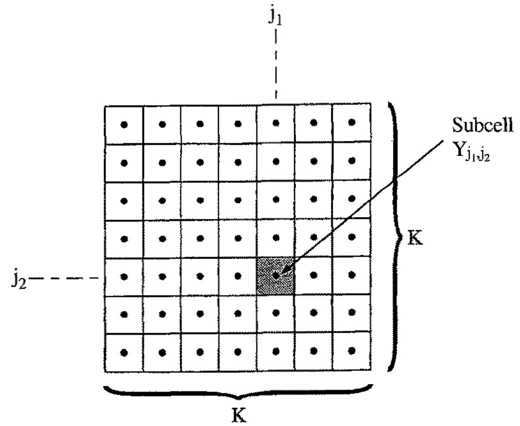

Fig. 3. Reference cell KY composed of $K \times K$ subcells $(Y_{j_1, j_2})_{\substack{0 \leq j_1 \leq K-1 \\ 0 \leq j_2 \leq K-1}}$ (in 2 dimensions)

constant and equal to $s_j(x)$ in each subcell Y_j. (In other words, the space $[L^2(\Omega)^N]^{K^N}$ is isomorphic to the subspace of $L^2(\Omega \times KY)^N$ consisting of piecewise constant functions in y.) Thus, according to our intuition, the new operator S_ε^K should capture in the limit more oscillating eigenvectors than $\tilde{S}_\varepsilon$.

To give a precise definition of this operator S_ε^K, we have to introduce, as in Section 2.1, two linear maps, P_ε^K and E_ε^K linking $[L^2(\Omega)^N]^{K^N}$ and $\mathbb{R}^{Nn(\varepsilon)}$ such that $S_\varepsilon^K = E_\varepsilon^K S_\varepsilon P_\varepsilon^K$. To do so, we first introduce some notations connecting the two indices p (indexing constant vectors in $\mathbb{R}^{Nn(\varepsilon)}$ and j (indexing vector functions in $[L^2(\Omega)^N]^{K^N}$).

Definition 3.1.1. Let $j = (j_1, \ldots, j_N)$ be the multi-integer which enumerates all the tubes in the periodic reference cell KY. Each component of j belongs to $\{0, 1, \ldots, K-1\}$, and we use the notation $0 \leqq j \leqq K-1$ to indicate its range. Let $p = (p_1, \ldots, p_N\}$ be the multi-integer which enumerates all the tubes in Ω_ε (see (3)). We do not precisely describe the range of each of its components, and for simplicity the range of p is denoted by $1 \leqq p \leqq n(\varepsilon)$. We define a third multi-integer $l = (l_1, \ldots, l_N)$ which enumerates all the periodic reference cells $\varepsilon(KY)$ in Ω_ε. For simplicity its range is denoted by $1 \leqq l \leqq n_K(\varepsilon)$. These three indexes are assumed to be related by the one-to-one map

$$l_m = E\left(\frac{p_m}{K}\right), \quad j_m = p_m - Kl_m \quad \forall m = 1, \ldots, N \tag{49}$$

where $E(\cdot)$ denotes the integer-part function.

The projection operator P_ε^K, which maps $[L^2(\Omega)^N]^{K^N}$ onto $\mathbb{R}^{Nn(\varepsilon)}$, associates with any family of functions $(s_j(x))_{0 \leqq j \leqq K-1}$ a collection of constant vectors $(s_p)_{0 \leqq p \leqq n(\varepsilon)}$, each of them obtained by simply taking the average of $s_j(x)$ on the cell $\varepsilon(KY)_l$, where p is related to (l, j) through Definition 3.1.1. More precisely, we have

$$\begin{aligned} &P_\varepsilon^K : [L^2(\Omega)^N]^{K^N} \to \mathbb{R}^{Nn(\varepsilon)}, \\ &(s_j(x))_{0 \leqq j \leqq K-1} \to (s_p)_{1 \leqq p \leqq n(\varepsilon)} \end{aligned} \tag{50}$$

where

$$s_i = \frac{1}{|\varepsilon(KY)_l|} \int_{\varepsilon(KY)_l} s_j(x)\, dx \tag{51}$$

with p related to (l, j) by formula (49).

On the other hand, the extension operator E_ε^K is defined by

$$\begin{aligned} &E_\varepsilon^K : \mathbb{R}^{Nn(\varepsilon)} \to [L^2(\Omega)^N]^{K^N}, \\ &(s_p)_{1 \leqq p \leqq n(\varepsilon)} \to (s_j(x))_{0 \leqq j \leqq K-1} \end{aligned} \tag{52}$$

where

$$s_j(x) = \sum_l \chi_\varepsilon(KY)_l(x) s_p \tag{53}$$

since p is related to (l, j) by formula (49), and where, by convention, s_p is taken equal to zero if, near the boundary of Ω, some values of l and j yield an index p which corresponds to a subcell containing no tube (or outside the domain Ω).

Then, S_ε^K is defined by the composition rule

$$\begin{aligned} S_\varepsilon^K &: [L^2(\Omega)^N]^{K^N} \to [L^2(\Omega)^N]^{K^N}, \\ (s_j)_{0 \leq j \leq K-1} &\to E_\varepsilon^K S_\varepsilon P_\varepsilon^K (s_j)_{0 \leq j \leq K-1} \end{aligned} \tag{54}$$

where S_ε is the original operator defined by (8) on $\mathbb{R}^{Nn(\varepsilon)}$.

One can easily check that the adjoint operator of P_ε^K, denoted by $(P_\varepsilon^K)^*$, is nothing but $(\varepsilon K)^{-N} E_\varepsilon^K$, and that $P_\varepsilon^K E_\varepsilon^K$ is equal to the identity in $\mathbb{R}^{Nn(\varepsilon)}$. Therefore, S_ε^K is also self-adjoint and its spectrum is just the same as that of S_ε, except for the eigenvalue 0, which is again an eigenvalue of infinite multiplicity. It is also worth noting that our old extended operator $\tilde{S}_\varepsilon$ is nothing but S_ε^K with $K = 1$.

The convergence analysis of this sequence of extensions S_ε^K is amenable to the two-scale convergence method by appropriately choosing $\varepsilon K Y$ as the period of the oscillating test functions. It turns out that the corresponding limit operator S^K has a quite complicated form since it mixes all the variables $s_j(x)$. However, S^K can be diagonalized, and thus considerably simplified, by introducing the so-called Bloch-wave decomposition of the family $(s_j)_{0 \leq j \leq K-1}$. Our main results will be presented with this diagonal form of S^K (each diagonal term being an operator from $L^2(\Omega)^N$ into itself). We call this blend of the two-scale convergence method and the Bloch-wave analysis *a Bloch-wave homogenization procedure.*

For the sake of completeness, we conclude this section by recalling the ad hoc discrete version of the Bloch-wave decomposition we shall use in the sequel (we call it *discrete* in contrast with the usual *continuous* Bloch-wave decomposition of functions in $L^2(\mathbb{R}^N)$; see [7, 14, 24, 39]). Such a decomposition was already introduced for the same problem in a different context in [1]. With each family $(s_j)_{0 \leq j \leq K-1} \in (\mathbb{C}^N)^{K^N}$, we associate a KY-periodic function, constant and equal to s_j in each subcell Y_j. In other words, denoting by $\chi_{Y_j}(y)$ the characteristic function of Y_j, we consider the function

$$s(y) = \sum_{j=0}^{K-1} s_j \chi_{Y_j}(y) \quad \forall y \in KY.$$

The Bloch-wave decomposition of $s(y)$ is given by

Lemma 3.1.2. *There exists a unique family of constant vectors $(t_j)_{0 \leq j \leq K-1}$ in $\mathbb{C}^N$ such that*

$$s(y) = \sum_{j=0}^{K-1} t_j e^{2\pi i \theta_j \cdot E(y)} \quad \forall y \in KY \quad \text{with } \theta_j = \frac{j}{K} \tag{55}$$

where $E(\cdot)$ denotes the integer-part function. Moreover, Parseval's identity holds:

$$\frac{1}{K^N} \int_{KY} |s(y)|^2 \, dy = \frac{1}{K^N} \sum_{j=0}^{K-1} |s_j|^2 = \sum_{j=0}^{K-1} |t_j|^2. \tag{56}$$

Let $\mathscr{B}$ (for Bloch decomposition) denote the linear map which gives $K^{N/2}(t_j)$ in terms of (s_j). The map $\mathscr{B}$ defines an isometry on $(\mathbb{C}^N)^{K^N}$. In particular, the adjoint $\mathscr{B}^*$ of $\mathscr{B}$ is simply $\mathscr{B}^{-1}$.

Proof. For each $j = 0, \ldots, K-1$, let us define t_j by

$$t_j = \frac{1}{K^N} \sum_{j=0}^{K-1} s_{j'} e^{-2\pi i j' \cdot \theta_j}. \tag{57}$$

It suffices now to check that (55) holds with this definition of t_j:

$$\frac{1}{K^N} \sum_{j=0}^{K-1} \sum_{j'=0}^{K-1} s_{j'} e^{2\pi i \theta_j \cdot (E(y) - j')} = \sum_{j'=0}^{K-1} s_{j'} \left(\frac{1}{K^N} \sum_{j=0}^{K-1} e^{2\pi i \theta_j \cdot (E(y) - j')} \right). \tag{58}$$

If $E(y) = j'$, the expression in parentheses on the right-hand side of (58) is equal to 1. If $E(y) \neq j'$, it is equal to 0, thanks to a well-known property of the K-th roots of 1 in the complex plane (see [1] for details). Thus, this term is equal to $\chi_{Y_{j'}}(y)$, which proves (55). The proof of (56) is similar, so we omit it.

The net effect of the discrete Bloch-wave decomposition is to allow us to work with the family of frequencies $(t_j)_{0 \leq j \leq K-1}$ instead of the family of displacements $(s_j)_{0 \leq j \leq K-1}$. In the next section, we shall see that the operator $T^K = \mathscr{B} S^K \mathscr{B}^*$, defined on $[L^2(\Omega)^N]^{K^N}$, is diagonal, i.e., $T^K(t_j) = (T_j^K t_j)$, where each component T_j^K is an operator on $L^2(\Omega)^N$. Thus, in the present context, it is easier to work with frequencies than displacements. Note that, as usual, the above Bloch-wave decomposition holds for families of complex-valued vectors. Even if the original family (s_j) is real, its image (t_j) under $\mathscr{B}$ is complex. Therefore, in the sequel we shall sometimes need to consider complex-valued functions. To simplify the exposition, we shall use the same notations for the usual Sobolev spaces considered as vector spaces of complex-valued, or real-valued, functions, according to the context. As usual, if u is any complex number, then by $\bar{u}$ we mean the conjugate of u.

3.2. Main results

Since S_ε^K has been introduced, we can now study its asymptotic behavior as ε goes to zero. In Section 3.3 we shall prove the following generalization of Theorem 2.1.1.

Theorem 3.2.1. *For each fixed $K \in \mathbb{N}^*$, the sequence S_ε^K converges strongly to a limit S^K in $[L^2(\Omega)^N]^{K^N}$, i.e., for any family $(s_j(x))_{0 \leq j \leq K-1}$,*

$$S_\varepsilon^K(s_j) \to S^K(s_j) \text{ in } [L^2(\Omega)^N]^{K^N} \text{ strongly.} \tag{59}$$

By using the Bloch-wave decomposition operator $\mathscr{B}$ defined in Lemma 3.1.2, the limit operator S^K can be diagonalized, i.e.,

$$S^K = \mathscr{B}^* T^K \mathscr{B}, \quad \text{with } T^K = \operatorname{diag}[(T_j^K)_{0 \leq j \leq K-1}] \tag{60}$$

where the entries T_j^K are self-adjoint continuous operators in $L^2(\Omega)^N$, defined by

$$T_j^K t = \begin{cases} (A - I)\nabla u - (A - \theta I)t & \text{if } j = 0, \\ A^j t & \text{if } j \neq 0, \end{cases} \tag{61}$$

where A is the usual homogenized matrix (defined by (15)), $\theta = |Y^*|$ the volume fraction of fluid, u the unique solution of the usual homogenized problem

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= \operatorname{div}((I - A)t) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{62}$$

and A^j the Bloch-homogenized matrix with components $(A^j_{mm'})_{1 \leqq m,m' \leqq N}$ defined by

$$\bar{A}^j_{mm'} = \int_{Y^*} \nabla w^j_m(y) \cdot \nabla \bar{w}^j_{m'}(y)\, dy \tag{63}$$

where $(w^j_m)_{1 \leqq m \leqq N}$ are solutions of the so-called cell problem at the Bloch frequency $\theta_j = (j/K)$:

$$\begin{aligned} -\Delta w^j_m &= 0 && \text{in } Y^*, \\ (\nabla w^j_m - e_m)\cdot n &= 0 && \text{on } \partial T, \\ y \to e^{-2\pi i\,\theta_j \cdot y}\, w^j_m(y) &\text{ is } Y^*\text{-periodic.} \end{aligned} \tag{64}$$

Remark 3.2.2. Observe that the component T_0^K of T^K is nothing but the macroscopic limit operator S defined in Section 2. The other components of T^K are simple linear multiplication operators that represent the microscopic limit behavior of the sequence S_ε^K. Note also that the homogenized problem (62) is independent of K, so that the macroscopic fluid potential u is also independent of K.

Remark 3.2.3. A function $w(y)$ satisfying the periodicity condition of the cell problem (64) is said to be $(e^{2\pi i\theta_j}, Y^*)$-periodic; its properties have been extensively studied in [1, 12]. We briefly recall their basic results. This class of functions can equivalently be characterized by the (generalized) periodicity condition

$$w(y + j') = e^{2\pi i j' \cdot \theta_j} w(y) \quad \forall y \in Y^*, \quad \forall j' \in \mathbb{Z}^N.$$

The subspace of $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ consisting of such $(e^{2\pi i\theta_j}, Y^*)$-periodic functions is a Hilbert space for the usual inner product in $H^1(Y^*)$ (see Theorem 2.1 of [1]). We denote it by $H^1_\#(e^{2\pi i\theta_j}, Y^*)$. Then, the cell problem (64) is well-posed in this space $H^1_\#(e^{2\pi i\theta_j}, Y^*)$ if $j \neq 0$, and in the quotient space $[H^1_\#(Y^*)/\mathbb{C}]$ if $j = 0$ (see Lemma 3.1 [1]).

Let us now describe the spectrum $\sigma(S^K)$ of S^K. Since the Bloch-wave decomposition operator $\mathscr{B}$ is an isometry on $\mathbb{C}^N$, the spectrum of S^K is exactly that of T^K, which is a diagonal operator whose diagonal entries $(T_j^K)_{0 \leqq j \leqq K-1}$ are linear, self-adjoint, continuous operators in $L^2(\Omega)^N$. Therefore its spectrum is nothing else than the union of the spectra of the operators T_j^K for $j = 0, \ldots, K-1$, i.e.,

$$\sigma(S^K) = \sigma(S) \cup \bigcup_{j \neq 0} \sigma(T_j^K). \tag{65}$$

Recall that T_0^K is equal to S, which has a purely essential spectrum (see Theorems 2.1.4 and 2.1.5). If $j \neq 0$, T_j^K is simply the *multiplication operator* by the Bloch-homogenized matrix A^j defined by (63). The description of $\sigma(T_j^K)$ follows easily from the definition (63) of A^j.

Proposition 3.2.4. *For each fixed $j \neq 0$, $0 \leqq j \leqq K-1$, the Bloch-homogenized matrix A^j is Hermitian and positive definite. Thus, the operator T_j^K defined on $L^2(\Omega)^N$ by* (61) *is self-adjoint and nonnegative, and its spectrum coincides with the eigenvalues of A^j. All the elements of $\sigma(T_j^K)$ are eigenvalues of infinite multiplicity, and T_j^K is therefore non-compact.*

Collecting the above results concerning the spectra of the components T_j^K, we deduce that S^K is a non-compact, self-adjoint operator whose spectrum is purely essential. Since T^K is diagonal, it is very easy to compute its spectral family in terms of the spectral families of its components. Indeed, if $\mathscr{F}_j^K(\lambda)$ denotes the spectral family of T_j^K and $\mathscr{E}^K(\lambda)$ denotes that of S^K, then, for any $(s_j)_{0 \leqq j \leqq K-1} \in [L^2(\Omega)^N]^{K^N}$, $\mathscr{E}^K(\lambda)$ is simply given by

$$\mathscr{E}^K(\lambda)(s_j) = \mathscr{B}^* \operatorname{diag}[\mathscr{F}_j^K(\lambda)] \, \mathscr{B}(s_j).$$

Recall that for the isotropic case (i.e., $A = \alpha I$) we were able to obtain an explicit formula for $\mathscr{F}_0^K(\lambda)$ (see (17)). In order to describe $\mathscr{F}_j^K(\lambda)$ for $j \neq 0$, let us denote by $0 < \lambda_1^j \leqq \cdots \leqq \lambda_N^j$ the eigenvalues, and by $(e_m^j)_{1 \leqq m \leqq N}$ the corresponding eigenvectors of the Bloch-homogenized matrix A^j. The spectral family $\mathscr{F}_j^K(\lambda)$ (for $j \neq 0$) is defined by

$$\mathscr{F}_j^K(\lambda) t(x) = \sum_{\{m \mid \lambda_m^j \leqq \lambda\}} (t(x) \cdot e_m^j) e_m^j.$$

As in Section 2, we can deduce from the strong convergence of S_ε^K to S^K the convergence of the spectral family of S_ε^K to that of S^K. We are again in a non-standard situation where neither S_ε^K nor its resolvent converges uniformly to their limits. Thus, we cannot expect pointwise convergence of the eigenvalues and we must content ourselves with the following result, which is an immediate consequence of Rellich's Theorem.

Theorem 3.2.5. *Let S_ε^K and S^K be the operators on $[L^2(\Omega)^N]^{K^N}$ defined by* (54) *and* (60) *respectively. For all $\lambda \in \mathbb{R}$ such that λ is not an eigenvalue of S^K, the spectral family $\mathscr{E}_\varepsilon^K(\lambda)$ of S_ε^K converges strongly to that of S^K in the following sense: For each $(s_j)_{0 \leqq j \leqq K-1} \in [L^2(\Omega)^N]^{K^N}$,*

$$\mathscr{E}_\varepsilon^K(\lambda)(s_j) \to \mathscr{E}^K(\lambda)(s_j) \quad \textit{in } [L^2(\Omega)^N]^{K^N} \textit{ strongly as } \varepsilon \to 0.$$

Since the main goal of this paper is to study the pointwise convergence of the spectrum $\sigma(S_\varepsilon) = \sigma(S_\varepsilon^K)\backslash\{0\}$, we can infer some partial results from the strong convergence of the sequence S_ε^K, as we did in Section 2. More precisely, we define the limit set σ_∞ of $\sigma(S_\varepsilon)$ as the set of all cluster points of sequences λ_ε of eigenvalues of S_ε, i.e.,

$$\sigma_\infty = \{\lambda \in \mathbb{R} \text{ such that, up to a subsequence, } \exists \lambda_\varepsilon \in \sigma(S_\varepsilon), \lambda_\varepsilon \to \lambda\}. \tag{66}$$

In other words, σ_∞ is nothing but the Γ-limit of the sequence $\sigma(S_\varepsilon)$ in $\mathbb{R}$ (in the sense of the Γ-convergence of DE GIORGI [13]). Then, by application of Proposition 2.1.11 we have

$$\sigma(S^K) \subset \sigma_\infty, \tag{67}$$

but the inclusion is generically strict. Note that, by virtue of (65), the spectrum of the macroscopic limit operator S is included in the spectrum of the "macro-microscopic" limit operator S^K. Thus, the above results improve on those obtained in Section 2. Of course, we have the liberty of choosing any $K \in \mathbb{N}^*$, and by varying the cell size K we obtain a wealth of information on σ_∞. Indeed, as K goes to infinity, the Bloch frequencies $\theta_j = (j/K)$ become dense in $]0,1[^N$. It is then natural to introduce a continuous vector parameter $\theta \in \mathbb{R}^N$, and to see if we can define a "Bloch-homogenized matrix" function $A(\theta)$ which coincides with the previous matrices A^j for $\theta = \theta_j$, and whose entries $(A(\theta)_{mm'})_{1 \leq m,m' \leq N}$ are defined by

$$\overline{A_{mm'}}(\theta) = \int_{Y^*} \nabla w_m^\theta(y) \cdot \nabla \bar{w}_{m'}^\theta(y)\, dy \tag{68}$$

where $(w_m^\theta)_{1 \leq m \leq N}$ are the unique solutions of the following cell problems at the Bloch frequency θ:

$$\begin{aligned} &-\Delta w_m^\theta = 0 && \text{on } Y^*,\\ &(\nabla w_m^\theta - \boldsymbol{e}_m)\cdot \boldsymbol{n} = 0 && \text{on } \partial T,\\ &y \to e^{-2\pi i \theta \cdot y} w_m^\theta(y) \text{ is } Y^*\text{-periodic.} \end{aligned} \tag{69}$$

As for the discrete case, any function satisfying the above periodicity condition is said to be $(e^{2\pi i\theta}, Y^*)$-periodic, and the space of all functions of $H^1_{\text{loc}}(\mathbb{R}^N)$ which are $(e^{2\pi i\theta}, Y^*)$-periodic is a Hilbert space, denoted by $H^1_\#(e^{2\pi i\theta}, Y^*)$ for the inner product in $H^1(Y^*)$) (see [1] for the details). Furthermore, problem (69) is well-posed in $H^1_\#(e^{2\pi i\theta}, Y^*)$ for $\theta \neq 0$, and in $[H^1_\#(Y^*)/\mathbb{C}]$ for $\theta = 0$. In Section 3.4 we shall prove

Proposition 3.2.6. *For any value of θ, the Bloch-homogenized matrix $A(\theta)$ defined by* (68) *is Hermitian and positive-definite. Furthermore, as a function of θ, it is a $[0,1]^N$-periodic, bounded function which is continuous in $]0,1[^N$, but discontinuous at the origin $\theta = 0$.*

Since the main properties of the Bloch matrices have been established, we can now introduce a continuous family $T(\theta)$ of multiplication operators by these matrices acting on $L^2(\Omega)^N$. More precisely, for each θ in $]0,1[^N$, we define $T(\theta)$ by

$$T(\theta)t = A(\theta)t \quad \forall t \in L^2(\Omega)^N.$$

From Proposition 3.2.6 and the very definition of $T(\theta)$, it follows that $T(\theta)$ is self-adjoint and non-compact and that its spectrum coincides with the N real positive eigenvalues of $A(\theta)$ that we denote by

$$0 < \lambda_1(\theta) \leqq \lambda_2(\theta) \leqq \cdots \leqq \lambda_N(\theta),$$

where each eigenvalue is repeated as many times as its multiplicity. Since $A(\theta)$ is continuous in θ, each eigenvalue $\lambda_m(\theta)$ is also a continuous function of θ on $]0,1[^N$ (see, e.g., [18], or, in the present situation, Chapter III in [11]).

Corollary 3.2.7. *For each $m = 1, \ldots, N$, the function $\lambda_m(\theta)$ is continuous on $]0,1[^N$, and bounded on $[0,1]^N$. Thus, the closure of the images of $]0,1[^N$ under the maps $\lambda_m(\cdot)$ are connected, closed bounded intervals in $]0; +\infty[$ denoted by*

$$[a_m, b_m] = \overline{\lambda_m(]0,1[^N)}, \quad m = 1, \ldots, N,$$

where the bar denotes the closure in $\mathbb{R}$ and where

$$a_m = \inf_{\theta \in]0,1[^N} \lambda_m(\theta), \quad b_m = \sup_{\theta \in]0,1[^N} \lambda_m(\theta).$$

Remark 3.2.8. A detailed study of the behavior of $A(\theta)$ at the origin will be provided by Proposition 3.4.4. In particular, although $A(\theta)$ is not continuous at zero, we shall prove that all the eigenvalues of $A(0)$ are actually included in the so-called Bloch spectrum $\bigcup_{m=1}^N [a_m, b_m]$ (see Remark 3.4.5). For the moment, let us simply point out that, for the Bloch frequency $\theta = 0$, the cell problem involves the usual periodicity condition, and problems (69) and (16) coincide. Comparing the definitions (15) of A and (68) of $A(\theta)$, we easily check that $A(0) = |Y^*|I - A$. Thus, all the eigenvalues of $A(0)$ are also eigenvalues of S, but some of S are not in $\sigma(A(0))$. For example, in the isotropic case $A = \alpha I$ (see Lemma 2.1.3), S has two eigenvalues: $|Y^*| - \alpha$, which is also the unique eigenvalue of $A(0)$, and $|Y^*| + (1-2\alpha)/\alpha$ which is specific to S. We remark in passing that, if the cubic symmetry of Y^* implies that $A(0)$ and A are multiples of the identity, then in general this is not true any longer for $A(\theta)$ with $\theta \neq 0$.

It is now clear by the very construction of the operator $T(\theta)$ that, for all $K \geqq 1$ and for all $j \neq 0$, $0 \leqq j \leqq K-1$, the spectrum $\sigma(T_j^K)$ of the j-th component of T^K is contained in the union of all the intervals $[a_m, b_m]$, and by continuity, the closure of the union of all the spectra of the limit operators T_j^K, as K goes to infinity, is nothing else than the Bloch spectrum $\bigcup_{m=1}^N [a_m, b_m]$. Of course, since the Γ-limit sets are closed, we deduce from (67) that

$$\lim_{K \to +\infty} \sigma(S^K) = \left(\sigma(S) \cup \bigcup_{m=1}^{N} [a_m, b_m]\right) \subset \sigma_\infty. \tag{70}$$

The question is now to see whether equality in (70) is achieved, which would imply that our limiting procedure recovers the whole limit spectrum in the limit when K goes to infinity. In other words, we seek a characterization of σ_∞, i.e., a so-called *completeness* result. However, there is another potential source of limit spectrum which is not taken into account by our analysis, namely, that of eigenvectors oscillating on the ε-scale while concentrating near the boundary $\partial\Omega$. The reason for expecting such a *boundary-layer* spectrum is the following. We know from Remarks 2.1.12 and 3.2.13 that the Bloch spectrum can also be obtained by rescaling the original ε-network of tubes to size 1. Then, the boundary goes to infinity, and the limit domain is an infinite periodic arrangement of unit tubes (see [12]). The point is that it is also possible to center this rescaling procedure on some part of the boundary (rather than strictly inside the domain). In this case the limit domain is easily seen to be a half space filled with a periodic arrangement of tubes. It turns out that there actually exist sequences of eigenvectors concentrating on $\partial\Omega$. To account for this situation, we introduce σ_{boundary}, the subset of σ_∞ corresponding to this boundary-layer spectrum, defined by

$$\sigma_{\text{boundary}} = \{\lambda \in \mathbb{R}/\exists(\lambda_\varepsilon, s_\varepsilon) \text{ such that } \tilde{S}_\varepsilon s_\varepsilon = \lambda_\varepsilon s_\varepsilon, \lambda_\varepsilon \to \lambda,$$
$$\|s_\varepsilon\|_{L^2(\Omega)} = 1, \forall \omega \text{ with } \bar{\omega} \subset \Omega, \|s_\varepsilon\|_{L^2(\omega)} \to 0\}. \tag{71}$$

The main result of this paper states that, apart from this boundary-layer spectrum, the Bloch-wave homogenization method recovers all the limit spectrum.

Theorem 3.2.9. *The limit set of the spectrum of the operator S_ε is precisely made of three parts: the homogenized, the Bloch, and the boundary-layer spectrum, i.e.,*

$$\lim_{\varepsilon \to 0} \sigma(S_\varepsilon) = \sigma_\infty = \sigma(S) \cup \bigcup_{m=1}^{N} [a_m, b_m] \cup \sigma_{\text{boundary}}.$$

The proof of Theorem 3.2.9 is quite delicate and is the focus of Section 3.4. It involves two key ingredients. First, a careful analysis of the partial continuity of the matrix $A(\theta)$ near 0 is made (this is crucial for studying the asymptotic behavior of eigenvectors which oscillate on an intermediate length scale between ε and 1; see Proposition 3.4.4). Second, for any sequence of eigenvectors, we introduce two successive measures which quantify its amplitude and direction of oscillations. The first so-called *Bloch measure* selects only those oscillations having a length scale of order ε, and decomposes them into Bloch frequencies. It can be seen as an ad hoc version of the well-known Wigner, or semi-classical, measure (see [16] and [20]). The second so-called *rescaled Bloch measure* keeps track of all oscillations having a length scale much larger than ε, but still smaller than 1, and sorts them out according to their individual directions. This last type of measure is very similar, although specific to our context, to the recently introduced H-measures of P. GÉRARD [15] and L. TARTAR [37].

Remark 3.2.10. Theorem 3.2.9 is optimal in the sense that, in general, the boundary-layer spectrum σ_{boundary} is neither empty, nor included in $\sigma(S) \cup \bigcup_{m=1}^{N} [a_m, b_m]$.

The study of $\sigma_{\mathrm{boundary}}$, which is the focus of our next paper [3], is a highly non-trivial problem. For a general domain Ω (with a smooth boundary not necessarily aligned with the periodic structure), we are unable to characterize $\sigma_{\mathrm{boundary}}$. We suspect its definition depends on the possible subsequence of periods ε (a fact which is reminiscent of a recent work of F. SANTOSA & M. VOGELIUS [34]). However, when the domain is exactly made of a finite number of entire periodic cells, we shall characterize $\sigma_{\mathrm{boundary}}$ completely and prove that it is not included in the homogenized and Bloch parts of the limit spectrum (see [3]). By introducing a notion of two-scale convergence for boundary layers, we are able to adapt our Bloch-wave homogenization method in order to obtain a limit problem in a half space filled with a periodic arrangement of tubes.

Of course, if the domain Ω is a torus, there is no boundary, and thus no contribution of boundary layers in the limit spectrum! In this case, Theorem 3.2.9 can be further improved:

Corollary 3.2.11. *Let Ω be a parallelepiped, $]0, L_1[\times]0, L_2[\times \cdots]0, L_N[$, where the $(L_m)_{1 \leqq m \leqq N}$ are positive integers. Define the sequence of periods $\varepsilon_n = 1/n$. Assume that the unit tube in the periodic cell has cubic symmetry, or replace the Dirichlet boundary condition in the spectral problem* (1) *by a periodicity condition. Then, the limit spectrum reduces to*

$$\sigma_\infty = \sigma(S) \cup \bigcup_{m=1}^{N} [a_m, b_m].$$

Remark 3.2.12. Theorem 3.2.9 is a completeness result since it gives a decomposition of the limit spectrum into three parts: the homogenized or macroscopic spectrum $\sigma(S)$, the Bloch or microscopic spectrum $\bigcup_{m=1}^{N} [a_m, b_m]$, and the boundary-layer spectrum $\sigma_{\mathrm{boundary}}$. From the point of view of "physical" intuition, this result is a little surprising because it excludes a "potential" part of the limit spectrum corresponding to *mesoscopic* oscillations (which are intermediate between microscopic and macroscopic oscillations). To explain this, note that $\sigma(S)$ corresponds to macroscopic eigenvectors, while the Bloch spectrum is obtained for ε-microscopic eigenvectors. One might wonder if some eigenvectors oscillating on an intermediate scale (such as $\sqrt{\varepsilon}$) would produce different limit eigenvalues. It turns out that this *mesoscopic* case is simply a limiting case of the Bloch-wave analysis when the number of subcells K goes to infinity. In other words, the mesoscopic spectrum is already included in the Bloch spectrum. (The proof of this fact involves a careful analysis of the behavior of the matrix $A(\theta)$ near zero; see Proposition 3.4.4 and step 4 in the proof of Theorem 3.2.9.)

Remark 3.2.13. We feel that it is now appropriate to say a few words about the previous results obtained by C. CONCA & M. VANNINATHAN in [12]. In Remark 2.1.12 we recalled that they studied the same problem as ours, but rescaled by a factor ε^{-1} (this means that the tube size is fixed while the domain boundary goes to infinity, see (18)). By using a standard Bloch-decomposition method in an infinite domain, they only partially recover the limit spectrum. It turns out that they proved that

$$\sigma_\infty \supset \bigcup_{m=1}^{N} [a_m, b_m].$$

Our approach improves on theirs in two different respects: First, we give a more explicit characterization of the limit spectrum (see in particular Remark 3.4.5), and second we obtain the complete limit spectrum.

Remark 3.2.14. Theorem 3.2.9 can be interpreted as a Γ-convergence type result for the spectrum of S_ε. Let us briefly explain why. For example, take the maximum eigenvalue $\lambda_\varepsilon(Nn(\varepsilon))$ of S_ε (our argument works essentially in the same way for the minimum eigenvalue). It has the characterization

$$\lambda_\varepsilon(Nn(\varepsilon)) = \max_{u \in H_0^1(\Omega)} \frac{\varepsilon^{-N} \sum_{p=1}^{n(\varepsilon)} \left(\int_{\Gamma_p^\varepsilon} \boldsymbol{u n}\right)^2}{\int_{\Omega_\varepsilon} |\nabla u|^2}. \tag{72}$$

This maximization problem is easily seen to have at least one solution, since the embedding of $H_0^1(\Omega)$ in $L^2(\partial\Omega_\varepsilon)$ is compact, and any such solution satisfies the original spectral equation (7) with the maximal eigenvalue $\lambda_\varepsilon(Nn(\varepsilon))$. As ε goes to zero, one can study the Γ-convergence of the variational problem (72). By adapting the ideas of [8] and [21], we can prove that the Γ-limit of (72) is simply

$$\lambda_{\max} = \sup_{K \geqq 1} \sup_{u, u_1} \frac{\int_\Omega \sum_{0 \leqq j \leqq K-1} \left(\int_{T_j} (\nabla_x u + \nabla_y u_1)\, dy\right)^2 dx}{\int_\Omega \int_{KY} |\nabla_x u + \nabla_y u_1|^2\, dx\, dy}, \tag{73}$$

with

$$u(x) \in H_0^1(\Omega), u_1(x, y) \in L^2(\Omega; H_\#^1(KY)/\mathbb{R}).$$

This type of result is classical for the homogenization of periodic non-convex energies: It implies that the usual "local cell problem" is not necessarily posed in a single period Y, but rather in the union of an (a priori unknown) number of periods KY. Applying the Bloch-wave decomposition (see Lemma 3.1.2) in (73), we check that the value $\lambda_{\max}$ of the supremum coincides with the maximum value of $\lim_{K \to +\infty} \sigma(S^K)$. Therefore, this Γ-convergence method allows us to study the pointwise convergence of the minimum and maximum eigenvalues of S_ε. However, we do not know if it can succeed for the intermediate eigenvalues (in this case we have to rely on the min-max principle, whose asymptotic behavior is unclear). On the other hand, our Bloch-wave homogenization technique shows that any sequence of intermediate eigenvalues converges to a limit which belongs either to $\lim_{K \to +\infty} \sigma(S^K)$ or to the boundary-layer spectrum. In this sense, we can say that Theorem 3.2.9 is a Γ-convergence-type result. On the link between Γ-convergence and Bloch waves, related results have been obtained in [17].

Remark 3.2.15. Let us discuss some concrete consequences of our main results. Theorem 3.2.9 shows that the limit of the spectrum of S_ε always has a band structure. In the anisotropic case (i.e., when Y^* does not have a cubic symmetry), there are, at least, two series of bands (which may well intersect): One is due to the

Bloch-wave or microscopic part of the limit operator, the other arises from the macroscopic part S (see Remark 2.1.6). In the isotropic case, the spectrum of S is simply made of two points, and some simplifications arise in the number of independent "Bloch-wave" bands (see Remark 3.4.5). However, in all cases it is useless to compute numerically all the eigenvalues of S_ε (for small ε), since in the limit $\sigma(S_\varepsilon)$ is dense in σ_∞! Rather, it is preferable to compute the upper and lower bounds of these bands, which are amenable to simple computations in the unit cell Y.

From a physical point of view, one is usually interested in the lowest reasonance frequency ω_ε of the original vibration problem for a tubes bundle immersed in a fluid. Recall that we have rescaled the frequencies by

$$\lambda_\varepsilon = \frac{k - m\omega_\varepsilon^2}{\varepsilon^N \rho\omega_\varepsilon^2}.$$

Thus, low frequencies correspond to large λ_ε. An interesting open problem is to see if the maximal value of σ_∞ is attained in $\sigma(S)$ (i.e., corresponds to a macroscopic displacement of the tubes) or in the Bloch-wave microscopic part. (Note that it is never attained in the boundary-layer spectrum by virtue of Remark 3.2.14). In the latter case, another problem is to find the values of the Bloch frequency θ for which the eigenvalue $\lambda_N(\theta)$ is maximal. The numerical computations in [1] suggest that, at least in the isotropic case in two dimensions, it is maximal when one of the components of θ is equal to 0 and the other to $\frac{1}{2}$.

3.3. *Convergence analysis*

This section is devoted to the proof of Theorem 3.2.1 on the strong convergence of the sequence S_ε^K. This proof is very similar to that of Theorem 2.1.1, although a little more tedious since the reference cell KY contains K^N tubes instead of a single one. It relies on the homogenization of the problem

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} &= P_\varepsilon^K(s_j)\cdot \boldsymbol{n} && \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon), \\ u_\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{74}$$

where the family of displacements $(s_j)_{0 \leqq j \leqq K-1}$ belong to $[L^2(\Omega)]^{K^N}$. This problem has a unique solution $u_\varepsilon \in H^1(\Omega_\varepsilon)$, which, by virtue of Lemma 2.2.3, satisfies the a priori estimate

$$\| u_\varepsilon \|_{H^1(\Omega)}^2 \leqq C \sum_{j=0}^{K-1} \| s_j \|_{L^2(\Omega)^N}^2 \tag{75}$$

where the constant C does not depend on ε. (As in Section 2, we use the same notation for a function in $H^1(\Omega_\varepsilon)$ and its bounded extension in $H^1(\Omega)$; see Lemma 2.2.2.)

To homogenize problem (74), we again use the two-scale convergence method and we obtain

Proposition 3.3.1. *The sequence u_ε of solutions of* (74) *converges weakly in $H_0^1(\Omega)$ to a limit u, and its gradient ∇u_ε two-scale converges to a limit $\nabla_x u(x) + \nabla_y u^1(x, y)$, where (u, u^1) is the unique solution in $H_0^1(\Omega) \times L^2(\Omega; H_\#^1(KY^*)/\mathbb{R})$ of the two-scale homogenized problem*

$$
\begin{aligned}
- \operatorname{div}_x \left(\int_{KY^*} (\nabla u(x) + \nabla_y u^1(x,y))\, dy \right) &= |T| \sum_{0 \leq j \leq K-1} \operatorname{div}_x \boldsymbol{s}_j \quad \text{in } \Omega, \\
- \Delta_y u^1(x,y) &= 0 \quad \text{in } \Omega \times KY^*, \\
u(x) &= 0 \quad \text{on } \partial\Omega, \\
(\nabla u(x) + \nabla_y u^1(x,y) - \boldsymbol{s}_j(x)) \cdot \boldsymbol{n} &= 0 \quad \text{on } \Omega \times \partial T_j, \\
y \to u^1(x,y) &\text{ is } KY^*\text{-periodic.}
\end{aligned}
\tag{76}
$$

Furthermore,

$$
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, dx = \frac{1}{K^N} \int_\Omega \int_{KY^*} |\nabla_x u(x) + \nabla_y u^1(x,y)|^2\, dx\, dy. \tag{77}
$$

The proof of Proposition 3.3.1 requires the following technical lemma.

Lemma 3.3.2. *Let $(\boldsymbol{s}_j^\varepsilon)_{0 \leq j \leq K-1}$ be a family of bounded sequences in $L^2(\Omega)^N$ which converges weakly to a limit family $(\boldsymbol{s}_j)_{0 \leq j \leq K-1}$. Then, the piecewise constant function $E_\varepsilon P_\varepsilon^K(\boldsymbol{s}_j^\varepsilon) \in L^2(\Omega)^N$ defined by*

$$
E_\varepsilon P_\varepsilon^K(\boldsymbol{s}_j^\varepsilon)(x) = \sum_{l,j} \left(\frac{1}{|\varepsilon(KY)_l|} \int_{\varepsilon(KY)_l} \boldsymbol{s}_j^\varepsilon(x)\, dx \right) \chi_{Y_{lj}^\varepsilon}(x),
$$

where $\chi_{Y_{lj}^\varepsilon}(x)$ is the characteristic function of the jth subcell of the periodic cell $\varepsilon(KY)_l$, two-scale converges to $\boldsymbol{s}(x,y) \in L^2(\Omega \times KY)^N$ defined by

$$
\boldsymbol{s}(x,y) = \sum_{j=0}^{K-1} \boldsymbol{s}_j(x) \chi_{Y_j}(y).
$$

Moreover, if $\boldsymbol{s}_j^\varepsilon$ converges strongly to $\boldsymbol{s}_j$ in $L^2(\Omega)^N$ for all j, then

$$
\| P_\varepsilon^K(\boldsymbol{s}_j^\varepsilon)(x) \|^2_{L^2(\Omega)^N} \to \frac{1}{K^N} \| \boldsymbol{s}(x,y) \|^2_{L^2(\Omega \times KY)^N}. \tag{78}
$$

Proof. Let $\varphi \in \mathcal{D}(\Omega; C_\#^\infty(KY)^N)$ be given. We check the definition of two-scale convergence:

$$
\begin{aligned}
\int_\Omega E_\varepsilon P_\varepsilon^K(\boldsymbol{s}_j^\varepsilon)(x) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx &= \sum_{l,j} \left(\frac{1}{(\varepsilon K)^N} \int_{\varepsilon(KY)_l} \boldsymbol{s}_j^\varepsilon(x)\, dx \right) \cdot \int_{Y_{lj}^\varepsilon} \varphi\left(x, \frac{x}{\varepsilon}\right) dx \\
&= \frac{1}{K^N} \sum_{j=0}^{K-1} \int_\Omega \boldsymbol{s}_j^\varepsilon(x) \cdot \left[\sum_l \left(\frac{1}{\varepsilon^N} \int_{Y_{lj}^\varepsilon} \varphi\left(x, \frac{x}{\varepsilon}\right) dx \right) \chi_{\varepsilon(KY)_l}(x) \right] dx.
\end{aligned}
$$

It is easily seen that for each fixed j the term between brackets converges strongly to $\int_{Y_j} \varphi(x,y)\,dy$. Thus we can pass to the limit and obtain

$$\frac{1}{K^N}\sum_{j=0}^{K-1}\int_\Omega s_j(x)\cdot\int_{Y_j}\varphi(x,y)\,dy,$$

which is the desired result. In the case where s_j^ε converges strongly to s_j, (78) can be proved similarly by replacing in the above computations the test function $\varphi(x,x/\varepsilon)$ by $E_\varepsilon P_\varepsilon^K(s_j^\varepsilon)(x)$.

Proof of Proposition 3.3.1. By virtue of Proposition 2.2.1 there exists (u,u^1) in $H_0^1(\Omega)\times L^2[\Omega;\, H^1_\#(KY)]$ such that (up to a subsequence) u_ε and ∇u_ε two-scale converge to $u(x)$ and $\nabla u(x)+\nabla_y u^1(x,y)$ respectively. Next, we multiply equation (74) by $\phi(x)+\varepsilon\phi_1(x,x/\varepsilon)$ where $\phi\in\mathscr{D}(\Omega)$ and $\phi_1\in\mathscr{D}[\Omega; C^\infty_\#(Y)]$. Integrating by parts in Ω_ε, we get

$$\int_\Omega \chi\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\left(\nabla\phi(x)+\nabla_y\phi_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi_1\left(x,\frac{x}{\varepsilon}\right)\right)dx$$

$$=\sum_{p=1}^{n(\varepsilon)}(P_\varepsilon^K(s_j))_p\cdot\left(\int_{\Gamma_p^\varepsilon}\left(\phi(x)+\varepsilon\phi_1\left(x,\frac{x}{\varepsilon}\right)\right)\boldsymbol{n}\,ds\right), \tag{79}$$

where $\chi(y)$ is the characteristic function of Y^*. Applying Green's formula to the boundary integral on Γ_p^ε, we obtain (26) and, since $E_\varepsilon P_\varepsilon^K(s_j)$ is piecewise constant, we can rewrite the right-hand side of (79) as

$$\int_\Omega\left(\chi\left(\frac{x}{\varepsilon}\right)-1\right)E_\varepsilon P_\varepsilon^K(s_j)(x)\cdot\left(\nabla_x\phi(x)+\nabla_y\phi_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\phi_1\left(x,\frac{x}{\varepsilon}\right)\right)dx. \tag{80}$$

By application of Lemma 3.3.2, $E_\varepsilon P_\varepsilon^K(s_j)$ two-scale converges, and passing to the limit in (80) yields

$$\frac{1}{K^N}\int_\Omega\int_{KY}(\chi(y)-1)\sum_{0\leq j\leq K-1}s_j(x)\chi_{Y_j}(y)\cdot(\nabla\phi(x)+\nabla_y\phi_1(x,y))\,dx\,dy.$$

On the other hand, the left-hand side of (79) passes easily to the limit using the definition of two-scale convergence. We obtain

$$\frac{1}{K^N}\int_\Omega\int_{KY^*}(\nabla u(x)+\nabla_y u^1(x,y))\cdot(\nabla\phi(x)+\nabla_y\phi_1(x,y))\,dx\,dy$$

$$=\frac{|T|}{K^N}\int_\Omega\phi(x)\sum_j \operatorname{div}_x s_j(x)\,dx+\frac{1}{K^N}\int_\Omega\left(\sum_j s_j(x)\cdot\int_{\Gamma_j}\phi_1(x,y)\boldsymbol{n}_y\,ds\right)dx. \tag{81}$$

We recognize in this identity the variational formulation for (u,u^1) in the space $H_0^1(\Omega)\times L^2[\Omega;\, H^1_\#(KY^*)]$ of the two-scale homogenized problem (76). A standard application of the Lax-Milgram lemma shows existence and uniqueness of (u, u^1), and hence the whole sequence u_ε converges to its limit.

To obtain the convergence (77), we multiply equation (74) by u_ε to obtain

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \int_{\Omega} \left(\chi\left(\frac{x}{\varepsilon}\right) - 1 \right) E_\varepsilon P_\varepsilon^K(s_j)(x) \cdot \nabla u_\varepsilon \, dx. \tag{82}$$

We can pass to the limit in the right-hand side of (82) since, by Lemma 3.3.2, $E_\varepsilon P_\varepsilon^K(s_j^\varepsilon)$ two-scale converges strongly (see Part 3 of Proposition 2.2.1). Thus

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = -\frac{1}{K^N} \int_{\Omega} \int_{KT} \sum_j s_j(x) \chi_{Y_j}(y) \cdot (\nabla u(x) + \nabla_y u^1(x, y)) \, dx \, dy.$$

Finally, using the homogenized variational formulation (81) yields the desired result.

Proposition 3.3.1 yields implicitly the existence of a weak limit S^K in $[L^2(\Omega)^N]^{K^N}$ of the sequence S_ε^K. Indeed, since $S_\varepsilon^K = E_\varepsilon^K S_\varepsilon P_\varepsilon^K$ and $(E_\varepsilon^K)^* = (\varepsilon K)^N P_\varepsilon^K$, we have

$$\langle S_\varepsilon^K(s_j), (s_j) \rangle = K^N \sum_{p=1}^{n(\varepsilon)} \left(\int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n} \, ds \right) \cdot (P_\varepsilon^K(s_j))_p$$

where the brackets $\langle \cdot, \cdot \rangle$ indicate the standard inner product in $[L^2(\Omega)^N]^{K^N}$. Hence, from the energy identity (82), it follows that

$$\langle S_\varepsilon^K(s_j), (s_j) \rangle = K^N \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx. \tag{83}$$

Letting ε go to zero in this identity, we see that Proposition 3.3.1 tells us that the limit of the right-hand side of (83) is a quadratic form in (s_j). Therefore, S_ε^K has a weak limit S^K defined by

$$\langle S^K(s_j), (s_j) \rangle = \int_{\Omega} \int_{KY^*} |\nabla_x u(x) + \nabla_y u^1(x, y)|^2 \, dx \, dy, \tag{84}$$

where $(u(x), u^1(x, y))$ is the solution of (76) which depends linearly on (s_j). Our next task is to show that the convergence of the sequence S_ε^K is not merely weak but strong.

Proposition 3.3.3. *The sequence* S_ε^K converges strongly in $[L^2(\Omega)^N]^{K^N}$ *to its limit* S^K *defined by* (84).

Proof. Let (r_j^ε) be any sequence of families of functions that converges weakly to a family (r_j) in $[L^2(\Omega)^N]^{K^N}$. We have to prove that

$$\lim_{\varepsilon \to 0} \langle S_\varepsilon^K(s_j), (r_j^\varepsilon) \rangle = \langle S^K(s_j), (r_j) \rangle. \tag{85}$$

To this end, we introduce a function v_ε defined as the unique solution in $H^1(\Omega_\varepsilon)$ of problem (74) in which the right-hand side is precisely $P_\varepsilon^K(\boldsymbol{r}_j)_{0 \leqq j \leqq K-1}$, i.e.,

$$\begin{aligned} -\Delta v_\varepsilon = 0 \quad & \text{in } \Omega_\varepsilon, \\ \frac{\partial v_\varepsilon}{\partial n} = P_\varepsilon^K(\boldsymbol{r}_j^\varepsilon)\cdot \boldsymbol{n} \quad & \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon), \\ v_\varepsilon = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{86}$$

Multiplying the differential equation in (86) by u_ε and integrating by parts in Ω_ε, we easily obtain

$$\langle S_\varepsilon^K(\boldsymbol{t}_j), (\boldsymbol{r}_j^\varepsilon)\rangle = K^N \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, dx. \tag{87}$$

To prove the desired result, we have to pass to the limit in the right-hand side of (87). First, we need to homogenize problem (86). This is very similar to what we did in Proposition 3.3.1, except that here the right-hand side of (86) involves weakly converging sequences. For the sake of brevity, we simply sketch the main argument. Multiplying the differential equation in (86) by a test function $\phi(x) + \varepsilon\phi_1(x, x/\varepsilon)$, and integrating by parts yields

$$\begin{aligned} &\int_\Omega \chi\left(\frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \left(\nabla\phi(x) + \nabla_y \phi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx \\ &= \int_\Omega \left(\chi\left(\frac{x}{\varepsilon}\right) - 1\right) E_\varepsilon P_\varepsilon^K(\boldsymbol{r}_j^\varepsilon)(x) \cdot \left(\nabla_x\phi(x) + \nabla_y \phi_1\left(x, \frac{x}{\varepsilon}\right)\right) dx + o(1), \end{aligned} \tag{88}$$

where $o(1)$ is a term which goes to zero as ε does. By application of Lemma 3.3.2, the piecewise constant function $E_\varepsilon P_\varepsilon^K(\boldsymbol{r}_j^\varepsilon)(x)$ two-scale converges to $\boldsymbol{r}(x, y) \in L^2(\Omega \times KY)^N$ defined by

$$\boldsymbol{r}(x, y) = \sum_{j=0}^{K-1} \boldsymbol{r}_j(x)\, \chi_{Y_j}(y).$$

Thus, we can pass to the two-scale limit in (88), and it is easily seen that v_ε and ∇v_ε two-scale converge towards $v(x)$ and $\nabla v(x) + \nabla_y v^1(x, y)$ respectively, where (v, v^1) is the unique solution in $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(KY^*)]$ of the same two-scale homogenized system (76) where the right-hand sides $\boldsymbol{s}_j$ are replaced by $\boldsymbol{r}_j$.

Now, to pass to the limit in (87), we use the strong two-scale convergence of ∇u_ε (see part 3 of Proposition 2.2.1), which is a consequence of the energy convergence (77). This yields

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, dx = \frac{1}{K^N} \int_\Omega \int_{KY^*} (\nabla_x u + \nabla_y u^1) \cdot (\nabla_x v + \nabla_y v^1) \, dx\, dy.$$

An easy integration by parts in the two-scale homogenized system (76) combined with the definition (84) shows that

$$\int_{\Omega}\int_{KY^*} (\nabla_x u + \nabla_y u^1)\cdot(\nabla_x v + \nabla_y v^1)\,dx\,dy = \langle S^K(s_j),(r_j)\rangle,$$

which is the desired result.

To complete the proof of Theorem 3.2.1, it remains to give a definition of the limit operator S^K more explicit than (84), by using the Bloch-wave decomposition operator $\mathscr{B}$ introduced in Lemma 3.1.2. Since the solution $u^1(x,y)$ of the two-scale homogenized problem (76) is KY^*-periodic, it is natural to decompose it into its Bloch components (as we did for the family (s_j)), and to try to reduce the homogenized problem (76) into K^N sub-problems posed in Y^*. To this end, we recall the following lemma on the "Bloch-wave" orthogonal decomposition of $H^1_\#(KY^*)$ (see Theorem 2.2 in [1]).

Lemma 3.3.4. *For any multi-index $0 \leqq j \leqq K-1$, let $H^1_\#(e^{2\pi i\theta_j}, Y^*)$ be the complex Hilbert subspace of $H^1_\#(KY^*)$ consisting of functions satisfying the so-called $(e^{2\pi i\theta_j}, Y^*)$-periodicity condition:*

$$w(y+j') = e^{2\pi i j'\cdot\theta_j} w(y) \quad \forall y \in Y^*, \quad \forall j' \in \mathbb{Z}^N \quad \textit{with } \theta_j = \frac{j}{K}.$$

They form an orthogonal decomposition of $H^1_\#(KY^*)$ with respect to both inner products of $L^2(KY^*)$ and $H^1(KY^*)$, i.e.,

$$H^1_\#(KY^*) = \bigoplus_{0\leqq j\leqq K-1} H^1_\#(e^{2\pi i\theta_j}, Y^*).$$

Therefore, any function $w(y) \in H^1_\#(KY^*)$ can be uniquely decomposed in

$$w(y) = \sum_{j=0}^{K-1} w_j(y) e^{2\pi i\theta_j\cdot y},$$

with $w_j(y) \in H^1_\#(Y^*)$ satisfying the Parseval identity

$$\frac{1}{K^N}\int_{KY^*} |w(y)|^2\,dy = \sum_{j=0}^{K-1}\int_{Y^*} |w_j(y)|^2\,dy. \tag{89}$$

Proof of Theorem 3.2.1. By Lemma 3.3.4, the solution $u^1(x,y)$ of the two-scale homogenized problem (76) can be written as

$$u^1(x,y) = \sum_{j=0}^{K-1} u_j(x,y)\, e^{2\pi i\theta_j\cdot y} \tag{90}$$

where each Bloch component $u_j(x,y)$ belongs to $L^2(\Omega; H^1_\#(Y^*))$. Furthermore, by Lemma 3.1.2, the right-hand side of (76) is

$$s(y) = \sum_{j=0}^{K-1} s_j\chi_{Y_j}(y) = \sum_{j=0}^{K-1} t_j e^{2\pi i\theta_j\cdot E(y)}. \tag{91}$$

Substituting (90) and (91) into (76) and identifying coefficients corresponding to the same Bloch frequency θ_j (thanks to the orthogonality property of the Bloch-wave decomposition), we find that each u_j is a solution of a boundary-value problem in Y^* involving only $\boldsymbol{t}_j$.

If $j = 0$, it is easily seen that $u_0(x, y)$ satisfies a system of equations which also involves $u(x)$. More precisely, (u, u_0) is solution of the two-scale homogenized problem

$$
\begin{gathered}
-\operatorname{div}_x\left(\textstyle\int_{Y^*}(\nabla u(x) + \nabla_y u_0(x,y))\,dy\right) = |T|\operatorname{div}_x \boldsymbol{t}_0 \quad \text{in } \Omega,\\
-\Delta_y u_0(x,y) = 0 \quad \text{in } \Omega \times Y^*,\\
u(x) = 0 \quad \text{on } \partial\Omega, \qquad\qquad (92)\\
(\nabla u(x) + \nabla_y u_0(x,y) - \boldsymbol{t}_0(x))\cdot\boldsymbol{n} = 0 \quad \text{on } \Omega \times \partial T,\\
y \to u^0(x,y) \text{ is } Y^*\text{-periodic.}
\end{gathered}
$$

Observe that this problem is nothing but the usual two-scale homogenized problem obtained in Section 2.2 (see problem (30)). We recall that its solution $u_0(x, y)$ can be written in terms of $\nabla u(x)$, $\boldsymbol{t}_0(x)$ and the solutions $(w_m(y))_{1 \leq m \leq N}$ of the usual cell problems (16) (see (31)). Eliminating the y variable in (92) shows that u is the unique solution of the homogenized problem (62).

On the other hand, if $j \neq 0$, then the average of $u_j e^{2\pi i\theta_j \cdot y}$ on KY^* is zero, yielding a zero contribution of u_j in the first equation of (76). Therefore, the subproblem for u_j reduces to

$$
\begin{aligned}
&\Delta_y(u_j(x,y)e^{2\pi i\theta_j\cdot y}) = 0 && \text{in } Y^*,\\
&(\nabla_y(u_j(x,y)e^{2\pi i\theta_j\cdot y}) - \boldsymbol{t}_j(x))\cdot\boldsymbol{n} = 0 && \text{on } \partial T, \qquad (93)\\
&y \to u_j(x,y) \text{ is } Y^*\text{-periodic.}
\end{aligned}
$$

By linearity, the solution u_j of (93) can be computed in terms of $\boldsymbol{t}_j(x)$ and of the solutions $(w_m^j(y))_{1 \leq m \leq N}$ of the cell problem (64) at the Bloch frequency θ_j:

$$
u_j(x,y) = \sum_{m=1}^{N} (\boldsymbol{t}_j(x)\cdot\boldsymbol{e}_m) w_m^j(y) e^{-2\pi i\theta_j\cdot y}.
$$

This completes the characterization of the homogenized solution (u, u^1).

We now use this information in order to diagonalize S^K. From Parseval identity (89), the characterization (84) of the limit operator S^K becomes

$$
\frac{1}{K^N}\langle S^K(\boldsymbol{s}_j)(\boldsymbol{s}_j)\rangle = \int_\Omega\int_{Y^*} |\nabla_x u(x) + \nabla_y u_0(x,y)|^2\,dx\,dy \qquad (94)
$$

$$
+ \sum_{0 \leq j \leq K-1,\, j \neq 0} \int_\Omega\int_{Y^*} |\nabla_y u_j(x,y)|^2\,dx\,dy, \qquad (95)
$$

and we have

$$\int_{\Omega}\int_{Y^*} |\nabla_x u(x) + \nabla_y u_0(x,y)|^2 \, dx\, dy = \int_{\Omega} S t_0(x)\cdot t_0(x)\, dx,$$

$$\int_{\Omega}\int_{Y^*} |\nabla_y u_j(x,y)|^2 \, dx\, dy = \int_{\Omega} A^j t_j(x)\cdot t_j(x)\, dx.$$

Since $K^{N/2}(t_j) = \mathscr{B}(s_j)$, it is easy to conclude the proof of Theorem 3.2.1 by checking that $S^K = \mathscr{B}^* T^K \mathscr{B}$, where T^K is the diagonal operator defined by (61).

3.4. Completeness of the limit spectrum

This section is devoted to the proof of the main result of this paper, namely, Theorem 3.2.9, which states the completeness of our analysis of the limit spectrum σ_∞. It also contains the proof of several auxiliary results that are required in the course of the proof of Theorem 3.2.9. In particular, we prove the continuity of the matrix $A(\theta)$ on $]0;1[^N$ (Proposition 3.2.6), and study in great detail its behavior near 0 (Proposition 3.4.4). Recall that the entries of $A(\theta)$ are defined by

$$\overline{A_{mm'}}(\theta) = \int_{Y^*} \nabla w_m^\theta(y)\cdot \nabla \bar{w}_{m'}^\theta(y)\, dy, \quad 1 \leqq m,\, m' \leqq N, \tag{96}$$

where $(w_m^\theta)_{1 \leqq m \leqq N}$ are the unique solutions of the cell problems at the Bloch frequency θ:

$$\begin{aligned} &-\Delta w_m^\theta = 0 && \text{in } Y^*,\\ &(\nabla w_m^\theta - e_m)\cdot n = 0 && \text{on } \partial T,\\ &y \to e^{-2\pi i\theta\cdot y} w_m^\theta(y) \text{ is } Y^*\text{-periodic.} \end{aligned} \tag{97}$$

Problem (97) admits the following variational formulation in the complex Hilbert space $H^1_\#(e^{2\pi i\theta}, Y^*)$ of $H^1_{\text{loc}}(\mathbb{R}^N)$ functions which satisfy the above so-called $(e^{2\pi i\theta}, Y^*)$-periodicity condition:

$$\int_{Y^*} \nabla w_m^\theta(y)\cdot \nabla\bar{\phi}^\theta(y)\, dy = \int_{\partial T} e_m\cdot n \bar{\phi}^\theta(s)\, ds \quad \forall \phi^\theta \in H^1_\#(e^{2\pi i\theta}, Y^*). \tag{98}$$

Remark 3.4.1. It is easily seen that adding any integer to any component of the parameter θ does not change the $(e^{2\pi i\theta}, Y^*)$-periodicity condition and thus the definition of the space $H^1_\#(e^{2\pi i\theta}, Y^*)$. Consequently the matrix $A(\theta)$ is $[0;1]^N$-periodic in θ. With no loss of generality, we can shift the domain of definition of $A(\theta)$ to the interval $[-\frac{1}{2};\frac{1}{2}]^N$. This has the advantage that now there is only one point of discontinuity of $A(\theta)$, which is 0, inside its domain of definition.

We begin by proving a Poincaré inequality in $H^1_\#(e^{2\pi i\theta}, Y^*)$.

Lemma 3.4.2. *There exists a positive constant C such that*

$$\|\phi^\theta\|_{L^2(Y^*)} \leqq \frac{C}{|\theta|} \|\nabla\phi^\theta\|_{L^2(Y^*)^N}$$

for any non-zero $\theta \in [-\frac{1}{2};\frac{1}{2}]^N$, and for any $\phi^\theta \in H^1_\#(e^{2\pi i\theta}, Y^*)$.

Proof. With no loss of generality we assume that the first component θ_1 of θ is the largest one in absolute value. It is non-zero since θ is different from 0. Let us prove the Poincaré inequality for a function ϕ belonging to $H^1_\#(e^{2\pi i\theta}, Y)$. For any point $y \in Y$,

$$\phi(y + e_1) - \phi(y) = e^{2\pi i\theta_1}\phi(y) - \phi(y) = \int_0^1 \frac{\partial\phi}{\partial y_1}(y + te_1)\,dt.$$

By the Schwarz inequality and integration over the cell Y, one obtains

$$\|\phi\|_{L^2(Y)} \leqq \frac{1}{|e^{2\pi i\theta_1} - 1|} \|\nabla\phi^\theta\|_{L^2(Y)^N}.$$

It is easily seen that the constant in this equation is bounded by $C/|\theta|$. To obtain the same result in Y^*, one needs to introduce some extension operator from $H^1_\#(e^{2\pi i\theta}, Y^*)$ into $H^1_\#(e^{2\pi i\theta}, Y)$. It turns out that the usual extension operator from $H^1_\#(Y^*)$ into $H^1_\#(Y)$ (introduced in [9]) also works in the present context. We leave the details to the reader.

To study the continuity of the matrix $A(\theta)$, we need a result of [1] which, roughly speaking, means that the space $H^1_\#(e^{2\pi i\theta}, Y^*)$ is continuous in θ. For the sake of completeness, we briefly sketch its proof.

Lemma 3.4.3. *Let θ_n be a sequence converging to a (possibly zero) limit θ in $[-\frac{1}{2};\frac{1}{2}]^N$. Then, for any $\phi^\theta \in H^1_\#(e^{2\pi i\theta}, Y^*)$, there exists a sequence $\phi^{\theta_n} \in H^1_\#(e^{2\pi i\theta_n}, Y^*)$ such that ϕ^{θ_n} converges strongly to ϕ^θ in $H^1(Y^*)$. Conversely, if $\phi^{\theta_n} \in H^1_\#(e^{2\pi i\theta_n}, Y^*)$ is a sequence which converges strongly to ϕ^θ in $H^1(Y^*)$, then this limit ϕ^θ belongs to $H^1_\#(e^{2\pi i\theta}, Y^*)$.*

Proof. Any function $\phi^\theta \in H^1_\#(e^{2\pi i\theta}, Y^*)$ can be written as

$$\phi^\theta(y) = \phi(y)e^{2\pi i\theta\cdot y}, \tag{99}$$

where $\phi(y)$ belongs to $H^1_\#(Y^*)$. We define the sequence ϕ^{θ_n} by

$$\phi^{\theta_n}(y) = \phi(y)e^{2\pi i\theta_n\cdot y}.$$

One can easily check that ϕ^{θ_n} converges strongly to ϕ^θ in $H^1(Y^*)$. Conversely, if a sequence ϕ^{θ_n}, defined by

$$\phi^{\theta_n}(y) = \phi_n(y)e^{2\pi i\theta_n\cdot y}, \tag{100}$$

converges strongly to ϕ^θ in $H^1_\#(Y^*)$, then the sequence ϕ_n also converges strongly to a limit ϕ in $H^1_\#(Y^*)$. One can pass to the limit in (100) to obtain (99), which proves that ϕ^θ belongs to $H^1_\#(e^{2\pi i\theta}, Y^*)$.

We are now in a position to prove the continuity of the matrix $A(\theta)$ on $[-\frac{1}{2};\frac{1}{2}]^N\backslash\{0\}$.

Proof of Proposition 3.2.6. Let θ_n be a sequence converging to a non-zero limit θ in $[-\frac{1}{2};\frac{1}{2}]^N$. Let us prove that the sequence of solutions $w_m^{\theta_n}$ of the cell problem (97) converges to the solution w_m^θ. By the very definition (96) of $A(\theta)$, this is enough to prove its continuity. The variational formulation (98) yields the energy estimate

$$\|\nabla w_m^{\theta_n}\|^2_{L^2(Y^*)} = \int_{\partial T} \boldsymbol{e}_m \cdot \boldsymbol{n} \bar{w}_m^{\theta_n}, \tag{101}$$

which, by using a standard extension operator, implies that the sequence $\nabla w_m^{\theta_n}$ is bounded in $L^2(Y^*)^N$. Up to a subsequence, it converges weakly in $L^2(Y^*)^N$, but the convergence is indeed strong by using (101) again.

Since θ is not 0, $|\theta_n|$ is bounded away from 0 for sufficiently large n, and we can apply the Poincaré inequality (Lemma 3.4.2) to a subsequence of $w_m^{\theta_n}$ which thus converges strongly in $H^1(Y^*)$. By Lemma 3.4.3 this limit belongs to $H^1_\#(e^{2\pi i\theta}, Y^*)$, and we can pass to the limit in the variational formulation for $w_m^{\theta_n}$. This proves that the limit is nothing but w_m^θ, the solution of the cell problem (97).

Note that this argument for proving the continuity of $A(\theta)$ does not work at $\theta = 0$. Indeed, in this case it may well happen that the $L^2(Y^*)$ norm of the sequence $w_m^{\theta_n}$ goes to infinity. It turns out that the limit of $w_m^{\theta_n}$ is not necessarily w_m^0 when θ_n goes to 0, and thus $A(\theta)$ is not continuous at 0. However, we now prove that $A(\theta_n)$ has a limit when θ_n goes to zero along rays of constant direction. More precisely, we have

Proposition 3.4.4. *Let θ_n be a sequence converging to* 0, *and such that $\theta_n/|\theta_n|$ converges to a unit vector ξ. Then, the matrix $A(\theta_n)$ converges to a real, positive-definite matrix $\hat{A}(\xi)$ which is defined by its entries*

$$\hat{A}_{mm'}(\xi) = \int_{Y^*} \nabla v_m^\xi(y) \cdot \nabla v_{m'}^\xi(y)\, dy, \quad 1 \leq m, m' \leq N, \tag{102}$$

where $(v_m^\xi)_{1\leq m\leq N}$ are the unique solutions in $H^1_\#(Y^*)/\mathbb{R}$ of the problems

$$\begin{aligned} -\Delta v_m^\xi &= 0 && \textit{in } Y^*, \\ \frac{\partial v_m^\xi}{\partial n} &= \boldsymbol{e}_m \cdot \boldsymbol{n} + (\boldsymbol{\xi} \cdot \textstyle\int_{Y^*} \nabla v_m^\xi + |T|\boldsymbol{\xi} \cdot \boldsymbol{e}_m)\boldsymbol{\xi} \cdot \boldsymbol{n} && \textit{on } \partial T, \\ y &\to v_m^\xi(y) \textit{ is } Y^*\textit{-periodic.} \end{aligned} \tag{103}$$

Furthermore, this matrix $A(\xi)$ is related to $A(0)$ by

$$\hat{A}(\xi) = A(0) + (1 - A(0)\xi\cdot\xi)^{-1}(A(0)\xi + |T|\xi)\otimes(A(0)\xi + |T|\xi), \quad (104)$$

which proves that $\hat{A}(\xi)$ is a continuous function of ξ and that $A(\theta)$ is not continuous at $\theta = 0$.

Proof. Let $w_m^{\theta_n}$ be the solution of (97) (to simplify the notation, we drop the index m in the sequel). We rewrite w^{θ_n} as

$$w^{\theta_n}(y) = (w_n(y) + m_n)\,e^{2\pi i\theta_n\cdot y}$$

where w_n belongs to $H^1_\#(Y^*)$ and has zero average, and where m_n is a complex number. We choose a sequence of test function $\phi^{\theta_n} \in H^1_\#(e^{2\pi i\theta_n}, Y^*)$ defined by

$$\phi^{\theta_n}(y) = (\phi(y) + \mu_n)\,e^{2\pi i\theta_n\cdot y}$$

where ϕ belongs to $H^1_\#(Y^*)$ and has zero average, and μ_n is a complex number which goes as $\mu/|\theta_n|$ when n goes to infinity (with $\mu \in \mathbb{C}$). Plugging these expressions into the variational formulation (98), we obtain

$$\int_{Y^*} (\nabla w_n + 2\pi i\theta_n(w_n + m_n))\cdot(\nabla\bar{\phi} - 2\pi i\theta_n(\bar{\phi} + \bar{\mu}_n)) = \int_{\partial T} \boldsymbol{e}\cdot\boldsymbol{n}(\bar{\phi} + \bar{\mu}_n)e^{-2\pi i\theta_n\cdot y}. \quad (105)$$

Since w_n has mean value zero in Y^*, the Poincaré-Wirtinger inequality yields the a priori estimate

$$\|w_n\|_{H^1(Y^*)} \leqq C,$$

while Lemma 3.4.2 gives

$$|\theta_n m_n| \leqq C.$$

Consequently, up to a subsequence, w_n converges weakly to w in $H^1_\#(Y^*)$ and $\theta_n m_n$ converges to ξm for some $\xi \in \mathbb{C}$. Thus, we can pass to the limit in (105):

$$\int_{Y^*} \nabla w\cdot\nabla\bar{\phi} + 4\pi^2 m\bar{\mu} + 2\pi i m\xi\cdot\int_{Y^*}\nabla\bar{\phi} - 2\pi i\bar{\mu}\xi\cdot\int_{Y^*}\nabla w = -\int_T \boldsymbol{e}\cdot(\nabla\bar{\phi} - 2\pi i\bar{\mu}\xi).$$

This holds for any $\phi \in H^1_\#(Y^*)$ with zero average and for any $\mu \in \mathbb{C}$. By varying μ, we obtain

$$m = \frac{2\pi i\xi}{4\pi^2}\cdot\left(|T|\boldsymbol{e} + \int_{Y^*}\nabla w\right),$$

which in turn yields

$$\int_{Y^*}\nabla w\cdot\nabla\bar{\phi} - \left(\xi\cdot\int_{Y^*}\nabla w\right)\left(\xi\cdot\int_{Y^*}\nabla\bar{\phi}\right) = \int_{\partial T}(\boldsymbol{e} + |T|(\xi\cdot\boldsymbol{e})\xi)\cdot\boldsymbol{n}\bar{\phi} \quad (106)$$

for any $\phi \in H^1_\#(Y^*)$ with zero average. It is easy to check that the left-hand side of (106) is coercive on $H^1_\#(Y^*)/\mathbb{C}$, uniformly in ξ, and thus that there exists a unique solution of (106) in $H^1_\#(Y^*)/\mathbb{C}$. By taking ϕ to be real, we check that the solution w is purely real and that it coincides with v^ξ, the solution of (103). By linearity, v^ξ can be computed in terms of $w^0(\boldsymbol{e})$ and $w^0(\xi)$, solutions of the cell problem (97) at the Bloch frequency 0, and with right-hand side $\boldsymbol{e}$ and ξ respectively. A simple calculation yields

$$v^\xi = w^0(\boldsymbol{e}) + \frac{(A(0)\boldsymbol{e} + |T|\boldsymbol{e})\cdot\xi}{1 - A(0)\xi\cdot\xi} w^0(\xi),$$

$$\begin{aligned}\lim_{\theta_n\to 0,\, \theta_n/|\theta_n|\to\xi} A(\theta_n)\boldsymbol{e}\cdot\boldsymbol{e} &= \hat{A}(\xi)\boldsymbol{e}\cdot\boldsymbol{e} \\ &= A(0)\boldsymbol{e}\cdot\boldsymbol{e} + \frac{((A(0)\boldsymbol{e} + |T|\boldsymbol{e})\cdot\xi)^2}{1 - A(0)\xi\cdot\xi},\end{aligned} \tag{107}$$

which is the desired result.

Remark 3.4.5. Proposition 3.4.4 tells us that $\hat{A}(\xi)$ differs from $A(0)$ by a positive rank-one matrix. Let us point out two consequences of this: First, the eigenvalues of $\hat{A}(\xi)$ are always greater than or equal to those of $A(0)$. Second, any eigenvalue λ_0 of $A(0)$ (with a corresponding eigenvector ξ_0) is also an eigenvalue of $\hat{A}(\xi)$ for any choice of ξ orthogonal to ξ_0. Thus, the eigenvalues of $A(0)$ are always included in the Bloch spectrum $\bigcup_{m=1}^N [a_m, b_m]$ (see (70)). In the isotropic case (i.e., $A(0) = a^0\mathrm{Id}$), for any value of ξ, $\hat{A}(\xi)$ has $N-1$ eigenvalues equal to a^0 and one equal to $a^0 + (a^0 + |T|)^2/(1 - a^0)$. In particular, this proves that even if $\hat{A}(\xi)$ is not a constant function of ξ, its eigenvalues do not depend on ξ. It also proves that the a priori N distinct bands in the Bloch-wave spectrum collapse in at most two distinct bands since $N-1$ of them intersect.

We now turn to the proof of Theorem 3.2.9 on the completeness of the limit spectrum σ_∞. As in Section 2, we consider the extended operator $\tilde{S}_\varepsilon$ acting on $L^2(\Omega)^N$. Let λ be any value in the limit set σ_∞ of the spectra $\sigma(\tilde{S}_\varepsilon)$. By definition, there exists a subsequence (still denoted by ε) of eigenvalues λ_ε and eigenvectors $\boldsymbol{s}_\varepsilon$ such that

$$\begin{aligned}&\tilde{S}_\varepsilon \boldsymbol{s}_\varepsilon = \lambda_\varepsilon \boldsymbol{s}_\varepsilon \quad \text{with } \|\boldsymbol{s}_\varepsilon\|_{L^2(\Omega)^N} = 1,\\ &\lambda_\varepsilon \to \lambda.\end{aligned} \tag{108}$$

Our goal is to prove that λ does indeed belong to $\sigma(S)\cup(\bigcup_{m=1}^N [a_m, b_m])\cup\sigma_{\text{boundary}}$. We recall that $\tilde{S}_\varepsilon \boldsymbol{s}_\varepsilon$ is defined by

$$\tilde{S}_\varepsilon \boldsymbol{s}_\varepsilon = \sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N}\left(\int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n}\right)\chi_{Y_p^\varepsilon}(x),$$

where u_ε is the solution of

$$\begin{aligned}-\Delta u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon,\\ \frac{\partial u_\varepsilon}{\partial n} &= \varepsilon^{-N}\left(\int_{Y_p^\varepsilon} \boldsymbol{s}_\varepsilon(x)\,dx\right)\cdot\boldsymbol{n} && \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq n(\varepsilon),\\ u_\varepsilon &= 0 && \text{on } \partial\Omega.\end{aligned} \tag{109}$$

The following lemma establishes the connection between s_ε and u_ε.

Lemma 3.4.6. *The solution u_ε of* (109) *satisfies the estimates*

$$0 < c \leqq \|u_\varepsilon\|_{H^1(\Omega)} \leqq C,$$

where c and C are two positive constants independent of ε. Furthermore, the sequence s_ε converges weakly to zero in $L^2(\Omega)^N$ if and only if u_ε converges weakly to zero in $H_0^1(\Omega)$.

Proof. The estimates for u_ε are deduced from Lemma 2.2.3 since $\|s_\varepsilon\|_{L^2(\Omega)^N} = 1$. The second statement is a direct consequence of the proof of Proposition 2.2.6 on the homogenization of problem (109).

Proof of Theorem 3.2.9. The proof is divided in four steps corresponding to four different behaviors of the sequence of eigenvectors s_ε which cover all possible cases.
Step 1: Macroscopic convergence of the eigenvectors. Since s_ε is bounded in $L^2(\Omega)^N$, up to a subsequence, it converges weakly to a limit s. Assume that this limit is not zero, i.e., that the sequence s_ε has macroscopic oscillations. Then, multiplying (108) by a test function, one can pass to the limit thanks to the strong convergence of the operator $\tilde{S}_\varepsilon$. Since $s \neq 0$, this shows that s is an eigenvector associated with the eigenvalue λ for the macroscopic limit operator S. This proves that λ belongs to $\sigma(S)$.
Step 2: Concentration of the eigenvectors on the boundary. Assume that for any open subset ω such that $\bar{\omega} \subset \Omega$, the sequence s_ε satisfies

$$\lim_{\varepsilon \to 0} \|s_\varepsilon\|_{L^2(\omega)} = 0.$$

Then, by definition (71) of the boundary-layer spectrum, the corresponding sequence of eigenvalues λ_ε converges to a limit $\lambda \in \sigma_{\text{boundary}}$.
Step 3: ε-Microscopic convergence of the eigenvectors. We now consider a sequence s_ε which converges to 0 weakly in $L^2(\Omega)^N$ and such that there exists at least one open subset ω, with $\bar{\omega} \subset \Omega$, and a strictly positive constant $c > 0$ satisfying

$$\lim_{\varepsilon \to 0} \|s_\varepsilon\|_{L^2(\omega)} \geqq c > 0. \tag{110}$$

Let $\phi \in \mathscr{D}(\Omega)$ be a smooth function with compact support in Ω and identically equal to 1 in ω. The first key idea is to construct a "quasi-eigenvector", with fixed compact support in Ω, by multiplying the true eigenvector s_ε by the cut-off function ϕ. To simplify the sequel, it is better to have a piecewise constant quasi-eigenvector in each cell Y_p^ε. Therefore, we apply to ϕs_ε the operator $E_\varepsilon P_\varepsilon$ which projects on such piecewise constant functions (E_ε and P_ε are defined by (10) and (11)), and we normalize it.

Lemma 3.4.7. *The function t_ε defined by*

$$t_\varepsilon = \frac{E_\varepsilon P_\varepsilon(\phi s_\varepsilon)}{\| E_\varepsilon P_\varepsilon(\phi s_\varepsilon)\|_{L^2(\Omega)^N}} \tag{111}$$

is said to be a quasi-eigenvector since it satisfies

$$\lim_{\varepsilon \to 0} \| \tilde{S}_\varepsilon t_\varepsilon - \lambda_\varepsilon t_\varepsilon \|_{L^2(\Omega)^N} = 0. \tag{112}$$

Proof. Let us define v_ε as the solution of

$$\begin{aligned}
-\Delta v_\varepsilon &= 0 && \text{in } \Omega_\varepsilon,\\
\frac{\partial v_\varepsilon}{\partial n} &= \varepsilon^{-N}\Bigg(\int_{Y_p^\varepsilon} t_\varepsilon(x)\, dx\Bigg)\cdot n && \text{on } \Gamma_p^\varepsilon, \text{ for } 1 \leqq p \leqq n(\varepsilon),\\
v_\varepsilon &= 0 && \text{on } \partial\Omega.
\end{aligned} \tag{113}$$

Let us first prove that $w_\varepsilon = v_\varepsilon - \phi u_\varepsilon / \| E_\varepsilon P_\varepsilon(\phi s_\varepsilon)\|$ converges to 0 strongly in $H_0^1(\Omega)$. To simplify the notation, we denote by a_ε the inverse of $\| E_\varepsilon P_\varepsilon(\phi s_\varepsilon)\|_{L^2(\Omega)^N}$, i.e.,

$$w_\varepsilon(x) = v_\varepsilon(x) - a_\varepsilon \phi(x) u_\varepsilon(x).$$

We have

$$\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2\, dx = \int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla w_\varepsilon\, dx - a_\varepsilon \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla(\phi w_\varepsilon)\, dx$$

$$+ a_\varepsilon \int_{\Omega_\varepsilon} \nabla\phi \cdot (w_\varepsilon \nabla u_\varepsilon - u_\varepsilon \nabla w_\varepsilon)\, dx. \tag{114}$$

By assumption, s_ε converges weakly to 0 in $L^2(\Omega)^N$, and the same is true for t_ε by construction. Thus, by Lemma 3.4.6, u_ε and v_ε converge weakly to 0 in $H^1(\Omega)$, and strongly in $L^2(\Omega)$ by the Rellich theorem. Moreover, in view of (110), the scalar a_ε is bounded. Therefore the last term in the right-hand side of (114) tends to 0. On the other hand, by integration by parts and by using (109) and (113), the two first terms are equal to

$$\sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N} \Bigg\{\Bigg(\int_{Y_p^\varepsilon} a_\varepsilon E_\varepsilon P_\varepsilon(\phi s_\varepsilon)\, dx\Bigg)\cdot\Bigg(\int_{\Gamma_p^\varepsilon} w_\varepsilon n\, ds\Bigg) - a_\varepsilon\Bigg(\int_{Y_p^\varepsilon} s_\varepsilon\, dx\Bigg)\cdot\Bigg(\int_{\Gamma_p^\varepsilon} \phi w_\varepsilon n\, ds\Bigg)\Bigg\}.$$

Recalling that s_ε is constant in each cell Y_p^ε, and replacing ϕ by its average value in each elementary integral, we find that the difference in this expression cancels out up to a remainder term which is easily seen to be bounded by

$$\varepsilon a_\varepsilon \| \phi \|_{C^1(\Omega)} \left| \sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N} \Bigg(\int_{\Gamma_p^\varepsilon} w_\varepsilon n\, ds\Bigg)\cdot\Bigg(\int_{Y_p^\varepsilon} s_\varepsilon\, dx\Bigg)\right|$$

$$\leqq C\varepsilon \| \phi \|_{C^1(\Omega)} \| w_\varepsilon \|_{H_0^1(\Omega)} \| s_\varepsilon \|_{L^2(\Omega)} \leqq C\varepsilon \| \phi \|_{C^1(\Omega)}.$$

This proves that ∇w_ε converges to 0 strongly in $L^2(\Omega)^N$ (as usual, we identify functions defined in Ω_ε and their extensions to Ω defined by Lemma 2.2.2). Now, recalling the definition of $\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon$, we obtain

$$\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon = a_\varepsilon \sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N} \left(\int_{\Gamma_p^\varepsilon} \phi u_\varepsilon \boldsymbol{n} \right) \chi_{Y_p^\varepsilon}(x) + \sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N} \left(\int_{\Gamma_p^\varepsilon} w_\varepsilon \boldsymbol{n} \right) \chi_{Y_p^\varepsilon}(x). \tag{115}$$

The last term in the right-hand side of (115) is easily seen to be bounded by $\| \nabla w_\varepsilon \|_{L^2(\Omega)^N}$, which tends to 0, while the first term can be rewritten as

$$a_\varepsilon \sum_{p=1}^{n(\varepsilon)} \left(\varepsilon^{-N} \int_{Y_p^\varepsilon} \phi \, dx \right) \left(\varepsilon^{-N} \int_{\Gamma_p^\varepsilon} u_\varepsilon \boldsymbol{n} \right) \chi_{Y_p^\varepsilon}(x) = a_\varepsilon E_\varepsilon P_\varepsilon(\phi(x) \tilde{S}_\varepsilon \boldsymbol{s}_\varepsilon(x)) = \lambda_\varepsilon \boldsymbol{t}_\varepsilon(x),$$

up to a small remainder term bounded by $\varepsilon \| \phi \|_{C^1(\Omega)}$. This completes the proof of Lemma 3.4.7.

In order to be able to compute explicitly the term $\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon$, the second key idea is to replace the domain of integration Ω by a larger cube with a periodic boundary condition where a Bloch-wave decomposition of $\boldsymbol{t}_\varepsilon$ could be performed. Let $K(\varepsilon)$ be the smallest integer such that the cube $\varepsilon K(\varepsilon) Y$ contains the domain Ω. Let L be the size of the smallest cube containing Ω. As ε goes to 0, $\varepsilon K(\varepsilon)$ converges to L. The quasi-eigenvector $\boldsymbol{t}_\varepsilon$ is extended by 0 in $\varepsilon K(\varepsilon) Y \setminus \Omega$.

Lemma 3.4.8. *Let $\boldsymbol{t}_\varepsilon$ be the quasi-eigenvector defined by Lemma 3.4.7. Let v_ε be the solution of*

$$\begin{aligned} -\Delta v_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \frac{\partial v_\varepsilon}{\partial n} &= \varepsilon^{-N} \left(\int_{Y_p^\varepsilon} \boldsymbol{t}_\varepsilon(x) \, dx \right) \cdot \boldsymbol{n} && \text{on } \Gamma_p^\varepsilon, \text{for } 1 \leqq p \leqq n(\varepsilon), \\ v_\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{116}$$

and w_ε that of

$$\begin{aligned} -\Delta w_\varepsilon &= 0 && \text{in } \varepsilon K(\varepsilon) Y, \\ \frac{\partial w_\varepsilon}{\partial n} &= \varepsilon^{-N} \left(\int_{Y_p^\varepsilon} \boldsymbol{t}_\varepsilon(x) \, dx \right) \cdot \boldsymbol{n} && \text{on } \Gamma_p^\varepsilon \text{ for } 1 \leqq p \leqq K(\varepsilon)^N, \\ & x \to w_\varepsilon(x) \text{ is } (\varepsilon K(\varepsilon) Y)\text{-periodic.} \end{aligned} \tag{117}$$

Then, the difference $(v_\varepsilon - w_\varepsilon)$ converges to 0 strongly in $H^1(\Omega)$. Therefore,

$$\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon = \sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N} \left(\int_{\Gamma_p^\varepsilon} w_\varepsilon \boldsymbol{n} \right) \chi_{Y_p^\varepsilon}(x) + \boldsymbol{r}_\varepsilon,$$

where $\boldsymbol{r}_\varepsilon$ is a remainder term which goes to 0 strongly in $L^2(\Omega)^N$.

Proof. Let us define the difference $\delta_\varepsilon = w_\varepsilon - v_\varepsilon$. By combining (116) and (117), it is easily seen that δ_ε realizes the minimum in $H^1(\Omega_\varepsilon)$ of

$$\min_{\substack{\delta \in H^1(\Omega_\varepsilon) \\ \delta = w_\varepsilon \text{ on } \partial\Omega}} \int_{\Omega_\varepsilon} |\nabla \delta|^2 \, dx. \tag{118}$$

Recall that the sequence $\boldsymbol{t}_\varepsilon$ has compact support in a fixed compact subset of Ω. Therefore, there exists a smooth function $\psi \in C^\infty(\mathbb{R}^N)$ such that $\psi \equiv 1$ in $\mathbb{R}^N \backslash \Omega$ and $\psi \equiv 0$ on the compact support of all $\boldsymbol{t}_\varepsilon$. Then, ψw_ε is an admissible test function in the minimization problem (118), and we have

$$\int_{\Omega_\varepsilon} |\nabla \delta_\varepsilon|^2 \, dx \leqq \int_{\Omega_\varepsilon} \psi^2 |\nabla w_\varepsilon|^2 \, dx + 2 \int_{\Omega_\varepsilon} \psi w_\varepsilon \nabla w_\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_\varepsilon} w_\varepsilon^2 |\nabla \psi|^2 \, dx. \tag{119}$$

The two last terms in the right-hand side of (119) tend to 0 since, by Lemma 3.4.6, w_ε converges to 0 weakly in $H^1(\Omega)$, and strongly in $L^2(\Omega)$. On the other hand, multiplying equation (117) by $\phi^2 w_\varepsilon$ and integrating by parts yield

$$\int_{\Omega_\varepsilon} \nabla w_\varepsilon \cdot \nabla(\psi^2 w_\varepsilon) \, dx = \sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N} \left(\int_{Y_p^\varepsilon} \boldsymbol{t}_\varepsilon(x) \, dx \right) \cdot \left(\int_{\Gamma_p^\varepsilon} \psi^2 w_\varepsilon \boldsymbol{n} \, ds \right) = 0,$$

because the intersection of the supports of ψ and $\boldsymbol{t}_\varepsilon$ is empty. Thus, we deduce that

$$\int_{\Omega_\varepsilon} \psi^2 |\nabla w_\varepsilon|^2 \, dx = -2 \int_{\Omega_\varepsilon} \psi w_\varepsilon \nabla w_\varepsilon \cdot \nabla \psi dx,$$

which tends to 0. This implies that all three terms in the right-hand side of (119) tend to 0, which proves the desired result.

We are now in a position to compute $\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon$ explicitly by decomposing the solution w_ε of (117) in Bloch waves on the cube $\varepsilon K(\varepsilon) Y$. To do so, we first decompose the quasi-eigenvector $\boldsymbol{t}_\varepsilon$, which is piecewise constant in each subcell Y_p^ε. It is thus amenable to the Bloch-wave decomposition given by Lemma 3.1.2. As already noted in Remark 3.4.1, we can shift the range of the Bloch frequency $j/K(\varepsilon)$ to the interval $[-\frac{1}{2}; \frac{1}{2}]^N$. This has the advantage of concentrating all the difficulties near $\boldsymbol{0}$. From now on, we replace the summation $\sum_{j=0}^{K(\varepsilon)-1}$ by $\sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2}$ (assuming with no loss of generality that $K(\varepsilon)$ is odd). Finally, by Lemma 3.1.2, we get

$$\boldsymbol{t}_\varepsilon(x) = \sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} \boldsymbol{t}_\varepsilon^j e^{2\pi i \theta_j(\varepsilon) \cdot E(x/\varepsilon)} \quad \text{with } \theta_j(\varepsilon) = \frac{j}{K(\varepsilon)} \tag{120}$$

where each $\boldsymbol{t}_\varepsilon^j$ is a constant vector in $\mathbb{C}^N$, and where $E(\cdot)$ denotes the integer part function. On the other hand, the Bloch-wave decomposition in $H^1(\varepsilon K(\varepsilon) Y)$ (as described by Lemma 3.3.4) yields

$$w_\varepsilon(x) = \sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} w_\varepsilon^j \left(\frac{x}{\varepsilon} \right), \tag{121}$$

where each component $w_\varepsilon^j(y)$ belongs to $H^1_\#(e^{2\pi i\theta_j(\varepsilon)}, Y^*)$. Then, using (117) and the orthogonality of the Bloch waves, and setting $\theta = j/K(\varepsilon)$ and $\boldsymbol{t}_\varepsilon^\theta = \boldsymbol{t}_\varepsilon^j$, we deduce that each w_ε^j is given by

$$w_\varepsilon^j\left(\frac{x}{\varepsilon}\right) = \varepsilon w_\varepsilon^\theta\left(\frac{x}{\varepsilon}\right).$$

where $w_\varepsilon^\theta(y)$ is the unique solution in $H^1_\#(e^{2\pi i\theta}, Y^*)$ of

$$\begin{aligned} -\Delta w_\varepsilon^\theta &= 0 \quad \text{in } Y^*, \\ \frac{\partial w_\varepsilon^\theta}{\partial n} &= \boldsymbol{t}_\varepsilon^\theta \cdot \boldsymbol{n} \quad \text{on } \partial T, \\ y \to w_\varepsilon^\theta(y)&e^{-2\pi i\theta\cdot y} \text{ is } Y^*\text{-periodic.} \end{aligned} \tag{122}$$

We can further decompose w_ε^θ in terms of the elementary solutions of the cell problem (64) at the Bloch frequency θ:

$$w_\varepsilon^\theta = \sum_{m=1}^{N} (\boldsymbol{t}_\varepsilon^\theta \cdot \boldsymbol{e}_m)\, w_m^\theta. \tag{123}$$

Note that only the constant vectors $\boldsymbol{t}_\varepsilon^\theta$ depend on ε, not the functions w_m^θ. Since w_ε^j can be written in terms of simple $(e^{2\pi i\theta}, Y^*)$-periodic functions, thanks to (123), we have

$$\sum_{p=1}^{n(\varepsilon)} \varepsilon^{-N}\left(\int_{\Gamma_p^\varepsilon} w_\varepsilon^j \boldsymbol{n}\right)\chi_{Y_p^\varepsilon}(x) = \left(\sum_{m=1}^{N} \boldsymbol{t}_\varepsilon^j\cdot\boldsymbol{e}_m \int_{\partial T} w_m^\theta \boldsymbol{n}\right) e^{2\pi i\theta_j(\varepsilon)\cdot E(x/\varepsilon)}.$$

On the other hand, the variational formulations (98) yields the following property of the matrix $A(\theta)$:

$$A(\theta)\boldsymbol{e}_m = \int_{\partial T} w_m^\theta \boldsymbol{n}.$$

Thus, by Lemma 3.4.8, we have

$$\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon = \sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} A(\theta_j(\varepsilon))\boldsymbol{t}_\varepsilon^j e^{2\pi i\theta_j(\varepsilon)\cdot E(x/\varepsilon)} + \boldsymbol{r}_\varepsilon(x). \tag{124}$$

Denoting by $(\lambda_m(\theta), \boldsymbol{e}_m(\theta))_{1\le m\le N}$ the eigenvalues and eigenvectors of the matrix $A(\theta)$, we decompose each vector $\boldsymbol{t}_\varepsilon^j$ as

$$\boldsymbol{t}_\varepsilon^j = \sum_{m=1}^{N} t_\varepsilon^{jm}\boldsymbol{e}_m(\theta_j(\varepsilon)),$$

and (124) becomes

$$\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon = \sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} \sum_{m=1}^{N} \lambda_m(\theta_j(\varepsilon)) t_\varepsilon^{jm} e^{2\pi i\theta_j(\varepsilon)\cdot E(x/\varepsilon)}\boldsymbol{e}_m(\theta_j(\varepsilon)) + \boldsymbol{r}_\varepsilon(x).$$

Let us recall that, in view of Lemma 3.4.7, $(\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon - \lambda_\varepsilon \boldsymbol{t}_\varepsilon)$ converges to 0 strongly in $L^2(\Omega)^N$. To conclude that, up to a subsequence, λ_ε converges to one of the $\lambda_m(\theta)$, the third key idea is to define a *modulation* of the sequence of quasi-eigenvectors, and to multiply it by the "quasi"-spectral equation satisfied by $\boldsymbol{t}_\varepsilon$. The modulation of $\boldsymbol{t}_\varepsilon$ is defined by

$$\mathscr{M}(\boldsymbol{t}_\varepsilon) = \sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} \sum_{m=1}^{N} \phi_m(\theta_j(\varepsilon)) t_\varepsilon^{jm} e^{2\pi i \theta_j(\varepsilon)\cdot E(x/\varepsilon)} \boldsymbol{e}_m(\theta_j(\varepsilon)),$$

where the functions $(\phi_m(\theta))_{1 \leqq m \leqq N}$ are continuous on $[-\frac{1}{2};\frac{1}{2}]^N$ and vanish for $\theta = 0$. Multiplying $\tilde{S}_\varepsilon \boldsymbol{t}_\varepsilon - \lambda_\varepsilon \boldsymbol{t}_\varepsilon$ by $\mathscr{M}(\boldsymbol{t}_\varepsilon)$ gives

$$\sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} \sum_{m=1}^{N} (\lambda_m(\theta_j(\varepsilon)) - \lambda_\varepsilon)\, \bar{\phi}_m(\theta_j(\varepsilon))\, |t_\varepsilon^{jm}|^2 = o(1), \tag{125}$$

where $o(1)$ is a remainder term which goes to zero. Let us define N positive real measures $(\mu_\varepsilon^m(\theta))_{1 \leqq m \leqq N}$ on $[-\frac{1}{2};\frac{1}{2}]^N$ by

$$\mu_\varepsilon^m(\theta) = (\varepsilon K(\varepsilon))^N \sum_{j=-(K(\varepsilon)-1)/2}^{(K(\varepsilon)-1)/2} |t_\varepsilon^{jm}|^2\, \delta_{\theta=\theta_j(\varepsilon)}, \tag{126}$$

where each $\delta_{\theta=\theta_j(\varepsilon)}$, with $\theta_j(\varepsilon) = j/K(\varepsilon)$ is a Dirac mass. We call these measures the *Bloch measures* associated with the sequence $\boldsymbol{t}_\varepsilon$ (they are very similar to the well-known Wigner, or semi-classical, measures; see [16] and [20]). The fundamental equation (125) can be stated as

$$\sum_{m=1}^{N} \int_{[-1/2;1/2]^N} (\lambda_m(\theta) - \lambda_\varepsilon)\bar{\phi}_m(\theta)\mu_\varepsilon^m(d\theta) = o(1). \tag{127}$$

Since $\sum_j (\varepsilon K(\varepsilon))^N\, |t_\varepsilon^j|^2 = \|\boldsymbol{t}_\varepsilon\|^2_{L^2(\Omega)} = 1$, each μ_ε^m is a bounded sequence of positive finite measures. Up to a subsequence, each converges in the sense of vague measures to a positive limit μ^m, and we have

$$\sum_{m=1}^{N} \int_{[-1/2;1/2]^N} \mu^m(d\theta) = \lim_{\varepsilon\to 0} \sum_{m=1}^{N} \int_{[-1/2;1/2]^N} \mu_\varepsilon^m(d\theta) = 1.$$

This implies that at least one of the μ^m is a non-zero finite measure. Since the eigenvalues $\lambda_m(\theta)$ are continuous functions of θ, except at 0 where $\phi_m(\theta)$ is purposely chosen equal to 0, we can pass to the limit in (127) to obtain

$$\int_{[-1/2;1/2]^N} (\lambda_m(\theta) - \lambda)\bar{\phi}_m(\theta)\mu^m(d\theta) = 0$$

for any m. By varying the continuous function $\phi_m(\theta)$, we obtain

$$\lambda_m(\theta) = \lambda \quad \mu^m\text{-almost everywhere on } [-\tfrac{1}{2};\tfrac{1}{2}]^N \setminus \{0\}. \tag{128}$$

Let us assume that, at least one of the limit Bloch measures μ^m is not a Dirac mass concentrated at 0 (this means that the sequence $\boldsymbol{s}_\varepsilon$ does actually oscillate on the microscopic scale ε). Then, (128) implies that there exists a point $\theta_0 \neq 0$ such that $\lambda_m(\theta_0) = \lambda$, which proves that λ belongs to the Bloch-wave spectrum $\bigcup_{m=1}^{N} [a_m, b_m]$.

Step 4: Microscopic oscillations of the eigenvectors on a length scale larger than ε. The last case to be considered after the first three steps is that of a sequence of eigenvectors $\boldsymbol{s}_\varepsilon$ satisfying the same assumptions as in the third step (i.e., converging weakly to zero in $L^2(\Omega)^N$ and such that there exist a constant $c > 0$ and a subset $\bar{\omega} \subset \Omega$ with $\lim_{\varepsilon\to 0} \|\boldsymbol{s}_\varepsilon\|_{L^2(\omega)} \geqq c > 0$), and furthermore having the property that the sequence of quasi-eigenvectors $\boldsymbol{t}_\varepsilon$ has all its Bloch measures μ^m equal to multiples of the Dirac mass at zero. This means that, in ω, the sequence $\boldsymbol{s}_\varepsilon$ oscillates on a scale much larger than ε, yet still smaller than 1. In this case, we do not care about the precise length scale, but we are only interested in the directions of oscillations, represented by the variable $\xi \in S^N$ (a fact reminiscent of the H-measures of P. GÉRARD [15] and L. TARTAR [37]). In the present situation some Bloch frequencies are negligible, as stated in the next lemma.

Lemma 3.4.9. *Since $\boldsymbol{t}_\varepsilon$ converges weakly to 0 in $L^2(\Omega)^N$, and since the Bloch measures $(\mu^m)_{1\leqq m\leqq N}$, which are limits of the sequences $(\mu_\varepsilon^m)_{1\leqq m\leqq N}$ defined from $\boldsymbol{t}_\varepsilon$ by* (126), *are reduced to multiples of the Dirac mass at zero, it follows that*

$$\lim_{\varepsilon\to 0} |\boldsymbol{t}_\varepsilon^{j(\varepsilon)}| = 0$$

for any sequence of multi-integers $j(\varepsilon)$ such that $\lim_{\varepsilon\to 0} j(\varepsilon)/K(\varepsilon) = \theta \neq 0$. Furthermore, there exists a sequence of integers $J(\varepsilon)$ such that

$$\lim_{\varepsilon\to 0} J(\varepsilon) = +\infty, \quad \lim_{\varepsilon\to 0} \frac{J(\varepsilon)}{K(\varepsilon)} = 0,$$

$$\lim_{\varepsilon\to 0} \sum_{J(\varepsilon)\leqq |j| \leqq K(\varepsilon)} |\boldsymbol{t}_\varepsilon^j|^2 = 0. \tag{129}$$

Proof. Let $j(\varepsilon)$ be a sequence of multi-indexes such that $j(\varepsilon)/K(\varepsilon)$ converges to a non-zero limit θ and that $|\boldsymbol{t}_\varepsilon^{j(\varepsilon)}|$ converges to a strictly positive limit. Then, it is easily seen that one of the Bloch measures μ^m must be supported at the point θ, which contradicts the hypothesis that it is a Dirac mass at the origin. Eventually, (129) is deduced from a standard diagonalization argument.

Lemma 3.4.9 shows that $\boldsymbol{t}_\varepsilon$ can be rewritten as

$$\boldsymbol{t}_\varepsilon(x) = \sum_{0<|j|\leqq J(\varepsilon)} \boldsymbol{t}_\varepsilon^j e^{2\pi i \theta_j(\varepsilon)\cdot E(x/\varepsilon)} + \boldsymbol{r}'_\varepsilon(x),$$

where $\boldsymbol{r}'_\varepsilon$ is another remainder term which goes to 0 strongly in $L^2(\Omega)^N$ (the term of order $j = 0$ is also incorporated into the remainder since $\boldsymbol{t}_\varepsilon$ converges weakly to 0 in

$L^2(\Omega)^N$). We can proceed as in the third step; multiplying $\tilde{S}_\varepsilon t_\varepsilon - \lambda_\varepsilon t_\varepsilon$ by a modulation of $t_\varepsilon - r'_\varepsilon$, we obtain

$$\sum_{0<|j|\leqq J(\varepsilon)} \sum_{m=1}^{N} (\lambda_m(\theta_j(\varepsilon)) - \lambda_\varepsilon)\bar{\phi}_m(\theta_j(\varepsilon))\, |t_\varepsilon^{jm}|^2 = o(1), \tag{130}$$

where each function $\phi_m(\theta)$ now depends only on the direction of the vector θ, namely,

$$\phi_m(\theta) = \hat{\phi}_m\,(\theta/|\theta|),$$

where $\hat{\phi}(\xi)$ is a continuous function of ξ on the unit sphere S^N. The essential ingredient for the sequel is the continuity of the matrix $A(\theta)$ near zero along the rays $\xi = \theta/|\theta|$, provided by Proposition 3.4.4. Since $J(\varepsilon)/K(\varepsilon)$ goes to zero with ε, we have

$$\lambda_m\,(\theta_j(\varepsilon)) = \hat{\lambda}_m \left(\frac{j}{|j|}\right) + o(1)$$

uniformly for $0 < |j| \leqq J(\varepsilon)$, where $\hat{\lambda}_m(\xi)$ are the eigenvalues of the continuous matrix $\hat{A}(\xi)$. In other words, the fundamental equation (130) can be rewritten as

$$\sum_{0<|j|\leqq J(\varepsilon)} \sum_{m=1}^{N} \left(\hat{\lambda}_m\left(\frac{j}{|j|}\right) - \lambda_\varepsilon\right) \overline{\hat{\phi}_m}\left(\frac{j}{|j|}\right) |t_\varepsilon^{jm}|^2 = o(1).$$

Let us define another family of positive real measures $(v_\varepsilon^m(\xi))_{1 \leqq m \leqq N}$ on S^N by

$$v_\varepsilon^m(\xi) = (\varepsilon K(\varepsilon))^N \sum_{0<|j|\leqq J(\varepsilon)} |t_\varepsilon^{jm}|^2\, \delta_{\xi = j/|j|}, \tag{131}$$

where each $\delta_{\xi=j/|j|}$ is a Dirac mass. We call these measures *rescaled Bloch measures.* Since $\sum_j (\varepsilon(K(\varepsilon))^N\, |t_\varepsilon^j|^2 = 1$, each v_ε^m is also a bounded sequence of positive finite measures on S^N. Up to a subsequence, they converge in the sense of vague measures to limits v^m which satisfy

$$\int_{S^N} \sum_{m=1}^{N} v^m(d\xi) = 1.$$

Since the eigenvalues $(\hat{\lambda}_m(\xi))_{1 \leqq m \leqq N}$ of $\hat{A}(\xi)$ are continuous by virtue of Proposition 3.4.4, we can pass to the limit to obtain

$$\int_{S^N} (\hat{\lambda}_m(\xi) - \lambda)\, \overline{\hat{\phi}_m(\xi)} v^m(d\xi) = 0$$

for any m. This proves that $\hat{\lambda}_m(\xi) = \lambda$, v^m-almost everywhere. Since all the rescaled Bloch measures v^m cannot be zero, there exists a direction ξ_0 such that $\hat{\lambda}_m(\xi_0) = \lambda$, which proves that λ belongs to the Bloch spectrum $\bigcup_{m=1}^{N} [a_m, b_m]$.

Proof of Corollary 3.2.11. If Ω is a parallelepiped, $]0,L_1[\times]0,L_2[\times\cdots]0,L_N[$, where the $(L_m)_{1\leqq m\leqq N}$ are positive integers, and if the sequence of periods is $\varepsilon_n = 1/n$, then it is easily seen in the previous proof that, for a periodic boundary condition, the case of a sequence of eigenvectors concentrating on the boundary cannot occur since the domain Ω is actually a torus with a finite number of entire cells. Furthermore, for the same parallelepiped with a Dirichlet boundary condition, this statement still holds at the price of assuming that the unit tube in the periodic cell has cubic symmetry. Indeed, by skew symmetry with respect to the boundary, one can extend the solution u_ε of (109) to a periodic solution of the same equation in a new domain of size twice that of Ω. A simpler proof is even available when Ω is the unit torus $[0,1]^N$ with $\varepsilon_n = 1/n$. In this case, Ω is precisely equal to $[0,\varepsilon_n K(\varepsilon_n)]^N$ with $K(\varepsilon_n) = n$. Thus, the operator S_{ε_n} obviously coincides, up to an isomorphism, with the homogenized operator $S^{K(\varepsilon_n)}$. Therefore, the limit spectrum σ_∞ reduces to $\lim_{K\to+\infty}\sigma(S^K)$ as expected.

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Commissariat à l' Energie Atomique
DRN/DMT/SERMA, C.E. Saclay
91191 Gif sur Yvette, France

and

Departamento de Ingeniería Matemática
Universidad de Chile, Casilla 170/3
Santiago

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