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EXPLICIT OPTIMAL BOUNDS ON THE ELASTIC ENERGY OF A TWO-PHASE COMPOSITE IN TWO SPACE DIMENSIONS

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Abstract. This paper is concerned with two-dimensional, linearly elastic, composite materials made by mixing two isotropic components. For given volume fractions and average strain, we establish *explicit* optimal upper and lower bounds on the effective energy quadratic form. There are two different approaches to this problem, one based on the "Hashin-Shtrikman variational principle" and the other on the "translation method". We implement both. The Hashin-Shtrikman principle applies only when the component materials are "well-ordered", i.e., when the smaller shear and bulk moduli belong to the same material. The translation method, however, requires no such hypothesis. As a consequence, our optimal bounds are valid even when the component materials are *not well-ordered*. Analogous results have previously been obtained by Gibianski and Cherkaev in the context of the plate equation.

0. Introduction. The macroscopic properties of a linearly elastic composite material are described by its tensor of effective moduli (Hooke's law) σ^*. This fourth-order tensor depends on the microgeometry of the mixture as well as on the elastic properties of the components. There is a large body of literature concerning the estimation of σ^* in terms of statistical information on the microstructure; see, e.g., [9, 39, 41].

Recently a related but somewhat different question has received much attention: given a fixed collection of component materials, can one describe all composites σ^* achievable by mixing these components in prescribed volume fraction? Known as the "G-closure problem", this question arises naturally from problems of structural optimization; see, e.g., [24, 26, 33]. A complete answer is available only in a few special cases; see, e.g., [13, 25, 27]. Much more is known about the analogous question

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for scalar phenomena (heat conduction, electrical resistance, etc.); see, e.g., [16, 28, 38].

While the full G-closure problem remains beyond reach in the context of elasticity, there has been remarkable progress concerning optimal bounds on the elastic energy quadratic form [1–6, 14, 15, 17, 21, 23, 30]. A specific consequence of that work is the following: consider two isotropic materials with bulk modulus κ_1, κ_2 and shear modulus μ_1, μ_2, which are to be mixed with volume fractions θ_1 and θ_2 respectively. Assume moreover that the components are "well-ordered", i.e., that either $\kappa_1 \le \kappa_2$, $\mu_1 \le \mu_2$ or $\kappa_2 \le \kappa_1$, $\mu_2 \le \mu_1$. Then one can identify the largest and smallest possible effective energy of σ^* as functions of the macroscopic strain. In other words, one can identify functions $f_\pm(\mu_1, \mu_2, \kappa_1, \kappa_2, \theta_1, \theta_2, \xi)$ such that

$$f_- \le \langle \sigma^* \xi, \xi \rangle \le f_+ \tag{0.1}$$

and such that both inequalities can be achieved (for any ξ) by suitable microstructures (which depend on ξ). We call $f_\pm$ "optimal bounds on the elastic energy", since f_- is evidently the smallest and f_+ the largest function for which (0.1) can hold. Clearly (0.1) improves upon the well-known harmonic and arithmetic mean bounds, known as Paul's or the Voigt-Reuss bounds:

$$\langle (\theta_1 \sigma_1^{-1} + \theta_2 \sigma_2^{-1})^{-1} \xi, \xi \rangle \le \langle \sigma^* \xi, \xi \rangle \le \langle (\theta_1 \sigma_1 + \theta_2 \sigma_2) \xi, \xi \rangle .$$

We emphasize that the symmetry of σ^* is not restricted in (0.1). As might be expected, the extremal composites are isotropic only when ξ is isotropic; in that case (0.1) reduces to the well-known Hashin-Shtrikman bound on the effective bulk modulus of an isotropic composite [19].

Energy bounds of the form (0.1) offer partial information about the G-closure problem, since they specify the extreme values of the linear functions $\sigma^* \to \langle \sigma^* \xi, \xi \rangle$ for every second-order tensor ξ. In addition, such bounds are of use in their own right. The best developed application is to structural optimization, where (0.1) or its analogue for complementary energy permits the solution of problems involving compliance as a design criterion [3, 7]. Other potential applications include coherent phase transitions [22] and modelling the accumulation of damage [11]. We remark that (0.1) represents only a special case of [4]: that paper actually identifies optimal upper and lower bounds for any sum of energies $\langle \sigma^* \xi_1, \xi_1 \rangle + \cdots + \langle \sigma^* \xi_n, \xi_n \rangle$.

The work just summarized has, alas, two significant shortcomings. First, the optimal upper and lower bounds $f_\pm$ are not given explicitly; rather, they are given as the extremal values of certain finite-dimensional optimizations. Second, the restriction that the component materials be well-ordered is unnatural: it is forced by the method of analysis (which is based on the Hashin-Shtrikman variational principle), not by anything intrinsic to the problem. The goal of the present work is to redress these difficulties in the special case of two space dimensions.

We implement this as follows. First, we review briefly the optimal energy bound from the Hashin-Shtrikman variational principle, specialized to the case of two well-ordered isotropic components in two space dimensions. Here we follow [2, 21, 23] rather than [4, 5], so $f_\pm$ is given as the extremal value of a convex but nonsmooth optimization problem. Then we solve this convex optimization to get an explicit formula for $f_\pm$; in the process, we also obtain an explicit description of the associated extremal microstructures. This makes everything explicit, but only for the well-ordered case. To handle non-well-ordered components, we turn to the "compensated compactness" or "translation" method (cf. [12, 14, 28, 29, 37, 38]). First, we calculate the translation bound on $\langle \sigma^* \xi, \xi \rangle$ explicitly, for a translation of the form $\lambda \det \xi$. Then we optimize over λ to get the "best" translation bound. Finally, we establish that the resulting bound is in fact optimal (i.e., is achieved by a microstructure) even when the component materials are not well-ordered.

The optimal bounds on elastic energy have previously been made explicit for mixtures of two incompressible materials in both two and three dimensions [23]. They have also been worked out for three-dimensional mixtures of an isotropic material with a rigid or totally degenerate material [15]. Our recent work on structural optimization makes use of an explicit lower bound on complementary energy [3]. Related results have also been obtained by others working on structural optimization [7, 15, 26]. The explicit lower bound f_- presented here as Proposition 1.3 was simultaneously and independently obtained by Francfort and Marigo [11].

Our use of the translation method is fundamentally the same as made by Gibianski and Cherkaev in [14]. Indeed, that paper addresses the analogue for Kirchhoff plate theory of the problem considered here. There is an isomorphism between plate theory and two-dimensional elasticity, so our results could basically be read off from those of [14]. The bounds presented here appear explicitly in [15]. Unfortunately, that work has not been published in any (Soviet or western) scientific journal and therefore will be unavailable to many readers. (A summary appears in [26].) There is one significant difference between our treatment and that of [14, 15]. They establish the optimality of their bound by displaying an explicit (laminated or sequentially laminated) microstructure that saturates the bound; we prove optimality instead as an easy by-product of the Hashin-Shtrikman calculation (even in the non-well-ordered case!).

When $\xi = I$ in (0.1), our bounds $f_\pm$ agree with the Hashin-Shtrikman bulk modulus bounds [19], which were first extended to the non-well-ordered case by Walpole [40]. These bulk modulus bounds have more recently been derived using the translation method [12]. One could say that (0.1) extends the Hashin-Shtrikman-Walpole bulk modulus bounds to an analogous result on the effective energy at any strain ξ. Even at $\xi = I$, however, our result is in a sense stronger than that of [19]: that paper bounds $\langle \sigma^* \xi, \xi \rangle$ under the hypothesis that σ^* is isotropic; we bound it without assuming that σ^* is isotropic. Related bounds on the "generalized bulk modulus" for possibly anisotropic σ^* are also given in [12, 17, 20, 30, 42].

This paper is exclusively devoted to two space dimensions. It is natural to ask what the prospects are for similar results in three space dimensions. As for the explicit evaluation of the optimal energy bounds for mixtures of two well-ordered isotropic materials: here there is no conceptual obstacle, just a larger number of different regimes. Extending the bounds to the non-well-ordered case, however, is a more subtle matter. For example, in 3D there are 9 linearly independent translations of the form $\tau_{ij}(\xi) = \xi_{ij}^2 - \xi_{ii}\xi_{jj}$ which are quasi-convex on strains. Moreover, even after using the best linear combination of these there is no guarantee, it seems, that the result will be optimal. Nevertheless, we have recently extended the optimal lower bound to the non-well-ordered case in 3 (or more) dimensions [1], without computing its value explicitly!

The remainder of this introduction is devoted to establishing notation and reviewing basic facts about composite materials. Since each component material is isotropic, it is characterized by a bulk modulus κ_i and a shear modulus μ_i (all moduli are positive, $i = 1, 2$). The Hooke's law σ_i is defined by

$$\sigma_i\xi = 2\mu_i(\xi - \tfrac{1}{2}(\mathrm{tr}\,\xi)I_2) + \kappa_i(\mathrm{tr}\,\xi)I_2 \tag{0.2}$$

for any symmetric second-order tensor ξ, where I_2 is the identity in the space of all second-order tensors. (Notice that we are working in two space dimensions.) We may and shall assume without loss of generality that

$$\mu_1 \le \mu_2 . \tag{0.3}$$

The well-ordered case is thus $\kappa_1 \le \kappa_2$, and the non-well-ordered case is $\kappa_1 > \kappa_2$.

By a composite made from materials σ_1 and σ_2 we mean a mixture having fine scale structure, with perfect bonding at all material interfaces. To give a mathematical definition one can use the theory of random composites (see, e.g., [18, 34]), the theory of H-convergence (also known as G-convergence; see, e.g., [12, 32, 36, 37, 43]), or the spatially periodic theory (see, e.g., [8, 35]). However, the last point of view is the easiest to work with and is sufficient for proving bounds (for a rigorous proof of this point, see [18] in the random case and [10] in the general case of H-convergence). Consequently, there is no restriction in considering spatially periodic composites as we shall do henceforth.

Let ε be a small positive number (it will tend to zero in the sequel), and let $Q = (0;1)^2$ be the unit cell. The geometry of the composite is periodic of period $\varepsilon Q = (0;\varepsilon)^2$. We define a composite material σ_ε by

$$\sigma_\varepsilon(x) = \chi_1\left(\frac{x}{\varepsilon}\right)\sigma_1 + \chi_2\left(\frac{x}{\varepsilon}\right)\sigma_2$$

where $\chi_1(y)$ and $\chi_2(y)$ are Q-periodic functions such that

$$\chi_1(y) = 0 \text{ or } 1 \text{ a.e. in } Q, \qquad \chi_2(y) = 1 - \chi_1(y).$$

The volume fraction of material σ_i is thus

$$\theta_i = \int_Q \chi_i(y)\,dy \quad \text{for } i = 1, 2.$$

Assume that the composite occupies some domain Ω in R^2, is loaded by some body force f, and satisfies a prescribed boundary condition (e.g., zero displacement on $\partial\Omega$,—actually, the precise nature of the boundary condition is irrelevant). The displacement u_ε (a vector in R^2) and the strain $e(u_\varepsilon)$ (a 2×2 matrix) are solutions of the elasticity equations

$$\begin{gathered} e(u_\varepsilon) = \tfrac{1}{2}(\nabla u_\varepsilon + {}^t\nabla u_\varepsilon), \\ \nabla \cdot [\sigma_\varepsilon e(u_\varepsilon)] = f \quad \text{in } \Omega, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The fundamental convergence theorem of homogenization (see, e.g., [8, 35]) says that, as ε goes to zero, the solutions u_ε converge to the solution u of the constant coefficient system

$$\begin{gathered} e(u) = \tfrac{1}{2}(\nabla u + {}^t\nabla u), \\ \nabla \cdot [\sigma^* e(u)] = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where the tensor σ^* is the effective Hooke's law of the mixture. It is independent of Ω, f, and the boundary condition, and, for any symmetric second-order tensor ξ, it is characterized by the formula

$$\langle \sigma^*\xi, \xi\rangle = \inf \int_Q \langle \sigma(y)[\xi + e(\phi)],\ [\xi + e(\phi)]\rangle\, dy$$

where the infimum is taken over all Q-periodic functions ϕ, $e(\phi) = \frac{1}{2}(\nabla\phi + {}^t\nabla\phi)$, and the "local" Hooke's law is $\sigma(y) = \chi_1(y)\sigma_1 + \chi_2(y)\sigma_2$.

Our goal is to maximize or minimize $\langle \sigma^*\xi, \xi\rangle$, with μ_1, μ_2, κ_1, κ_2, θ_1, θ_2, and ξ held fixed, as the microstructure varies.

1. Lower bounds. This section deals with the optimal lower bound $\langle \sigma^*\xi, \xi\rangle \geq f_-$. Addressing first the well-ordered case, we recall the bound as presented by [2], in the form of a concave maximization. Evaluation of the maximum leads to an explicit formula, with three distinct regimes. In two of the regimes the bound is achieved by a rank-one laminate; the third requires a rank-two laminate. We then give an alternative proof of the bound using the translation method. This new proof has the advantage of applying even in the non-well-ordered case.

The following result is proved in [2] using the Hashin-Shtrikman variational principle.

Proposition 1.1. When the two materials are well-ordered, we have for any symmetric second-order tensor ξ

$$\langle \sigma^*\xi, \xi\rangle \geq \langle \sigma_1\xi, \xi\rangle + \theta_2 \sup_\eta [2\langle \xi, \eta\rangle - \langle (\sigma_2 - \sigma_1)^{-1}\eta, \eta\rangle - \theta_1 g(\eta)], \tag{1.1}$$

where the supremum is taken over all symmetric constant second-order tensors η and $g(\eta)$ is given by

$$g(\eta) = \sup_{|\nu|=1} \left[\frac{1}{\mu_1}(|\eta\nu|^2 - \langle \eta\nu, \nu\rangle^2) + \frac{1}{\mu_1 + \kappa_1}\langle \eta\nu, \nu\rangle^2 \right]. \tag{1.2}$$

Furthermore the bound (1.1) is optimal in the sense that there exists a sequentially laminated composite which achieves equality in (1.1).

The first step toward making (1.1) explicit is the evaluation of $g(\eta)$. This is done in [2], where the following result is proved.

LEMMA 1.2. If we label the eigenvalues of η so that $\eta_1 \leq \eta_2$, then the function $g(\eta)$ defined by (1.2) is equal to

$$g(\eta) = \begin{cases} \dfrac{\kappa_1(\eta_1 - \eta_2)^2 + \mu_1(\eta_1 + \eta_2)^2}{4\mu_1\kappa_1} & \text{if } \eta_2 \geq \dfrac{\mu_1 + \kappa_1}{2\kappa_1}(\eta_2 + \eta_1) \geq \eta_1, \\ \dfrac{\eta_1^2}{\mu_1 + \kappa_1} & \text{if } \eta_1 \geq \dfrac{\mu_1 + \kappa_1}{2\kappa_1}(\eta_2 + \eta_1), \\ \dfrac{\eta_2^2}{\mu_1 + \kappa_1} & \text{if } \dfrac{\mu_1 + \kappa_1}{2\kappa_1}(\eta_2 + \eta_1) \geq \eta_2. \end{cases} \tag{1.3}$$

We now compute the explicit form of the bound (1.1). An equivalent calculation has been done independently by Francfort and Marigo [11].

PROPOSITION 1.3. Denoting by ξ_1 and ξ_2 the eigenvalues of ξ, the explicit formula for the bound (1.1) is

$$\begin{aligned} \langle\sigma^*\xi, \xi\rangle &\geq \frac{\kappa_1\kappa_2}{\theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 + \frac{\mu_1\mu_2}{\theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2)^2 \\ &\quad \text{if } (\kappa_2 - \kappa_1)(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| \leq (\mu_2 - \mu_1)(\theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2|; \\ \langle\sigma^*\xi, \xi\rangle &\geq \langle(\theta_1\sigma_1 + \theta_2\sigma_2)\xi, \xi\rangle - \theta_1\theta_2\frac{[(\kappa_2 - \kappa_1)|\xi_1 + \xi_2| + (\mu_2 - \mu_1)|\xi_1 - \xi_2|]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)} \\ &\quad \text{if } (\kappa_2 - \kappa_1)(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| \geq (\mu_2 - \mu_1)(\theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \\ &\quad \text{and } (\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \geq \theta_1(\kappa_2 - \kappa_1)|\xi_1 + \xi_2|; \\ \langle\sigma^*\xi, \xi\rangle &\geq \mu_1(\xi_1 - \xi_2)^2 + \frac{\kappa_1\kappa_2 + \mu_1(\theta_1\kappa_1 + \theta_2\kappa_2)}{\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 \\ &\quad \text{if } (\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \leq \theta_1(\kappa_2 - \kappa_1)|\xi_1 + \xi_2|. \end{aligned} \tag{1.4}$$

Proof. For simplicity, we adopt the notation $\delta\mu = \mu_2 - \mu_1$ and $\delta\kappa = \kappa_2 - \kappa_1$. It is well known that the maximum of $\langle\xi, \eta\rangle$ is obtained when η and ξ are simultaneously diagonal (see, e.g., [31]). Furthermore $\langle(\sigma_2 - \sigma_1)^{-1}\eta, \eta\rangle$ and $g(\eta)$ depend only on the eigenvalues of η. Thus, denoting by ξ_1 and ξ_2 the eigenvalues of ξ, maximizing the right-hand side of (1.1) over all tensors η is equivalent to maximizing over all real numbers η_1 and η_2 the concave function

$$F(\eta_1, \eta_2) = 2(\xi_1\eta_1 + \xi_2\eta_2) - \frac{1}{2\delta\mu}(\eta_1^2 + \eta_2^2) - \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2)^2 - \theta_1 g(\eta_1, \eta_2). \tag{1.5}$$

Here $g(\eta_1, \eta_2)$ is defined by (1.3) when $\eta_1 < \eta_2$ and by its symmetric counterpart obtained by interchanging η_1 and η_2 when $\eta_1 > \eta_2$. The function $g(\eta_1, \eta_2)$ is continuously differentiable on each domain $\eta_1 < \eta_2$ and $\eta_1 > \eta_2$ but is merely continuous on R^2. We therefore proceed in two steps. In the first we assume that the maximizer of (1.5) satisfies $\eta_1 \neq \eta_2$, and we investigate the three regimes of $g(\eta_1, \eta_2)$. In the second step we consider the case where the maximum of (1.5) is reached on the line $\eta_1 = \eta_2$.

Step 1. We assume that $\eta_1 \neq \eta_2$, so $g(\eta_1, \eta_2)$ is differentiable. We first consider the case $\eta_1 < \eta_2$; the case $\eta_1 > \eta_2$ will be obtained later, by symmetry. The maximizer of (1.5) is obtained by solving the linear system $\nabla F(\eta_1, \eta_2) = 0$.

(1) Assume that $\eta_2 \geq (\mu_1 + \kappa_1)(\eta_2 + \eta_1)/2\kappa_1 \geq \eta_1$. Then

$$\begin{aligned} \frac{1}{2\delta\mu}\eta_1 + \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2) + \theta_1 \frac{\kappa_1(\eta_1 - \eta_2) + \mu_1(\eta_1 + \eta_2)}{4\mu_1\kappa_1} &= \xi_1, \\ \frac{1}{2\delta\mu}\eta_2 + \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2) + \theta_1 \frac{-\kappa_1(\eta_1 - \eta_2) + \mu_1(\eta_1 + \eta_2)}{4\mu_1\kappa_1} &= \xi_2, \end{aligned} \tag{1.6}$$

which is equivalent to

$$\begin{aligned} &\eta_1[\mu_1\kappa_1\delta\mu + \mu_1\kappa_1\delta\kappa + \theta_1(\mu_1 + \kappa_1)\delta\mu\delta\kappa] \\ &\qquad + \eta_2[\mu_1\kappa_1\delta\mu - \mu_1\kappa_1\delta\kappa + \theta_1(\mu_1 - \kappa_1)\delta\mu\delta\kappa] = 4\mu_1\kappa_1\delta\mu\delta\kappa\xi_1, \\ &\eta_1[\mu_1\kappa_1\delta\mu - \mu_1\kappa_1\delta\kappa + \theta_1(\mu_1 - \kappa_1)\delta\mu\delta\kappa] \\ &\qquad + \eta_2[\mu_1\kappa_1\delta\mu + \mu_1\kappa_1\delta\kappa + \theta_1(\mu_1 + \kappa_1)\delta\mu\delta\kappa] = 4\mu_1\kappa_1\delta\mu\delta\kappa\xi_2. \end{aligned}$$

Let

$$\Delta = [\mu_1\kappa_1\delta\mu + \mu_1\kappa_1\delta\kappa + \theta_1(\mu_1 + \kappa_1)\delta\mu\delta\kappa]^2 - [\mu_1\kappa_1\delta\mu - \mu_1\kappa_1\delta\kappa + \theta_1(\mu_1 - \kappa_1)\delta\mu\delta\kappa]^2.$$

An easy but tedious computation shows that

$$\Delta = 4\mu_1\kappa_1\delta\mu\delta\kappa(\theta_1\kappa_2 + \theta_2\kappa_1)(\theta_1\mu_2 + \theta_2\mu_1).$$

Thus, the solution of (1.6) is

$$\begin{aligned} \eta_1 &= \frac{\kappa_1\delta\kappa}{\theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2) + \frac{\mu_1\delta\mu}{\theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2), \\ \eta_2 &= \frac{\kappa_1\delta\kappa}{\theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2) - \frac{\mu_1\delta\mu}{\theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2). \end{aligned} \tag{1.7}$$

The maximum of $F(\eta_1, \eta_2)$ is

$$\operatorname{Max} F(\eta_1, \eta_2) = \frac{\kappa_1\delta\kappa}{\theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 + \frac{\mu_1\delta\mu}{\theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2)^2. \tag{1.8}$$

The value of the bound in this case is

$$\langle\sigma^*\xi, \xi\rangle \geq \langle\sigma_1\xi, \xi\rangle + \theta_2 \operatorname{Max} F(\eta_1, \eta_2) = \frac{\kappa_1\kappa_2}{\theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 + \frac{\mu_1\mu_2}{\theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2)^2. \tag{1.9}$$

This is nothing but the harmonic mean bound. The bound (1.1) is equivalent to (1.9) if and only if the solution (1.7) satisfies $\eta_1 < \eta_2$ and $\eta_2 \geq (\mu_1 + \kappa_1)(2\kappa_1)^{-1}(\eta_2 + \eta_1) \geq \eta_1$, or in other words when

$$\begin{aligned} &\xi_1 < \xi_2, \\ &\delta\kappa(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| \leq \delta\mu(\theta_1\kappa_2 + \theta_2\kappa_1)(\xi_2 - \xi_1). \end{aligned} \tag{1.10}$$

(2) Assume $\eta_1 \geq (\mu_1 + \kappa_1)(\eta_2 + \eta_1)/2\kappa_1$. Then

$$\begin{aligned} \frac{1}{2\delta\mu}\eta_1 + \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2) + \frac{\theta_1}{\mu_1 + \kappa_1}\eta_1 &= \xi_1, \\ \frac{1}{2\delta\mu}\eta_2 + \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2) &= \xi_2, \end{aligned} \tag{1.11}$$

which is equivalent to

$$\eta_1[(\mu_1+\kappa_1)\delta\mu+(\mu_1+\kappa_1)\delta\kappa+4\theta_1\delta\mu\delta\kappa]+\eta_2(\mu_1+\kappa_1)(\delta\mu-\delta\kappa)=4(\mu_1+\kappa_1)\delta\mu\delta\kappa\xi_1,$$
$$\eta_1(\delta\mu-\delta\kappa)+\eta_2(\delta\mu+\delta\kappa)=4\delta\mu\delta\kappa\xi_2.$$

Let

$$\Delta=[(\mu_1+\kappa_1)\delta\mu+(\mu_1+\kappa_1)\delta\kappa+4\theta_1\delta\mu\delta\kappa][\delta\mu+\delta\kappa]-(\mu_1+\kappa_1)[\delta\mu-\delta\kappa]^2.$$

An easy computation shows that

$$\Delta=4\delta\mu\delta\kappa[\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)].$$

Thus the solution of (1.11) is

$$\eta_1=\frac{(\mu_1+\kappa_1)\delta\kappa(\xi_1+\xi_2)+(\mu_1+\kappa_1)\delta\mu(\xi_1-\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)},$$
$$\eta_2=\frac{[(\mu_1+\kappa_1)\delta\kappa+2\theta_1\delta\kappa\delta\mu](\xi_1+\xi_2)-[(\mu_1+\kappa_1)\delta\mu+2\theta_1\delta\kappa\delta\mu](\xi_1-\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}. \tag{1.12}$$

The maximum of $F(\eta_1, \eta_2)$ is

$$\operatorname{Max}F(\eta_1,\eta_2)=\frac{\delta\kappa(\mu_1+\kappa_1+\theta_1\delta\mu)(\xi_1+\xi_2)^2+\delta\mu(\mu_1+\kappa_1+\theta_1\delta\kappa)(\xi_1-\xi_2)^2-2\theta_1\delta\kappa\delta\mu(\xi_1^2-\xi_2^2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}. \tag{1.13}$$

The value of the bound in this case is

$$\begin{aligned}\langle\sigma^*\xi,\xi\rangle&\geq\langle\sigma_1\xi,\xi\rangle+\theta_2\operatorname{Max}F(\eta_1,\eta_2)\\&=\langle(\theta_1\sigma_1+\theta_2\sigma_2)\xi,\xi\rangle+\theta_2[\operatorname{Max}F(\eta_1,\eta_2)-\langle(\sigma_2-\sigma_1)\xi,\xi\rangle].\end{aligned}$$

An easy but tedious computation yields

$$\langle\sigma^*\xi,\xi\rangle\geq\langle(\theta_1\sigma_1+\theta_2\sigma_2)\xi,\xi\rangle-\theta_1\theta_2\frac{[\delta\kappa(\xi_1+\xi_2)+\delta\mu(\xi_1-\xi_2)]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}. \tag{1.14}$$

The bound (1.1) is equivalent to (1.14) if and only if the solution (1.12) satisfies $\eta_1<\eta_2$ and $\eta_1\geq(\mu_1+\kappa_1)(2\kappa_1)^{-1}(\eta_2+\eta_1)$, in other words when

$$\begin{aligned}\delta\kappa(\theta_1\mu_2+\theta_2\mu_1)(\xi_1+\xi_2)&\leq\delta\mu(\theta_1\kappa_2+\theta_2\kappa_1)(\xi_1-\xi_2),\\(\mu_1+\theta_1\kappa_2+\theta_2\kappa_1)(\xi_1-\xi_2)&<\theta_1\delta\kappa(\xi_1+\xi_2).\end{aligned} \tag{1.15}$$

(3) Assume $(\mu_1+\kappa_1)(\eta_2+\eta_1)/2\kappa_1\geq\eta_2$. This case is symmetric to the previous one. More precisely, defining $\eta_1'=-\eta_2$ and $\eta_2'=-\eta_1$, we find that (η_1', η_2') satisfies the hypothesis of the second case. The computations are the same via the correspondence $(\xi_1, \xi_2)\to(-\xi_2', -\xi_1')$. Therefore the maximizer of $F(\eta_1, \eta_2)$ is

$$\eta_2=\frac{(\mu_1+\kappa_1)\delta\kappa(\xi_1+\xi_2)-(\mu_1+\kappa_1)\delta\mu(\xi_1-\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)},$$
$$\eta_1=\frac{[(\mu_1+\kappa_1)\delta\kappa+2\theta_1\delta\kappa\delta\mu](\xi_1+\xi_2)+[(\mu_1+\kappa_1)\delta\mu+2\theta_1\delta\kappa\delta\mu](\xi_1-\xi_2)}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}, \tag{1.16}$$

which yields

$$\langle \sigma^* \xi , \xi \rangle \geq \langle (\theta_1 \sigma_1 + \theta_2 \sigma_2) \xi , \xi \rangle - \theta_1 \theta_2 \frac{[\delta\kappa(\xi_1 + \xi_2) - \delta\mu(\xi_1 - \xi_2)]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)} . \tag{1.17}$$

The bound (1.1) is equivalent to (1.17) if and only if the solution (1.16) satisfies $\eta_1 < \eta_2$ and $(\mu_1 + \kappa_1)(2\kappa_1)^{-1}(\eta_2 + \eta_1) \geq \eta_2$, i.e.,

$$\begin{aligned} &-\delta\kappa(\theta_1\mu_2 + \theta_2\mu_1)(\xi_1 + \xi_2) \leq \delta\mu(\theta_1\kappa_2 + \theta_2\kappa_1)(\xi_1 - \xi_2), \\ &(\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)(\xi_1 - \xi_2) < -\theta_1\delta\kappa(\xi_1 + \xi_2). \end{aligned} \tag{1.18}$$

Up to now, we have only considered the case $\eta_1 < \eta_2$. The other case $\eta_2 < \eta_1$ is obtained by symmetry, just interchanging η_1 and η_2 and ξ_1 and ξ_2. Regrouping the results of the two cases yields that (1.1) is the harmonic mean bound (1.9) when

$$\delta\kappa(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| \leq \delta\mu(\theta_1\kappa_2 + \theta_2\kappa_1)|\xi_2 - \xi_1| . \tag{1.19}$$

On the other hand, it is easily seen that (1.18) implies that $\xi_1 + \xi_2$ is positive while $\xi_1 - \xi_2$ is negative. Similarly (1.15) implies that $\xi_1 - \xi_2$ and $\xi_1 + \xi_2$ are negative. Thus the bounds (1.14) and (1.17) are equivalent to

$$\langle \sigma^* \xi , \xi \rangle \geq \langle (\theta_1 \sigma_1 + \theta_2 \sigma_2) \xi , \xi \rangle - \theta_1 \theta_2 \frac{[\delta\kappa|\xi_1 + \xi_2| + \delta\mu|\xi_1 - \xi_2|]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)} , \tag{1.20}$$

which is asserted under the condition

$$\begin{aligned} \delta\kappa(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| &\geq \delta\mu(\theta_1\kappa_2 + \theta_2\kappa_1)(\xi_2 - \xi_1), \\ \theta_1\delta\kappa|\xi_1 + \xi_2| &< (\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)(\xi_2 - \xi_1). \end{aligned}$$

The preceding applies when $\eta_1 < \eta_2$. Taking into account also the case $\eta_2 < \eta_1$, we see that (1.1) is equivalent to (1.20) whenever

$$\begin{aligned} \delta\kappa(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| &\geq \delta\mu(\theta_1\kappa_2 + \theta_2\kappa_1)|\xi_2 - \xi_1|, \\ \theta_1\delta\kappa|\xi_1 + \xi_2| &< (\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_2 - \xi_1|. \end{aligned} \tag{1.21}$$

Step 2. If the eigenvalues ξ_1 and ξ_2 satisfy neither of the "compatibility" conditions (1.19) and (1.21), then the maximum of $F(\eta_1, \eta_2)$ is attained where $g(\eta_1, \eta_2)$ is not differentiable, i.e., where $\eta_1 = \eta_2$. In this case the maximum is reached for

$$\eta_1 = \eta_2 = \frac{(\mu_1 + \kappa_1)\delta\kappa}{\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2), \tag{1.22}$$

and the corresponding value of the bound is

$$\begin{aligned} \langle \sigma^* \xi , \xi \rangle &\geq \langle \sigma_1 \xi , \xi \rangle + \theta_2 \operatorname{Max} F(\eta_1, \eta_2) \\ &= \langle \sigma_1 \xi , \xi \rangle + \theta_2 \frac{(\mu_1 + \kappa_1)\delta\kappa}{\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 . \end{aligned}$$

An easy computation yields

$$\langle \sigma^* \xi , \xi \rangle \geq \mu_1(\xi_1 - \xi_2)^2 + \frac{\kappa_1\kappa_2 + \mu_1(\theta_1\kappa_1 + \theta_2\kappa_2)}{\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 . \tag{1.23}$$

The bounds (1.9)–(1.19) and (1.20)–(1.21), combined with (1.23) in all the other cases, are the desired result. □

It is worthy of note that the first regime of the optimal bound is precisely the harmonic mean bound. Thus the well-known and elementary fact that

$$\langle\sigma^*\xi, \xi\rangle \geq \langle(\theta_1\sigma_1^{-1} + \theta_2\sigma_2^{-1})^{-1}\xi, \xi\rangle$$

is actually optimal for many choices of ξ—a set with interior in the space of symmetric tensors.

The general theory assures us that the bound (1.1) is optimal and indeed that it is saturated by a sequentially laminated microstructure. Our next goal is to describe the optimal microstructures explicitly.

We begin with a review of the theory. (See [2, Sec. 3] for a complete account.) The optimality condition for (1.1) is

$$2\xi - 2(\sigma_2 - \sigma_1)^{-1}\eta \in \theta_1\partial g(\eta), \tag{1.24}$$

where $\partial g(\eta)$ is the subdifferential of g at η. According to (1.2),

$$g(\eta) = \sup_{|\nu|=1}\langle f(\nu)\eta, \eta\rangle,$$

where $f(\nu)$ is the "degenerate Hooke's law"

$$f(\nu)\eta = \frac{1}{\mu_1}[(\eta\nu)\odot\nu - \langle\eta\nu, \nu\rangle\nu\odot\nu] + \frac{1}{\mu_1+\kappa_1}\langle\eta\nu, \nu\rangle\nu\odot\nu. \tag{1.25}$$

The subdifferential of g is simply the convex hull of the tensors $2f(\nu)\eta$ as ν ranges over extremals for (1.2). Hence (1.24) can be rewritten as

$$\xi - (\sigma_2 - \sigma_1)^{-1}\eta = \theta_1\sum_{i=1}^{p} m_i f(\nu_i)\eta, \tag{1.26}$$

in which $m_i \geq 0$, $\sum m_i = 1$, and each ν_i is extremal for (1.2). If g is differentiable at the optimal η then (1.26) becomes

$$\xi - (\sigma_2 - \sigma_1)^{-1}\eta = \theta_1 f(\nu)\eta$$

where ν is any extremal for (1.2), and the bound is achieved by a rank-one laminate with layer direction ν. If g is not differentiable at the optimal η then $p > 1$ in (1.26), and the bound is achieved by a rank-p laminate which is determined in a systematic way by the parameters $\{m_i\}$ and $\{\nu_i\}$.

The character of the optimal microstructures is now clear. The first two regimes of (1.4) correspond to rank-one laminates, since the optimal η has $\eta_1 \neq \eta_2$. The third regime corresponds to a higher-rank laminate (rank 2, as we shall show presently), since the optimal η is a multiple of the identity.

To specify the microstructures completely for the first two regimes we need to know, for given η, which vectors ν are extremal in (1.2). This information was obtained in the course of calculating $g(\eta)$, in Proposition 7.4 of [2]. The answer is as follows: Let e_1, e_2 be the eigenvectors of η, with associated eigenvalues η_1, η_2, ordered so that $\eta_1 \leq \eta_2$. Then:

(i) In the first regime of (1.3) the extremal ν are $\alpha_1 e_1 + \alpha_2 e_2$ with

$$\begin{aligned} \alpha_1^2 &= \frac{(\kappa_1+\mu_1)\eta_1 - (\kappa_1-\mu_1)\eta_2}{2\kappa_1(\eta_1-\eta_2)}, \\ \alpha_2^2 &= \frac{(\kappa_1+\mu_1)\eta_2 - (\kappa_1-\mu_1)\eta_1}{2\kappa_1(\eta_2-\eta_1)}. \end{aligned} \tag{1.27}$$

(ii) In the second regime of (1.3) the extremal ν are $\pm e_1$.

(iii) In the third regime of (1.3) the extremal ν are $\pm e_2$.

(iv) If $\eta_1 = \eta_2$ then the second and third regimes of (1.3) agree and every unit vector is extremal.

Of course, if ν is extremal then so is $-\nu$. If we identify ν and $-\nu$, then there are two extremal directions in case (i) but only one extremal direction in cases (ii) and (iii).

The first regime of (1.4), when the optimal bound is the harmonic mean, corresponds to case (i). Hence it is achieved by two different rank-one laminates. The associated layering directions are determined by (1.7) and (1.27); they vary with ξ_1 and ξ_2.

The second regime of (1.4) corresponds to cases (ii) or (iii). It is achieved by a unique rank-one laminate, whose layering direction is an eigenvector of ξ.

The analysis of the third regime of (1.4) is different, because we must deal with the nondifferentiability of g at multiples of the identity. If $\eta = \gamma I_2$ then *any* unit vector ν is extremal for (1.2), so there is no restriction on ν_i in (1.26). From (1.25) we have

$$f(\nu)\eta = \frac{\gamma}{\mu_1+\kappa_1}\nu \odot \nu$$

when $\eta = \gamma I_2$. Therefore,

$$\partial g(\eta) = \text{convex hull of } \left\{ \frac{2\gamma}{\mu_1+\kappa_1}\nu \odot \nu \right\}_{|\nu|=1}.$$

This is precisely the class of positive (if $\gamma > 0$) or negative (if $\gamma < 0$) semidefinite second-order tensors with trace $2\gamma/(\mu_1+\kappa_1)$.

To determine the optimal microstructure in this third regime, we must solve the optimality condition (1.26). From (1.22), the relevant value of η is $\eta = \gamma I$ with

$$\gamma = \frac{(\mu_1+\kappa_1)\delta\kappa}{\mu_1+\theta_1\kappa_2+\theta_2\kappa_1}(\xi_1+\xi_2).$$

One verifies that

$$\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1 \end{pmatrix} - \frac{1}{2\delta\kappa}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} = \frac{\theta_1}{2} \cdot \frac{2\gamma}{\mu_1+\kappa_1}\sum_{i=1}^{2} m_i e_i \odot e_i$$

when $e_1 = (1, 0)$, $e_2 = (0, 1)$, and

$$\begin{aligned} m_1 &= \frac{\xi_1(\mu_1+\kappa_1+2\theta_1\delta\kappa) - \xi_2(\mu_1+\kappa_1)}{2\theta_1\delta\kappa(\xi_1+\xi_2)}, \\ m_2 &= \frac{\xi_2(\mu_1+\kappa_1+2\theta_1\delta\kappa) - \xi_1(\mu_1+\kappa_1)}{2\theta_1\delta\kappa(\xi_1+\xi_2)}. \end{aligned} \tag{1.28}$$

It is clear that $m_1 + m_2 = 1$; one verifies that $m_1 \geq 0$, $m_2 \geq 0$ as a consequence of the relation which defines the third regime,

$$(\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \leq \theta_1\delta\kappa|\xi_1 + \xi_2|.$$

Thus the optimality condition (1.26) holds with $p = 2$: ν_1 and ν_2 are the eigenvectors of ξ, and m_1, m_2 are given by (1.28). According to the construction in [2, 12, 23], the bound is achieved by a second rank laminate. Specifically, we first layer σ_2 with σ_1 in volume fractions $\rho = 1 - \theta_1 m_1$ and $1 - \rho$ respectively, using layers orthogonal to ν_1, to get a composite C. Then we layer C with σ_1 in volume fractions $\rho' = \theta_2/(1 - \theta_1 m_1)$ and $1 - \rho'$ respectively, using layers orthogonal to ν_2. The resulting composite σ^* achieves equality in the bound.

We turn now to the translation method. Section 4 of [2] gives a general correspondence between the Hashin-Shtrikman variational principle and the translation method. Here, however, we proceed differently. Following [14], we use only multiples of the determinant as translations. When σ_1 and σ_2 are well-ordered, we recover the optimal bound (1.4). But this approach gives a valid bound even in the non-well-ordered case. We shall show that the resulting bound is optimal, i.e., is attainable by a microstructure. Thus the translation method gives an extension of Proposition 1.3 to the non-well-ordered case.

The essence of the translation method is the following result, proved, for example, in [29]:

PROPOSITION 1.4. Let τ be a constant fourth-order tensor. Assume that τ is quasiconvex on strains; i.e. for any Q-periodic function ϕ

$$\int_Q \langle \tau(\nabla\phi + {}^t\nabla\phi), (\nabla\phi + {}^t\nabla\phi)\rangle \geq 0. \tag{1.29}$$

Assume further that $\sigma(y) - \tau$ is positive on Q, i.e.,

$$\sigma_1 - \tau \geq 0 \quad \text{and} \quad \sigma_2 - \tau \geq 0. \tag{1.30}$$

Then we have

$$\sigma^* \geq \left(\int_Q (\sigma - \tau)^{-1}\right)^{-1} + \tau. \tag{1.31}$$

We call τ a "translation". The crucial issue, of course, is how to choose τ so that (1.31) becomes an interesting (and preferably optimal) bound. A fundamental understanding of this issue is at present lacking. However, for the problem at hand it will suffice to take $\tau = \lambda \cdot \det$, with the notation

$$\langle \det\xi, \xi\rangle = \det(\xi) = \xi_{11}\xi_{22} - \xi_{12}^2. \tag{1.32}$$

Following Milton [29], we define two fourth-order tensors Λ_s and Λ_h which project respectively to the subspace of tracefree tensors ("shears") and to the subspace of scalar tensors ("hydrostatic" tensors), i.e.,

$$\begin{aligned} \Lambda_s\xi &= \xi - \tfrac{1}{2}\operatorname{Tr}\xi I_2, \\ \Lambda_h\xi &= \tfrac{1}{2}\operatorname{Tr}\xi I_2. \end{aligned} \tag{1.33}$$

Being projection operators, Λ_s and Λ_h satisfy

$$\Lambda_s^2 = \Lambda_s, \qquad \Lambda_h^2 = \Lambda_h, \qquad \text{and} \qquad \Lambda_s + \Lambda_h = I_4. \tag{1.34}$$

The translation 2 det and the Hooke's law σ_i can be written in terms of Λ_s and Λ_h as

$$2\det = \Lambda_h - \Lambda_s \quad \text{and} \quad \sigma_i = 2\mu_i\Lambda_s + 2\kappa_i\Lambda_h. \tag{1.35}$$

LEMMA 1.5. Let σ_1 and σ_2 be two isotropic materials such that $\mu_1 \le \mu_2$ (the ordering of κ_1 and κ_2 is arbitrary). Then the assumptions of Proposition 1.4 are satisfied for the translation $\tau = 2\lambda\det$ precisely when $\lambda \in [-2\mu_1; 0]$.

Proof. Using Fourier analysis, it is easy to see that the definition (1.29) of the quasi-convexity on strains is equivalent to

$$\langle \tau(u \otimes \nu + \nu \otimes u), (u \otimes \nu + \nu \otimes u)\rangle \ge 0 \quad \text{for any vectors } u, \nu.$$

We observe that

$$\langle \det(u \otimes \nu + \nu \otimes u), (u \otimes \nu + \nu \otimes u)\rangle = \det(u \otimes \nu + \nu \otimes u) = -(u_1\nu_2 - \nu_1 u_2)^2.$$

Thus the translation $2\lambda\det$ is quasi-convex on strains when λ is negative. On the other hand, we have

$$\sigma_i - 2\lambda\det = (2\mu_i + \lambda)\Lambda_s + (2\kappa_i - \lambda)\Lambda_h.$$

Because Λ_s and Λ_h are nonnegative operators, $(\sigma_i - 2\lambda\det)$ is nonnegative for $i = 1, 2$ precisely when $\lambda \in [-2\mu_1; 0]$. □

Applying Proposition 1.4 with the translation $2\lambda\det$ and evaluating the resulting bound (1.31) at ξ, we obtain

$$\langle \sigma^*\xi, \xi\rangle \ge \left\langle \left[\int_Q (\sigma - 2\lambda\det)^{-1}\right]^{-1}\xi, \xi\right\rangle + 2\lambda\det(\xi). \tag{1.36}$$

The right-hand side of (1.36) is a differentiable function of λ in the interval $[-2\mu_1; 0]$. Thus, for a given tensor ξ, we can optimize the choice of λ in order to obtain the best possible bound from (1.36). We call the result the "translation bound".

PROPOSITION 1.6. Denoting by ξ_1 and ξ_2 the eigenvalues of ξ, the translation bound is

$$\begin{aligned}
\langle \sigma^*\xi, \xi\rangle &\ge \frac{\kappa_1\kappa_2}{\theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 + \frac{\mu_1\mu_2}{\theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2)^2 \\
&\quad \text{if } |\kappa_2 - \kappa_1|(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| \le |\mu_2 - \mu_1|(\theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2|; \\
\langle \sigma^*\xi, \xi\rangle &\ge \langle(\theta_1\sigma_1 + \theta_2\sigma_2)\xi, \xi\rangle - \theta_1\theta_2\frac{[|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2| + |\mu_2 - \mu_1|\,|\xi_1 - \xi_2|]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)} \\
&\quad \text{if } |\kappa_2 - \kappa_1|(\theta_1\mu_2 + \theta_2\mu_1)|\xi_1 + \xi_2| \ge |\mu_2 - \mu_1|(\theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \\
&\quad\quad \text{and } (\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \ge \theta_1|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|; \\
\langle \sigma^*\xi, \xi\rangle &\ge \mu_1(\xi_1 - \xi_2)^2 + \frac{\kappa_1\kappa_2 + \mu_1(\theta_1\kappa_1 + \theta_2\kappa_2)}{\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1}(\xi_1 + \xi_2)^2 \\
&\quad \text{if } (\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \le \theta_1|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|.
\end{aligned} \tag{1.37}$$

Proof. Let us compute the right-hand side of (1.36). We have

$$(\sigma_i - 2\lambda \det)^{-1} = (2\mu_i + \lambda)^{-1}\Lambda_s + (2\kappa_i - \lambda)^{-1}\Lambda_h ,$$

so

$$\left[\int_Q (\sigma - 2\lambda \det)^{-1}\right]^{-1} = \left(\frac{\theta_1}{2\mu_1 + \lambda} + \frac{\theta_2}{2\mu_2 + \lambda}\right)^{-1}\Lambda_s + \left(\frac{\theta_1}{2\kappa_1 - \lambda} + \frac{\theta_2}{2\kappa_2 - \lambda}\right)^{-1}\Lambda_h .$$

Thus (1.36) becomes

$$\begin{aligned}\langle \sigma^* \xi , \xi\rangle \geq & \left[\frac{(2\mu_1 + \lambda)(2\mu_2 + \lambda)}{\theta_1(2\mu_2 + \lambda) + \theta_2(2\mu_1 + \lambda)} - \lambda\right]\langle \Lambda_s \xi , \xi\rangle \\ & + \left[\frac{(2\kappa_1 - \lambda)(2\kappa_2 - \lambda)}{\theta_1(2\kappa_2 - \lambda) + \theta_2(2\kappa_1 - \lambda)} + \lambda\right]\langle \Lambda_h \xi , \xi\rangle .\end{aligned}$$

Note that in two dimensions $2\langle \Lambda_h \xi , \xi\rangle = (\xi_1 + \xi_2)^2$ and $2\langle \Lambda_s \xi , \xi\rangle = (\xi_1 - \xi_2)^2$. After some simplification, we find that (1.36) is equivalent to

$$\langle \sigma^* \xi , \xi\rangle \geq \langle (\theta_1\sigma_1 + \theta_2\sigma_2)\xi , \xi\rangle - \theta_1\theta_2 f(\lambda)$$

with

$$f(\lambda) = \frac{(\mu_2 - \mu_1)^2(\xi_1 - \xi_2)^2}{\theta_1\mu_2 + \theta_2\mu_1 + \lambda/2} + \frac{(\kappa_2 - \kappa_1)^2(\xi_1 + \xi_2)^2}{\theta_1\kappa_2 + \theta_2\kappa_1 - \lambda/2} . \tag{1.38}$$

Differentiating $f(\lambda)$, we easily find two possible roots of $f'(\lambda)$:

$$\lambda^+ = \frac{2(\theta_1\kappa_2 + \theta_2\kappa_1)|\mu_2 - \mu_1|\,|\xi_1 - \xi_2| - 2(\theta_1\mu_2 + \theta_2\mu_1)|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|}{|\mu_2 - \mu_1|\,|\xi_1 - \xi_2| + |\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|} ,$$

$$\lambda^- = \frac{2(\theta_1\kappa_2 + \theta_2\kappa_1)|\mu_2 - \mu_1|\,|\xi_1 - \xi_2| + 2(\theta_1\mu_2 + \theta_2\mu_1)|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|}{|\mu_2 - \mu_1|\,|\xi_1 - \xi_2| - |\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|} .$$

The second root λ^- is always outside the interval $[-2\mu_1 ; 0]$, and f is convex on $[-2\mu_1 ; 0]$. So there are three different cases:

(1) if $\lambda^+ < -2\mu_1$, then the minimum of $f(\lambda)$ is attained for $\lambda = -2\mu_1$;
(2) if $-2\mu_1 \leq \lambda^+ \leq 0$, then the minimum of $f(\lambda)$ is attained for $\lambda = \lambda^+$;
(3) if $\lambda^+ \geq 0$, then the minimum of $f(\lambda)$ is attained for $\lambda = 0$.

In the second case we easily compute the value of $f(\lambda^+)$:

$$f(\lambda^+) = \frac{(|\mu_2 - \mu_1|\,|\xi_1 - \xi_2| + |\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2|)^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)} .$$

The condition $\lambda^+ \leq 0$ is equivalent to

$$(\theta_1\kappa_2 + \theta_2\kappa_1)|\mu_2 - \mu_1|\,|\xi_1 - \xi_2| \leq (\theta_1\mu_2 + \theta_2\mu_1)|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2| ,$$

while the condition $-2\mu_1 \leq \lambda^+$ is equivalent to

$$(\mu_1 + \theta_1\kappa_2 + \theta_2\kappa_1)|\xi_1 - \xi_2| \geq \theta_1|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2| .$$

At this point very simple computations lead to the explicit formula (1.37) for the translation bound. □

This calculation highlights an error in [29]. There Milton asserts (p. 87) that among the translation bounds associated to $\tau = \lambda\tau_0$ with τ_0 fixed, the optimal result is obtained by taking λ at an end point of the interval of admissibility. This is clearly not correct, since one regime of the optimal bound corresponds to the interior value $\lambda = \lambda^+$.

In the well-ordered case (i.e., $\kappa_2 - \kappa_1 \geq 0$ and $\mu_2 - \mu_1 \geq 0$) the translation bound and the Hashin-Shtrikman bound (1.4) are clearly identical. In the non-well-ordered case, Proposition 1.6 yields a new bound which is algebraically almost identical to the Hashin-Shtrikman bound (up to absolute values for $\kappa_2 - \kappa_1$). Let us check that this new bound is optimal.

PROPOSITION 1.7. The translation bound (1.37) is also optimal in the non-well-ordered case. Specifically, there exists a sequentially laminated composite which achieves equality in the bound.

Proof. We review briefly how the argument goes in the well-ordered case. If η is optimal for the Hashin-Shtrikman variational principle (1.1), then it satisfies the first-order optimality condition (1.24). One shows that *if ξ and η are related by* (1.24), *then there exists a sequentially laminated composite whose effective tensor σ^* satisfies*

$$\langle \sigma^* \xi , \xi \rangle = \langle \sigma_1 \xi , \xi \rangle + \theta_2 [2\langle \xi , \eta \rangle - \langle (\sigma_2 - \sigma_1)^{-1} \eta , \eta \rangle - \theta_1 g(\eta)] .$$

This σ^* evidently achieves equality in the bound (1.1).

In the non-well-ordered case (1.1) is no longer a valid bound. However, the statement in italics remains valid (with the same proof). In other words, any critical value with respect to η of

$$\langle \sigma_1 \xi , \xi \rangle + \theta_2 [2\langle \xi , \eta \rangle - \langle (\sigma_2 - \sigma_1)^{-1} \eta , \eta \rangle - \theta_1 g(\eta)] \tag{1.39}$$

is equal to $\langle \sigma^* \xi , \xi \rangle$ for some sequentially laminated microstructure.

We claim that (1.37) gives the unique critical value of (1.39) in the non-well-ordered case. Indeed, the proof of Proposition 1.3 applies equally in this setting, with one minor change. When $\kappa_2 - \kappa_1 \geq 0$, the result of the symmetry between the cases $\eta_1 < \eta_2$ and $\eta_1 > \eta_2$ was the introduction in all formulae of absolute values for the terms $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$. In the present case, when $\kappa_2 - \kappa_1 < 0$, we must also introduce absolute values for $\kappa_2 - \kappa_1$. Therefore, the critical value of (1.39) is given by (1.37) rather than (1.4). □

2. Upper bounds. We turn now to the optimal bound $\langle \sigma^* \xi , \xi \rangle \leq f_+$. Our treatment is parallel to that of Sec. 1. Addressing first the well-ordered case, we recall the optimal bound as presented in [2]. We then derive an explicit formula, which again has three different regimes. One of the regimes is achieved by a rank-one laminate, while the other two require rank-two laminates. We then turn to the translation method. It gives an alternative proof of the bound, which applies even in the non-well-ordered case.

The following result is proved in [2] using the Hashin-Shtrikman variational principle.

PROPOSITION 2.1. When the two materials are well-ordered, we have for any symmetric second-order tensor ξ

$$\langle\sigma^*\xi\,,\xi\rangle \le \langle\sigma_2\xi\,,\xi\rangle - \theta_1 \sup_\eta [2\langle\xi\,,\eta\rangle - \langle(\sigma_2-\sigma_1)^{-1}\eta\,,\eta\rangle + \theta_2 h(\eta)], \tag{2.1}$$

where the supremum is taken over all symmetric constant second-order tensors η and $h(\eta)$ is given by

$$h(\eta) = \inf_{|\nu|=1}\left[\frac{1}{\mu_2}(|\eta\nu|^2 - \langle\eta\nu\,,\nu\rangle^2) + \frac{1}{\mu_2+\kappa_2}\langle\eta\nu\,,\nu\rangle^2\right]. \tag{2.2}$$

Furthermore the bound (2.1) is optimal in the sense that there exists a sequentially laminated composite which achieves equality in (2.1).

The first step toward making (2.1) explicit is the evaluation of $h(\eta)$. This is done in [2], where the following result is proved.

LEMMA 2.2. The function $h(\eta)$, defined by (2.2), is equal to

$$h(\eta) = \inf_{i=1,2}\frac{\eta_i^2}{\mu_2+\kappa_2} \tag{2.3}$$

where the $(\eta_1\,,\eta_2)$ are the eigenvalues of η.

We now compute the explicit form of the bound (2.1).

PROPOSITION 2.3. Denoting by ξ_1 and ξ_2 the eigenvalues of ξ, the explicit formula for the bound (2.1) is

$$\begin{aligned}
\langle\sigma^*\xi\,,\xi\rangle &\le \langle(\theta_1\sigma_1+\theta_2\sigma_2)\xi\,,\xi\rangle - \frac{\theta_1\theta_2[(\kappa_2-\kappa_1)|\xi_1+\xi_2| - (\mu_2-\mu_1)|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)}\\
&\quad \text{if } \theta_2(\kappa_2-\kappa_1)|\xi_1+\xi_2| \le (\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)|\xi_1-\xi_2|\\
&\quad \text{and } \theta_2(\mu_2-\mu_1)|\xi_1-\xi_2| \le (\theta_1\mu_2+\theta_2\mu_1+\kappa_2)|\xi_1+\xi_2|;\\
\langle\sigma^*\xi\,,\xi\rangle &\le \mu_2(\xi_1-\xi_2)^2 + \frac{\kappa_1\kappa_2+\mu_2(\theta_1\kappa_1+\theta_2\kappa_2)}{\mu_2+\theta_1\kappa_2+\theta_2\kappa_1}(\xi_1+\xi_2)^2\\
&\quad \text{if } \theta_2(\kappa_2-\kappa_1)|\xi_1+\xi_2| \ge (\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)|\xi_1-\xi_2|;\\
\langle\sigma^*\xi\,,\xi\rangle &\le \kappa_2(\xi_1+\xi_2)^2 + \frac{\mu_1\mu_2+\kappa_2(\theta_1\mu_1+\theta_2\mu_2)}{\kappa_2+\theta_1\mu_2+\theta_2\mu_1}(\xi_1-\xi_2)^2\\
&\quad \text{if } \theta_2(\mu_2-\mu_1)|\xi_1-\xi_2| \ge (\theta_1\mu_2+\theta_2\mu_1+\kappa_2)|\xi_1+\xi_2|.
\end{aligned} \tag{2.4}$$

Proof. For simplicity, we adopt the notation $\delta\mu = \mu_2-\mu_1$ and $\delta\kappa = \kappa_2-\kappa_1$. The maximum of $\langle\xi\,,\eta\rangle$ is attained when η and ξ are simultaneously diagonal (see, e.g., [31]), and $\langle(\sigma_2-\sigma_1)^{-1}\eta\,,\eta\rangle$ and $h(\eta)$ depend only on the eigenvalues of η. Therefore, maximizing the right-hand side of (2.1) over all tensors η is equivalent to maximizing over all real numbers η_1 and η_2 the concave function

$$F(\eta_1\,,\eta_2) = 2(\xi_1\eta_1+\xi_2\eta_2) - \frac{1}{2\delta\mu}(\eta_1^2+\eta_2^2) - \frac{1}{4}\left(\frac{1}{\delta\kappa}-\frac{1}{\delta\mu}\right)(\eta_1+\eta_2)^2 + \theta_2 h(\eta_1\,,\eta_2). \tag{2.5}$$

Here $h(\eta_1, \eta_2)$ is defined by

$$h(\eta_1, \eta_2) = \frac{1}{\mu_2 + \kappa_2} \begin{cases} \eta_1^2 & \text{if } |\eta_1| \le |\eta_2|, \\ \eta_2^2 & \text{if } |\eta_2| \le |\eta_1|. \end{cases} \tag{2.6}$$

We depart slightly from the approach used for Proposition 1.3, in that we do not assume that $\eta_1 \le \eta_2$. Clearly the function $h(\eta_1, \eta_2)$ is continuously differentiable everywhere except on the lines $\eta_1 = \eta_2$ and $\eta_1 = -\eta_2$.

(1) Assume $|\eta_1| < |\eta_2|$. Then the maximizer of F satisfies the Euler equation $\nabla F(\eta_1, \eta_2) = 0$, i.e.,

$$\begin{aligned} \frac{1}{2\delta\mu}\eta_1 + \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2) - \frac{\theta_2}{\mu_2 + \kappa_2}\eta_1 &= \xi_1, \\ \frac{1}{2\delta\mu}\eta_2 + \frac{1}{4}\left(\frac{1}{\delta\kappa} - \frac{1}{\delta\mu}\right)(\eta_1 + \eta_2) &= \xi_2, \end{aligned} \tag{2.7}$$

which is equivalent to

$$\begin{aligned} &\eta_1[(\mu_2 + \kappa_2)\delta\mu + (\mu_2 + \kappa_2)\delta\kappa - 4\theta_2\delta\mu\delta\kappa] \\ &\quad + \eta_2[(\mu_2 + \kappa_2)\delta\mu - (\mu_2 + \kappa_2)\delta\kappa] = 4(\mu_2 + \kappa_2)\delta\mu\delta\kappa\xi_1, \\ &\eta_1[\delta\mu - \delta\kappa] + \eta_2[\delta\mu + \delta\kappa] = 4\delta\mu\delta\kappa\xi_2. \end{aligned}$$

Let $\Delta = [(\mu_2 + \kappa_2)\delta\mu + (\mu_2 + \kappa_2)\delta\kappa - 4\theta_2\delta\mu\delta\kappa][\delta\mu + \delta\kappa] - (\mu_2 + \kappa_2)[\delta\mu - \delta\kappa]^2$. We compute that

$$\Delta = 4\delta\mu\delta\kappa[\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)].$$

Thus the solution of (2.7) is

$$\begin{aligned} \eta_1 &= (\mu_2 + \kappa_2)\frac{\delta\kappa(\xi_1 + \xi_2) + \delta\mu(\xi_1 - \xi_2)}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)}, \\ \eta_2 &= \frac{[(\mu_2 + \kappa_2)\delta\kappa - 2\theta_2\delta\mu\delta\kappa](\xi_1 + \xi_2) - [(\mu_2 + \kappa_2)\delta\mu - 2\theta_2\delta\mu\delta\kappa](\xi_1 - \xi_2)}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)}. \end{aligned} \tag{2.8}$$

The maximum of $F(\eta_1, \eta_2)$, therefore, is

$$\operatorname{Max} F(\eta_1, \eta_2) = \frac{\delta\kappa(\kappa_2 + \theta_1\mu_2 + \theta_2\mu_1)(\xi_1 + \xi_2)^2 + \delta\mu(\mu_2 + \theta_1\kappa_2 + \theta_2\kappa_1)(\xi_1 - \xi_2)^2 + 2\theta_2\delta\mu\delta\kappa(\xi_1^2 - \xi_2^2)}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)}. \tag{2.9}$$

The value of the bound in this case is

$$\begin{aligned} \langle\sigma^*\xi, \xi\rangle &\le \langle\sigma_2\xi, \xi\rangle - \theta_1 \operatorname{Max} F(\eta_1, \eta_2) \\ &\le \langle(\theta_1\sigma_1 + \theta_2\sigma_2)\xi, \xi\rangle - \frac{\theta_1\theta_2[\delta\kappa(\xi_1 + \xi_2) + \delta\mu(\xi_1 - \xi_2)]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)}. \end{aligned}$$

This bound is asserted only if the solution (2.8) satisfies $|\eta_1| < |\eta_2|$, in other words if

$$\begin{aligned} &(\mu_2 + \kappa_2)|\delta\kappa(\xi_1 + \xi_2) + \delta\mu(\xi_1 - \xi_2)| \\ &\quad < |[(\mu_2 + \kappa_2)\delta\kappa - 2\theta_2\delta\mu\delta\kappa](\xi_1 + \xi_2) - [(\mu_2 + \kappa_2)\delta\mu - 2\theta_2\delta\mu\delta\kappa](\xi_1 - \xi_2)|. \end{aligned} \tag{2.10}$$

Squaring both sides yields

$$\theta_2\delta\kappa[(\mu_2+\kappa_2)-\theta_2\delta\mu](\xi_1+\xi_2)^2+\theta_2\delta\mu[(\mu_2+\kappa_2)-\theta_2\delta\kappa](\xi_1-\xi_2)^2 \\ +[(\mu_2+\kappa_2)^2-\theta_2(\mu_2+\kappa_2)(\delta\kappa+\delta\mu)+2\theta_2^2\delta\mu\delta\kappa](\xi_1^2-\xi_2^2)<0,$$

which factors as

$$[\theta_2\delta\kappa(\xi_1+\xi_2)+(\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)(\xi_1-\xi_2)][\theta_2\delta\mu(\xi_1-\xi_2)+(\theta_1\mu_2+\theta_2\mu_1+\kappa_2)(\xi_1+\xi_2)]<0. \tag{2.11}$$

It is easy to check that

$$\theta_2^2\delta\kappa\delta\mu\le(\theta_1\mu_2+\theta_2\mu_1+\kappa_2)(\theta_1\kappa_2+\theta_2\kappa_1+\mu_2).$$

Thus (2.11) is equivalent to either

$$\begin{cases}\theta_2\delta\kappa(\xi_1+\xi_2)<-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)(\xi_1-\xi_2),\\ -\theta_2\delta\mu(\xi_1-\xi_2)<(\theta_1\mu_2+\theta_2\mu_1+\kappa_2)(\xi_1+\xi_2),\\ (\xi_1+\xi_2)\ge 0 \quad\text{and}\quad (\xi_1-\xi_2)\le 0\end{cases} \tag{2.12}$$

or

$$\begin{cases}\theta_2\delta\kappa(\xi_1+\xi_2)>-(\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)(\xi_1-\xi_2),\\ -\theta_2\delta\mu(\xi_1-\xi_2)>(\theta_1\mu_2+\theta_2\mu_1+\kappa_2)(\xi_1+\xi_2),\\ (\xi_1+\xi_2)\le 0 \quad\text{and}\quad (\xi_1-\xi_2)\ge 0.\end{cases} \tag{2.13}$$

(2) Assume $|\eta_2|<|\eta_1|$. This case is symmetric to the first one. It suffices to interchange the roles of ξ_1 and ξ_2.

Regrouping the "compatibility" conditions (2.12), (2.13), and their symmetric counterparts, we see that the value of the bound is

$$\langle\sigma^*\xi,\xi\rangle\ge\langle(\theta_1\sigma_1+\theta_2\sigma_2)\xi,\xi\rangle-\frac{\theta_1\theta_2[\delta\kappa|\xi_1+\xi_2|-\delta\mu|\xi_1-\xi_2|]^2}{\theta_1(\mu_2+\kappa_2)+\theta_2(\mu_1+\kappa_1)} \tag{2.14}$$

when ξ satisfies

$$\begin{aligned}\theta_2\delta\kappa|\xi_1+\xi_2|&<(\theta_1\kappa_2+\theta_2\kappa_1+\mu_2)|\xi_1-\xi_2|,\\ \theta_2\delta\mu|\xi_1-\xi_2|&<(\theta_1\mu_2+\theta_2\mu_1+\kappa_2)|\xi_1+\xi_2|.\end{aligned} \tag{2.15}$$

(3) Assume that condition (2.15) is not satisfied. Then the maximum of $F(\eta_1,\eta_2)$ is reached on one of the lines $\eta_1=\eta_2$ and $\eta_1=-\eta_2$. The maximum of $F(\eta,\eta)$ is reached at

$$\eta=\frac{(\mu_2+\kappa_2)\delta\kappa(\xi_1+\xi_2)}{\theta_1\kappa_2+\theta_2\kappa_1+\mu_2}, \tag{2.16}$$

and the corresponding value of the bound is

$$\langle\sigma^*\xi,\xi\rangle\le\mu_2(\xi_1-\xi_2)^2+\frac{\kappa_1\kappa_2+\mu_2(\theta_1\kappa_1+\theta_2\kappa_2)}{\mu_2+\theta_1\kappa_2+\theta_2\kappa_1}(\xi_1+\xi_2)^2. \tag{2.17}$$

The maximum of $F(\eta,-\eta)$ is reached at

$$\eta=\frac{(\mu_2+\kappa_2)\delta\mu(\xi_1-\xi_2)}{\theta_1\mu_2+\theta_2\mu_1+\kappa_2}, \tag{2.18}$$

and the corresponding value of the bound is

$$\langle \sigma^* \xi, \xi \rangle \le \kappa_2 (\xi_1 + \xi_2)^2 + \frac{\mu_1\mu_2 + \kappa_2(\theta_1\mu_1 + \theta_2\mu_2)}{\kappa_2 + \theta_1\mu_2 + \theta_2\mu_1}(\xi_1 - \xi_2)^2 . \tag{2.19}$$

It is easily seen that the bound (2.17) is better than (2.19) when $\theta_2 \delta\kappa |\xi_1 + \xi_2| \ge (\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2)|\xi_1 - \xi_2|$. Conversely, (2.19) is better than (2.17) when $\theta_2\delta\mu|\xi_1 - \xi_2| \ge (\theta_1\mu_2 + \theta_2\mu_1 + \kappa_2)|\xi_1 + \xi_2|$. Together with (2.14)–(2.15) this yields the desired result. □

It is interesting to compare the optimal upper bound with the more standard arithmetic-mean bound. One verifies that (2.4) is strictly below the arithmetic mean $\langle (\theta_1\sigma_1 + \theta_2\sigma_2)\xi, \xi\rangle$ unless $|\kappa_2 - \kappa_1|\,|\xi_1 + \xi_2| = |\mu_1 - \mu_2|\,|\xi_1 - \xi_2|$. Thus the arithmetic mean bound is optimal only for very special ξ—a set of codimension one in the space of symmetric tensors.

We turn now to a discussion of the optimal microstructures. The optimality condition for (2.1) is

$$-2\xi + 2(\sigma_2 - \sigma_1)^{-1}\eta \in \theta_2 \partial h(\eta). \tag{2.20}$$

According to (2.2),

$$h(\eta) = \inf_{|\nu|=1} \langle f(\nu)\eta, \eta\rangle ,$$

where f is the "degenerate Hooke's law"

$$f(\nu)\eta = \frac{1}{\mu_2}[(\eta\nu)\odot\nu - \langle \eta\nu, \nu\rangle \nu\odot\nu] + \frac{1}{\mu_2 + \kappa_2}\langle \eta\nu, \nu\rangle \nu\odot\nu . \tag{2.21}$$

The generalized gradient $\partial h(\eta)$ is the convex hull of the tensors $2f(\nu)\eta$ as ν ranges over extremals for (2.2). Hence (2.20) can be rewritten as

$$-\xi + (\sigma_2 - \sigma_1)^{-1}\eta = \theta_2 \sum_{i=1}^{p} m_i f(\nu_i)\eta , \tag{2.22}$$

in which $m_i \ge 0$, $\sum m_i = 1$, and each ν_i is extremal for (2.2). If h is differentiable at the optimal η then (2.22) becomes

$$-\xi + (\sigma_2 - \sigma_1)^{-1}\eta = \theta_2 f(\nu)\eta$$

where ν is any extremal for (2.2), and the bound is achieved by a rank-one laminate with layer direction ν. If h is not differentiable at the optimal η then $p > 1$ in (2.22), and the bound is achieved by a rank-p laminate.

To proceed, we need to know which vectors ν are extremal for (2.2), as a function of η. This information was obtained in the course of calculating $h(\eta)$, in Proposition 7.3 of [2]. The answer is as follows. Let e_1, e_2 be the eigenvectors of η, with associated eigenvalues η_1, η_2, ordered so that $|\eta_1| \le |\eta_2|$. Then:

(i) for $|\eta_1| < |\eta_2|$ the extremal ν are $\pm e_1$;
(ii) for $\eta_1 = -\eta_2$ the extremal ν are $\pm e_1$, $\pm e_2$;
(iii) for $\eta_1 = \eta_2$ any ν is extremal.

The first regime of (2.4) is easy. In this case h is differentiable at the optimal η. The optimal microstructure is a rank-one laminate, using layers orthogonal to the eigenvector associated to the eigenvalue of smaller absolute value.

The third regime of (2.4) corresponds to case (ii) above. From (2.21) we have

$$f(\nu) = \frac{\gamma}{\mu_2 + \kappa_2} \langle \eta\nu , \nu \rangle \nu \odot \nu$$

when $\eta = \gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\nu \in \{\pm e_1 , \pm e_2\}$. From (2.18), the relevant value of γ is

$$\gamma = \frac{(\mu_2 + \kappa_2)\delta\mu(\xi_1 - \xi_2)}{\theta_1\mu_2 + \theta_2\mu_1 + \kappa_2} . \tag{2.23}$$

It we take $\nu_1 = e_1$, $\nu_2 = e_2$, then the optimality condition (2.22) becomes

$$\begin{pmatrix} -\xi_1 & 0 \\ 0 & -\xi_2 \end{pmatrix} + \frac{\gamma}{2\delta\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\theta_2\gamma}{\mu_2 + \kappa_2} \begin{pmatrix} m_1 & 0 \\ 0 & -m_2 \end{pmatrix} .$$

This determines m_1 and m_2:

$$m_1 = \frac{\mu_2 + \kappa_2}{\theta_2} \left(\frac{-\xi_1}{\gamma} + \frac{1}{2\delta\mu} \right) ,$$
$$m_2 = \frac{\mu_2 + \kappa_2}{\theta_2} \left(\frac{\xi_2}{\gamma} + \frac{1}{2\delta\mu} \right) . \tag{2.24}$$

One verifies that $m_1 + m_2 = 1$ as a consequence of (2.23), and $m_1 \geq 0$, $m_2 \geq 0$ as a consequence of the inequality which defines this regime,

$$\theta_2\delta\mu|\xi_1 - \xi_2| \geq (\theta_1\mu_2 + \theta_2\mu_1 + \kappa_2)|\xi_1 + \xi_2| .$$

According to the construction in [2], the bound is achieved by a second-rank laminate. Specifically, we first layer σ_1 with σ_2 in volume fractions $\rho = 1 - \theta_2 m_1$ and $1 - \rho$ respectively, using layers orthogonal to e_1, to get a composite C. Then we layer C with σ_2 in volume fractions $\rho' = \theta_1/(1 - \theta_2 m_1)$ and $1 - \rho'$ respectively, using layers orthogonal to e_2. The resulting composite σ^* achieves equality in the bound.

The analysis of the middle regime of (2.4) is similar. It corresponds to case (iii) above. From (2.16) we have

$$\eta = \gamma I , \qquad \gamma = \frac{(\mu_2 + \kappa_2)\delta\kappa(\xi_1 + \xi_2)}{\theta_1\kappa_2 + \theta_2\kappa_1 + \mu_2} . \tag{2.25}$$

From (2.21) we have

$$f(\nu) = \frac{\gamma}{\mu_2 + \kappa_2} \nu \odot \nu$$

for any unit vector ν. If we once again choose $\nu_1 = e_1$, $\nu_2 = e_2$, then the optimality condition (2.22) becomes

$$\begin{pmatrix} -\xi_1 & 0 \\ 0 & -\xi_2 \end{pmatrix} + \frac{\gamma}{2\delta\kappa} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\theta_2\gamma}{\mu_2 + \kappa_2} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} .$$

This determines m_1 and m_2:

$$m_1 = \frac{\mu_2 + \kappa_2}{\theta_2} \left(\frac{-\xi_1}{\gamma} + \frac{1}{2\delta\kappa} \right) ,$$
$$m_2 = \frac{\mu_2 + \kappa_2}{\theta_2} \left(\frac{-\xi_2}{\gamma} + \frac{1}{2\delta\kappa} \right) . \tag{2.26}$$

One verifies that $m_1 + m_2 = 1$ as a consequence of (2.25), and $m_1 \geq 0$, $m_2 \geq 0$ as a consequence of the inequality that defines this regime,

$$\theta_2 \delta\kappa |\xi_1 + \xi_2| \geq (\theta_1 \kappa_2 + \theta_2 \kappa_1 + \mu_2)|\xi_1 - \xi_2| .$$

The construction of the optimal second-rank laminate is the same as before, except for using (2.26) in place of (2.24) to determine the values of m_1 and m_2.

We turn now to the translation method. Our goal is to recover the optimal bound (2.4) by the translation method, using only multiples of the determinant as translations. As before, this approach actually gives the optimal upper bound even in the non-well-ordered setting.

Our starting point is the analogue of Proposition 1.4 for complementary energy.

PROPOSITION 2.4. Let τ be a constant fourth-order tensor. Assume that τ is quasi-convex on stresses, i.e., for any Q-periodic, divergence-free, second-order tensor ψ with mean value equal to zero

$$\int_Q \langle \tau\psi , \psi \rangle \geq 0 . \tag{2.27}$$

Assume further that $\sigma^{-1}(y) - \tau$ is positive on Q, i.e.,

$$\sigma_1^{-1} - \tau \geq 0 \quad \text{and} \quad \sigma_2^{-1} - \tau \geq 0 . \tag{2.28}$$

Then we have

$$\sigma^* \leq \left(\left(\int_Q (\sigma^{-1} - \tau)^{-1} \right)^{-1} + \tau \right)^{-1} . \tag{2.29}$$

The proof of Proposition 2.4 is very similar to that of Proposition 1.4; it can be found, for example, in [2, 29]. We shall apply this proposition with τ a multiple of the translation det (defined in (1.32)).

LEMMA 2.5. Let σ_1 and σ_2 be two materials such that

$$\mu_1 \leq \mu_2 . \tag{2.30}$$

Whatever the ordering of κ_1 and κ_2, we set

$$\kappa_+ = \sup(\kappa_1 , \kappa_2) . \tag{2.31}$$

Then the assumptions of Proposition 2.4 are satisfied for all translations $\tau = \lambda \cdot \det$ with $\lambda \in [-\mu_2^{-1} ; \kappa_+^{-1}]$.

Proof. Using Fourier analysis, it is easy to see that the definition (2.27) of quasi-convexity on stresses is equivalent to

$$\langle \tau M , M \rangle \geq 0 \quad \text{for any symmetric matrix } M \text{ such that } \det(M) = 0 .$$

Thus for any $\lambda \in R$ the translation $\lambda \det$ is quasi-convex on stresses. On the other hand we have

$$\sigma_i^{-1} - \lambda \det = \frac{1}{2} \left(\frac{1}{\mu_i} + \lambda \right) \Lambda_s + \frac{1}{2} \left(\frac{1}{\kappa_i} - \lambda \right) \Lambda_h .$$

Because Λ_s and Λ_h are nonnegative operators (see their definitions (1.33)), $(\sigma_i^{-1} - \lambda \det)$ is positive for $i = 1, 2$ as long as $\lambda \in [-\mu_2^{-1} ; \kappa_+^{-1}]$. □

PROPOSITION 2.6. Denoting by ξ_1 and ξ_2 the eigenvalues of ξ, the translation method yields the bound

$$
\begin{aligned}
\langle \sigma^* \xi , \xi \rangle &\leq \langle (\theta_1 \sigma_1 + \theta_2 \sigma_2) \xi , \xi \rangle - \frac{\theta_1 \theta_2 [|\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2| - |\mu_2 - \mu_1| \, |\xi_1 - \xi_2|]^2}{\theta_1 (\mu_2 + \kappa_2) + \theta_2 (\mu_1 + \kappa_1)} \\
&\qquad \text{if } \theta_2 |\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2| \leq (\theta_1 \kappa_2 + \theta_2 \kappa_1 + \mu_2) |\xi_1 - \xi_2| \\
&\qquad \text{and } \theta_+ |\mu_2 - \mu_1| \, |\xi_1 - \xi_2| \leq (\theta_1 \mu_2 + \theta_2 \mu_1 + \kappa_+) |\xi_1 + \xi_2| ; \\
\langle \sigma^* \xi , \xi \rangle &\leq \mu_2 (\xi_1 - \xi_2)^2 + \frac{\kappa_1 \kappa_2 + \mu_2 (\theta_1 \kappa_1 + \theta_2 \kappa_2)}{\mu_2 + \theta_1 \kappa_2 + \theta_2 \kappa_1} (\xi_1 + \xi_2)^2 \\
&\qquad \text{if } \theta_2 |\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2| \geq (\theta_1 \kappa_2 + \theta_2 \kappa_1 + \mu_2) |\xi_1 - \xi_2| ; \\
\langle \sigma^* \xi , \xi \rangle &\leq \kappa_+ (\xi_1 + \xi_2)^2 + \frac{\mu_1 \mu_2 + \kappa_+ (\theta_1 \mu_1 + \theta_2 \mu_2)}{\kappa_+ + \theta_1 \mu_2 + \theta_2 \mu_1} (\xi_1 - \xi_2)^2 \\
&\qquad \text{if } \theta_+ |\mu_2 - \mu_1| \, |\xi_1 - \xi_2| \geq (\theta_1 \mu_2 + \theta_2 \mu_1 + \kappa_-) |\xi_1 + \xi_2| ,
\end{aligned}
\tag{2.32}
$$

where $\kappa_+ = \sup(\kappa_1 , \kappa_2)$ and θ_+ is the volume fraction of the material corresponding to κ_+.

Proof. Replacing τ by $\lambda \det$ in (2.29) leads after some elementary algebra to

$$
\sigma^* \leq \frac{2[\lambda^{-1}(\theta_1 \mu_1 + \theta_2 \mu_2) + \mu_1 \mu_2]}{\lambda^{-1} + (\theta_1 \mu_2 + \theta_2 \mu_1)} \Lambda_s + \frac{2[\lambda^{-1}(\theta_1 \kappa_1 + \theta_2 \kappa_2) - \kappa_1 \kappa_2]}{\lambda^{-1} - (\theta_1 \kappa_2 + \theta_2 \kappa_1)} \Lambda_h . \tag{2.33}
$$

Specializing (2.33) at energy ξ and recalling that in two dimensions $2\langle \Lambda_h \xi , \xi \rangle = (\xi_1 + \xi_2)^2$ and $2\langle \Lambda_s \xi , \xi \rangle = (\xi_1 - \xi_2)^2$, we obtain

$$
\langle \sigma^* \xi , \xi \rangle \leq \langle (\theta_1 \sigma_1 + \theta_2 \sigma_2) \xi , \xi \rangle - \theta_1 \theta_2 g(\lambda) \tag{2.34}
$$

with

$$
g(\lambda) = \frac{(\mu_2 - \mu_1)^2 (\xi_1 - \xi_2)^2}{\theta_1 \mu_2 + \theta_2 \mu_1 + \lambda^{-1}} + \frac{(\kappa_2 - \kappa_1)^2 (\xi_1 + \xi_2)^2}{\theta_1 \kappa_2 + \theta_2 \kappa_1 - \lambda^{-1}} . \tag{2.35}
$$

Notice that this function $g(\lambda)$ is very similar to the function $f(\lambda)$ introduced in Proposition 1.6 (see (1.38)). Differentiating $g(\lambda)$, we easily find two possible roots of $g'(\lambda)$:

$$
\begin{aligned}
\lambda^+ &= \frac{|\mu_2 - \mu_1| \, |\xi_1 - \xi_2| + |\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2|}{(\theta_1 \kappa_2 + \theta_2 \kappa_1) |\mu_2 - \mu_1| \, |\xi_1 - \xi_2| - (\theta_1 \mu_2 + \theta_2 \mu_1) |\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2|} , \\
\lambda^- &= \frac{|\mu_2 - \mu_1| \, |\xi_1 - \xi_2| - |\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2|}{(\theta_1 \kappa_2 + \theta_2 \kappa_1) |\mu_2 - \mu_1| \, |\xi_1 - \xi_2| + (\theta_1 \mu_2 + \theta_2 \mu_1) |\kappa_2 - \kappa_1| \, |\xi_1 + \xi_2|} .
\end{aligned}
$$

The first root λ^+ is always outside the interval $[-\mu_2^{-1} ; \kappa_+^{-1}]$, and g is convex on $[-\mu_2^{-1} ; \kappa_+^{-1}]$. So there are three different cases:

(1) if $\lambda^- < -\mu_2^{-1}$, then the minimum of $g(\lambda)$ is attained for $\lambda = -\mu_2^{-1}$;
(2) if $-\mu_2^{-1} \leq \lambda^- \leq \kappa_+^{-1}$, then the minimum of $g(\lambda)$ is attained for $\lambda = \lambda^-$;
(3) if $\lambda^- \geq \kappa_+^{-1}$, then the minimum of $g(\lambda)$ is attained for $\lambda = \kappa_+^{-1}$.

In the second case we easily compute the value of $g(\lambda^-)$:

$$g(\lambda^-) = \frac{[|\mu_2 - \mu_1||\xi_1 - \xi_2| - |\kappa_2 - \kappa_1||\xi_1 + \xi_2|]^2}{\theta_1(\mu_2 + \kappa_2) + \theta_2(\mu_1 + \kappa_1)}.$$

A little algebra leads directly to the desired result (2.32). □

In the well-ordered case the translation bound (2.32) and the Hashin-Shtrikman bound (2.4) are clearly identical. In the non-well-ordered case, Proposition 2.6 yields a new bound which is algebraically very similar to the Hashin-Shtrikman bound. Let us check that this new bound is optimal.

PROPOSITION 2.7. The translation bound (2.32) is also optimal in the non-well-ordered case. Specifically, there exists a sequentially laminated composite which achieves equality in the bound.

Proof. The argument is similar to that of Proposition 1.7. The proof that the Hashin-Shtrikman bound is optimal actually shows the following: *if ξ and η are related by the "optimality condition"* (2.20), *then there exists a sequentially laminated composite whose effective tensors σ^* satisfies*

$$\langle\sigma^*\xi, \xi\rangle = \langle\sigma_2\xi, \xi\rangle - \theta_1[2\langle\xi, \eta\rangle - \langle(\sigma_2 - \sigma_1)^{-1}\eta, \eta\rangle + \theta_2 h(\eta)].$$

This applies even in the non-well-ordered case. In other words, any critical value with respect to η of

$$\langle\sigma_2\xi, \xi\rangle - \theta_1[2\langle\xi, \eta\rangle - \langle(\sigma_2 - \sigma_1)^{-1}\eta, \eta\rangle + \theta_2 h(\eta)] \tag{2.36}$$

is equal to $\langle\sigma^*\xi, \xi\rangle$ for some sequentially laminated mixture of σ_1 and σ_2 in volume fractions θ_1 and θ_2, respectively. The associated microstructure consists of platelike inclusions of material 1 in a matrix of material 2.

Consider now the translation bound (2.32). We are in the non-well-ordered case, so $\kappa_+ = \kappa_1$. By inspection, the first two regimes are identical to the corresponding regimes of (2.4) except for replacing $\kappa_2 - \kappa_1$ with $|\kappa_2 - \kappa_1|$. The proof of Proposition 2.4 shows that these are critical values of (2.36). So the first two regimes of (2.32) are achieved, even in the non-well-ordered case, by a suitable microstructure of platelike inclusions of material 1 in a matrix of material 2.

The third regime is different: it corresponds to the third regime of (2.4) with the roles of σ_1 and σ_2 *reversed*. So it is a critical value of the analogue of (2.36) with σ_1 and σ_2 reversed. Hence this regime is optimal as well, but its microstructure consists of platelike inclusions of material 2 in a matrix of material 1. □

REMARK 2.8. We emphasize that there is a major difference between the non-well-ordered lower bound (1.37) and the non-well-ordered upper bound (2.32). The former is always achieved by a matrix-inclusion microstructure, in which the material with smaller shear modulus is the matrix, regardless of the value of ξ. The latter is also achieved by a matrix-inclusion microstructure. For the upper bound, however, the matrix must be the material with the larger shear modulus when $\theta_1|\mu_2 - \mu_1||\xi_1 - \xi_2| \le (\theta_1\mu_2 + \theta_2\mu_1 + \kappa_1)|\xi_1 + \xi_2|$, while it must be the material with the smaller shear modulus when this inequality is reversed. Thus for the upper bound in the non-well-ordered case the choice of "reference material" must depend on ξ.

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