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OPTIMAL LOWER BOUNDS ON THE ELASTIC ENERGY OF A COMPOSITE MADE FROM TWO NON-WELL-ORDERED ISOTROPIC MATERIALS

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Abstract. This paper is a continuation of our previous work [AK] concerning optimal bounds on the effective behavior of a mixture of two linearly elastic materials. While in [AK] we restricted our attention to the case of two well-ordered components, here we focus on the case of two *non-well-ordered and isotropic ones*, i.e., the case when the smaller shear and bulk moduli do not belong to the same material. For given volume fractions and average strain, we establish an *optimal lower bound* on the effective energy quadratic form. We give two proofs of this result: one based on the Hashin-Shtrikman-Walpole variational principle, the other on the translation method.

0. Introduction. This paper is a continuation of our previous work [AK], which was concerned with optimal bounds on the effective behavior of a mixture of two *well-ordered* elastic materials. Here, we establish an optimal lower bound on the elastic energy of a mixture of two *non-well-ordered, isotropic* elastic materials.

The macroscopic properties of a linearly elastic composite material are described by its tensor of effective moduli (Hooke's law) σ^*. This fourth-order tensor depends on the microgeometry of the mixture as well as on the elastic properties of the components.

Suppose that σ^* arises by mixing two materials σ_1 and σ_2 with prescribed volume fractions θ_1 and θ_2 respectively, but with an unknown microstructure. For any given strain ξ (a symmetric, second-order tensor), an *optimal* lower (resp. upper) bound is a function $f_- = f_-(\sigma_1, \sigma_2, \theta_1, \theta_2, \xi)$ (resp. f_+) such that

$$f_- \leq \langle \sigma^* \xi, \xi \rangle \leq f_+, \tag{0.1}$$

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and such that each inequality can be saturated (for any ξ) by a suitable microstructure (which depends on ξ).

When the component materials are *well ordered*, i.e., they satisfy

$$\langle \sigma_1 \eta , \eta \rangle \leq \langle \sigma_2 \eta , \eta \rangle \tag{0.2}$$

for every symmetric, second-order tensor η, such optimal bounds are known (see [Av, KL, AK]). They are sometimes called Hashin-Shtrikman bounds since they derive from the well-known Hashin-Shtrikman variational principle [HS]. They can be viewed as extensions of the well-known Hashin-Shtrikman bounds on the effective bulk modulus of an isotropic composite [HS]. When σ_1 and σ_2 are isotropic, condition (0.2) requires that the smaller bulk and shear moduli belong to the same material (σ_1).

The goal of this paper is to extend the optimal lower bound in (0.1) to the case of non-well-ordered, isotropic component materials. We achieve this goal for any pair of isotropic component materials σ_1 and σ_2, provided only that the bulk moduli κ_1, κ_2 and shear moduli μ_1, μ_2 are positive and satisfy, assuming with no loss of generality that $\mu_1 \leq \mu_2$, the condition $\kappa_2 - \frac{2}{N}\mu_1 \geq 0$, where N is the spatial dimension.

To state our result more precisely, we first recall the form of the lower bound in the well-ordered case $\sigma_1 \leq \sigma_2$ (see Proposition 2.1 in [AK]). It asserts that

$$\langle \sigma^* \xi , \xi \rangle \geq \langle \sigma_1 \xi , \xi \rangle + \theta_2 \sup_\eta \left[2\langle \xi , \eta \rangle - \langle (\sigma_2 - \sigma_1)^{-1} \eta , \eta \rangle - \theta_1 g(\eta) \right] , \tag{0.3}$$

with $g(\eta)$ an explicit, convex function of second-order tensors η. (See Remark 1.2 for the definition of $g(\eta)$.) This bound is optimal, and indeed an extremal sequentially laminated microstructure can be read off from the optimality condition for the maximization in η (see Theorem 3.5 in [AK]). Since $\sigma_1 < \sigma_2$, the function

$$F(\eta) = 2\langle \xi , \eta \rangle - \langle (\sigma_2 - \sigma_1)^{-1} \eta , \eta \rangle - \theta_1 g(\eta) \tag{0.4}$$

is strictly concave in η for given ξ; so it has a unique critical point $\eta^*(\xi)$ and critical value $F(\eta^*(\xi))$. We may therefore write (0.3) in the form

$$\langle \sigma^* \xi , \xi \rangle \geq \langle \sigma_1 \xi , \xi \rangle + \theta_2 \operatorname{crit}_\eta F(\eta) , \tag{0.5}$$

where $\operatorname{crit}_\eta F(\eta)$ represents the unique critical value of the function $\eta \to F(\eta)$. The main result of this paper is that (0.5) *gives the optimal lower bound even when σ_1 and σ_2 are not well ordered*. More precisely, we shall prove

THEOREM 0.1. Let σ_1 and σ_2 be two isotropic, non-well-ordered, elastic materials in $N \geq 2$ space dimensions, with bulk moduli κ_1, κ_2 and shear moduli μ_1, μ_2 respectively. Label them so that $\mu_1 < \mu_2$ and $\kappa_1 > \kappa_2$, and assume that $\kappa_2 - \frac{2}{N}\mu_1 \geq 0$. Then F , defined by (0.4), has a unique critical point $\eta^*(\xi)$, and (0.5) is a valid bound. This bound is moreover optimal, in the sense that for every ξ there exists a microstructure achieving equality in (0.5).

(It is worth noticing that, in 2D, Theorem 0.1 holds even without the assumption $\kappa_2 - \mu_1 \geq 0$; see [AK2].)

We give two different proofs of this theorem. The first one is based on the Hashin-Shtrikman-Walpole variational principle [Wa]; it is a generalization of the usual Hashin-Shtrikman variational principle involving a "mixed" reference material $\overline{\sigma}$ built from the weaker bulk and shear moduli κ_2 and μ_1. The second one (which requires slightly stronger assumptions) is based on the new "translation method", also known as the "compensated compactness method" (introduced in [Ta1, Ta2, FM, LC], and also presented in [Mi, AK]). Both proofs involve a few tedious computations, which "miraculously" succeed: in this respect we are not entirely happy with our analysis, since we know of no abstract reason that such complicated computations should yield a simple result. As a consequence, we do not know an analogue of Theorem 0.1 for the upper bound on $\langle\sigma^*\xi, \xi\rangle$ (see Sec. 3).

The attentive reader will see that our second proof of Theorem 0.1, making use of the translation method (Sec. 2), is more complicated than the first "variational" one (Sec. 1). Furthermore, it yields slightly weaker results (compare Theorem 0.1 vs. Theorem 2.5). It is natural to ask whether this second proof is worth the trouble. We believe the answer is yes. Indeed, the main idea behind that proof, which is to extract "extremal" translations from "old" or "classical" ones, may well have other potential applications. Let us briefly summarize this idea. In the well-ordered case, the optimal bound (0.3), usually obtained from the Hashin-Shtrikman variational principle [Av, Ko, AK], can also be proved by the translation method [Mi, AK], using the particular translation

$$\tau_\eta(\xi) = \langle\sigma_1\xi, \xi\rangle - g(\eta)^{-1}\langle\eta, \xi\rangle^2. \tag{0.6}$$

This translation may be decomposed as

$$\tau_\eta = \tilde{\tau}_\eta + \phi_\eta, \tag{0.7}$$

where ϕ_η is an explicit, convex quadratic form, and $\tilde{\tau}_\eta$ is an "extremal" translation (see Proposition 2.4 and Remark 2.9). Now suppose that σ_1 and σ_2 are not well ordered. Then, in general, the translation τ_η is not "admissible" (in the sense of Proposition 2.1). Nevertheless, the "extremal" translation $\tilde{\tau}_\eta$ turns out to be admissible and thus furnishes a new translation bound. Furthermore, considering a *critical point* $\eta^* = \eta^*(\xi)$ of (0.4) (which exists by virtue of Lemma 2.7), the translation bound obtained with $\tilde{\tau}_{\eta^*}$ coincides with (0.5).

As should be clear from the preceding summary, our analysis depends strongly on a proper understanding of the well-ordered case. Thus we make frequent references to our previous work [AK]. In Sec. 3 we also discuss several related issues, including so-called "trace" bounds in the non-well-ordered case (for a definition of trace bounds, see [MK, Mi]).

The remainder of this introduction is devoted to recalling basic notation as established in our preceding work [AK]. We shall consider only isotropic component materials. Their Hooke's laws have the form

$$\sigma_i = 2\mu_i\Lambda_s + N\kappa_i\Lambda_h, \tag{0.8}$$

where Λ_s and Λ_h are linear maps on symmetric second-order tensors that project respectively on "shear" (i.e., trace-free) tensors, and on "hydrostatic" (i.e., multiple of the identity) tensors. Namely, for any symmetric second-order tensor ξ, we have

$$\Lambda_s\xi = \xi - \frac{1}{N}(\operatorname{tr}\xi)I_2, \qquad \Lambda_h\xi = \frac{1}{N}(\operatorname{tr}\xi)I_2, \tag{0.9}$$

where I_2 is the identity second-order tensor. The constants μ_i and κ_i, both positive, are the shear and bulk moduli of the ith component, and N is the space dimension $(N \geq 2)$. We also consider only non-well-ordered materials. Thus, without loss of generality, we may and shall assume that

$$\mu_1 < \mu_2, \quad \kappa_1 > \kappa_2. \tag{0.10}$$

The cases $\kappa_1 = \kappa_2$ and $\mu_1 = \mu_2$ are excluded since they belong to the well-ordered case studied in [AK].

For simplicity, we consider spatially periodic composites, thus making use of the spatially periodic homogenization theory (see [BLP, Sp]). We emphasize that this point of view is sufficient for proving bounds that remain valid also for other types of composites (see [GP] and [DK] for a rigorous proof of this point).

Let $Q = (0;\, 1)^N$ be the unit cube, and let ε be the length scale of the microstructure (it will tend to zero presently). The microstructure is determined by Q-periodic functions $\chi_1(y)$ and $\chi_2(y)$, with

$$\chi_1(y) = 0 \text{ or } 1 \text{ almost everywhere,} \quad \text{and} \quad \chi_2(y) = 1 - \chi_1(y).$$

The local varying Hooke's law is

$$\sigma_\varepsilon(x) = \chi_1\left(\frac{x}{\varepsilon}\right)\sigma_1 + \chi_2\left(\frac{x}{\varepsilon}\right)\sigma_2.$$

The volume fraction of material σ_i is thus

$$\theta_i = \int_Q \chi_i(y)\,dy \quad \text{for } i = 1, 2.$$

As a consequence of homogenization theory (see e.g. [BLP, Sp, Ta3, ZK]), the effective Hooke's law σ^* of the composite, which describes its macroscopic behavior as the length scale ε tends to zero, is characterized by

$$\langle\sigma^*\xi, \xi\rangle = \inf \int_Q \langle\sigma(y)[\xi + e(\phi)], [\xi + e(\phi)]\rangle\,dy, \tag{0.11}$$

where the infimum is taken over all Q-periodic "elastic displacements" ϕ, $e(\phi)$ is the strain $\frac{1}{2}(\nabla\phi + {}^t\nabla\phi)$, and the "microscopic" Hooke's law is

$$\sigma(y) = \chi_1(y)\sigma_1 + \chi_2(y)\sigma_2.$$

Our goal is to establish an optimal lower bound on $\langle\sigma^*\xi, \xi\rangle$, with ξ, θ_1, θ_2, μ_1, μ_2, κ_1, and κ_2 held fixed, as the microstructure varies.

1. The Hashin-Shtrikman-Walpole variational principle. In this section we establish an optimal lower bound on the effective elastic energy $\langle\sigma^*\xi, \xi\rangle$. The two material components σ_1 and σ_2 are assumed to be *isotropic and not well ordered*; namely, their bulk and shear moduli satisfy (0.10). In a first (and easy) step, we use the Hashin-Shtrikman variational principle, as generalized by Walpole [Wa] for non-well-ordered materials, in order to obtain a lower bound. In a second (and more difficult) step, we prove that this bound is actually optimal, and that it coincides algebraically with the optimal lower bound obtained in the well-ordered case.

For any isotropic Hooke's law σ, with bulk and shear moduli κ, μ, and for any vector k, we define a degenerate Hooke's law $f_\sigma(k)$ by

$$f_\sigma(k)\eta = \frac{1}{\mu}[(\eta k)\otimes k - \langle \eta k, k\rangle k\otimes k] + \frac{1}{2\mu+\lambda}\langle \eta k, k\rangle k\otimes k, \tag{1.1}$$

where λ is the Lamé modulus of σ defined by

$$\lambda = \kappa - \frac{2}{N}\mu. \tag{1.2}$$

The notation $\otimes$ stands for the symmetrized tensorial product; i.e., $(k\otimes k')_{i,j} = \frac{1}{2}(k_i k'_j + k_j k'_i)$.

Let $\overline{\sigma}$ denote a "mixed" reference material, which is isotropic and made of the weakest moduli of σ_1 and σ_2, i.e.,

$$\overline{\sigma} = 2\mu_1\Lambda_s + N\kappa_2\Lambda_h. \tag{1.3}$$

Its Lamé modulus is also denoted $\overline{\lambda} = \kappa_2 - \frac{2}{N}\mu_1$.

THEOREM 1.1. When the two materials are not well ordered, the effective energy given by (0.11) satisfies

$$\langle \sigma^*\xi, \xi\rangle \geq \langle \overline{\sigma}\xi, \xi\rangle + \sup_\eta \Big[\theta_1[-2\langle \xi, \Lambda_h\eta\rangle - \langle (\sigma_1-\sigma_2)^{-1}\Lambda_h\eta, \Lambda_h\eta\rangle] \tag{1.4}$$
$$+\theta_2[2\langle \xi, \Lambda_s\eta\rangle - \langle (\sigma_2-\sigma_1)^{-1}\Lambda_s\eta, \Lambda_s\eta\rangle] - \theta_1\theta_2\overline{g}(\eta)\Big],$$

where the supremum is taken over all symmetric constant second-order tensors η, and $\overline{g}(\eta)$ is the homogeneous of degree two convex function given by

$$\overline{g}(\eta) = \sup_{|k|=1}\langle f_{\overline{\sigma}}(k)\eta, \eta\rangle. \tag{1.5}$$

REMARK 1.2. Since the proof of (1.4) is based on the Hashin-Shtrikman variational principle as revisited by Walpole, we call it the Hashin-Shtrikman-Walpole bound. Clearly (1.4) has some similarities with the optimal lower bound obtained in the well-ordered case (the so-called Hashin-Shtrikman bound), which reads

$$\langle \sigma^*\xi, \xi\rangle \geq \langle \sigma_1\xi, \xi\rangle + \theta_2\sup_\eta[2\langle \xi, \eta\rangle - \langle (\sigma_2-\sigma_1)^{-1}\eta, \eta\rangle - \theta_1 g(\eta)], \tag{1.6}$$

where $g(\eta)$ is defined by

$$g(\eta) = \sup_{|k|=1}\langle f_{\sigma_1}(k)\eta, \eta\rangle. \tag{1.7}$$

(We emphasize that (1.6) is not valid when the materials are not well ordered.)

Proof of Theorem 1.1. We start as in the Hashin-Shtrikman variational principle (see e.g. Proposition 2.1 in [AK]), but, following an idea of Walpole [Wa], we subtract from the effective energy defined by (0.11) a "mixed" reference energy:

$$\langle \sigma^*\xi, \xi\rangle = \inf_\phi\left[\int_Q \langle (\sigma(y)-\overline{\sigma})[\xi+e(\phi)], [\xi+e(\phi)]\rangle + \int_Q \langle \overline{\sigma}[\xi+e(\phi)], [\xi+e(\phi)]\rangle\right]. \tag{1.8}$$

Using the positivity of $\sigma(y)-\overline{\sigma}$ and convex duality, we rewrite the first term in the right-hand side of (1.8)

$$\begin{aligned}\int_Q &\langle(\sigma(y)-\overline{\sigma})[\xi+e(\phi)],[\xi+e(\phi)]\rangle\\ &=\sup_{\eta(y)}\int_Q[2\langle[\xi+e(\phi)],\eta(y)\rangle-\langle(\sigma(y)-\overline{\sigma})^{-1}\eta(y),\eta(y)\rangle].\end{aligned}\tag{1.9}$$

Here $\eta(y)$ ranges over periodic tensor fields, and one can get an *inequality* by choosing $\eta(y)$ of the form $\chi_2(y)\Lambda_s\eta-\chi_1(y)\Lambda_k\eta$, where η is any constant tensor. Thus (1.9) implies

$$\begin{aligned}\int_Q &\langle(\sigma(y)-\overline{\sigma})[\xi+e(\phi)],[\xi+e(\phi)]\rangle\\ &\geq -2\theta_1\langle\xi,\Lambda_k\eta\rangle+2\theta_2\langle\xi,\Lambda_s\eta\rangle\\ &\quad-\theta_1\langle(\sigma_1-\sigma_2)^{-1}\Lambda_k\eta,\Lambda_k\eta\rangle-\theta_2\langle(\sigma_2-\sigma_1)^{-1}\Lambda_s\eta,\Lambda_s\eta\rangle\\ &\quad+\int_Q 2\langle(\chi_2(y)\Lambda_s\eta-\chi_1(y)\Lambda_k\eta),e(\phi)\rangle.\end{aligned}$$

Together with (1.8) this yields

$$\begin{aligned}\langle\sigma^*\xi,\xi\rangle\geq{}&\langle\overline{\sigma}\xi,\xi\rangle-2\theta_1\langle\xi,\Lambda_k\eta\rangle+2\theta_2\langle\xi,\Lambda_s\eta\rangle\\ &-\theta_1\langle(\sigma_1-\sigma_2)^{-1}\Lambda_k\eta,\Lambda_k\eta\rangle-\theta_2\langle(\sigma_2-\sigma_1)^{-1}\Lambda_s\eta,\Lambda_s\eta\rangle\\ &+\inf_\phi\left[\int_Q 2\langle\chi_2(y)\eta,e(\phi)\rangle+\int_Q\langle\overline{\sigma}e(\phi),e(\phi)\rangle\right].\end{aligned}\tag{1.10}$$

The last term in (1.10) is the familiar "nonlocal" term, which is easily computed by means of Fourier analysis. This computation is by now classical (see Proposition 2.1 and Lemma 7.1 in [AK], or Lemmas 3.2 and 4.2 in [Ko]); the only difference here is that the reference material is $\overline{\sigma}$ instead of σ_1. We obtain a simple bound on this nonlocal term:

$$\begin{aligned}\inf_\phi&\left[\int_Q 2\langle\chi_2(y)\eta,e(\phi)\rangle+\int_Q\langle\overline{\sigma}e(\phi),e(\phi)\rangle\right]\\ &\geq-\theta_1\theta_2\sup_{|k|=1}\left[\frac{|\eta k|^2-\langle\eta k,k\rangle^2}{\mu_1}+\frac{\langle\eta k,k\rangle^2}{2\mu_1+\overline{\lambda}}\right],\end{aligned}$$

which, combined with (1.10), is the desired result. Q.E.D.

Now, the main difficulty is to prove the optimality of the lower bound established in Theorem 1.1. To be precise, the bound is optimal if, for any ξ, there exists a microstructure that achieves equality in (1.4). To exhibit such a microstructure is not an easy task, since checking equality in (1.4) amounts to computing explicitly the corresponding effective tensor σ^*. Fortunately, there exists a special class of composites, the so-called sequential laminates, for which explicit formulae (due to Francfort, Murat, and Tartar [FM, Ta2]) are available (for details, see also Proposition 3.2 in [AK]). In the well-ordered case, attainability of the bound (1.6) is established by combining the optimality condition for (1.6) and the layering formula. We emphasize that this procedure is very systematic since the optimality condition and the layering formula are closely related (see Theorem 3.5 in [AK] for details).

In the non-well-ordered case, the layering formula still holds, but it has no clear link with the optimality condition of the bound (1.4). Thus, there seems to be no systematic way of proving the optimality of (1.4), and more work is needed. Let us explain how we shall proceed. We start with the following functional:

$$F(\eta) = 2\langle \xi, \eta\rangle - \langle (\sigma_2 - \sigma_1)^{-1}\eta, \eta\rangle - \theta_1 g(\eta), \tag{1.11}$$

where $g(\eta)$ is defined by (1.7). Note that the supremum of $F(\eta)$ is involved in the well-ordered bound (1.6). In the non-well-ordered case, $F(\eta)$ is no longer concave, and its supremum may be equal to $+\infty$. It is not even clear that $F(\eta)$ admits a critical point. However, as a byproduct of Theorem 3.5 in [AK], we still have

Proposition 1.3. Suppose there exists a critical point η^* of $F(\eta)$, i.e., a point η^* satisfying

$$0 \in 2\xi - 2(\sigma_2 - \sigma_1)^{-1}\eta^* - \theta_1 \partial g(\eta^*), \tag{1.12}$$

where ∂g is the subdifferential of g (see [Cl] for an introduction to the subdifferential calculus). This means that there exists a family of positive reals m_i, with $\sum_{i=1}^n m_i = 1$, and vectors k_i, each extremal in the definition (1.7) of $g(\eta^*)$, such that

$$\xi = (\sigma_2 - \sigma_1)^{-1}\eta^* + \theta_1 \sum_{i=1}^n m_i f_{\sigma_1}(k_i)\eta^*. \tag{1.13}$$

Then there exists a sequentially laminated composite (determined by the parameters k_i and m_i) whose effective Hooke's law σ^* satisfies

$$\langle \sigma^*\xi, \xi\rangle = \langle \sigma_1\xi, \xi\rangle + \theta_2\langle \xi, \eta^*\rangle = \langle \sigma_1\xi, \xi\rangle + \theta_2 F(\eta^*). \tag{1.14}$$

Of course, in the strictly well-ordered case, $F(\eta)$ is strictly concave; so there exists a unique critical point of $F(\eta)$, and Proposition 1.3 clearly establishes the optimality of the bound (1.6). In the non-well-ordered case, we will still use Proposition 1.3, but with a different argument. Our strategy is as follows.

The right-hand side of (1.4) involves the supremum of the following functional:

$$\begin{aligned}\overline{F}(\eta) = \theta_1[&-2\langle \xi, \Lambda_h\eta\rangle - \langle (\sigma_1 - \sigma_2)^{-1}\Lambda_h\eta, \Lambda_h\eta\rangle] \\ &+ \theta_2[2\langle \xi, \Lambda_s\eta\rangle - \langle (\sigma_2 - \sigma_1)^{-1}\Lambda_s\eta, \Lambda_s\eta\rangle) - \theta_1\theta_2\overline{g}(\eta),\end{aligned} \tag{1.15}$$

which is easily seen to be strictly concave, so that it admits a unique critical point $\overline{\eta}^*$ (a maximizer). Superficially, the maximizer $\overline{\eta}^*$ of $\overline{F}$ has no apparent connection with the critical points η^* of F (if any), which in turn are linked to the layering formula. However, under a mild assumption on the moduli of the materials, a change of variables permits one to connect the optimality conditions of F and $\overline{F}$.

Proposition 1.4. Assume that the Lamé modulus of the mixed reference material $\overline{\sigma}$ is positive, i.e.,

$$\overline{\lambda} = \kappa_2 - \frac{2}{N}\mu_1 \geq 0. \tag{1.16}$$

Let $\overline{\eta}^*$ be the unique maximizer of $\overline{F}(\eta)$. Then, there exists a tensor $\eta^* = \overline{\eta}^* + cI_2$, with c a number depending on $\overline{\eta}^*$, such that

($*$) η^* is the unique critical point of $F(\eta)$,

($**$) $\langle \sigma_1 \xi, \xi\rangle + \theta_2 F(\eta^*) = \langle \overline{\sigma}\xi, \xi\rangle + \overline{F}(\eta^*)$.

Then, as a direct consequence of Propositions 1.3 and 1.4, we have the following:

THEOREM 1.5. Under assumption (1.16) the Hashin-Shtrikman-Walpole bound (1.4) is optimal, i.e., for any strain ξ, there exists a sequentially laminated composite that achieves equality in (1.4). Furthermore, the value of the bound (1.4) is exactly

$$\langle \sigma^* \xi, \xi\rangle \geq \langle \sigma_1 \xi, \xi\rangle + \theta_2 \operatorname{crit}_\eta F(\eta), \tag{1.17}$$

where $\operatorname{crit}_\eta F(\eta)$ represents the unique critical value of the function F (equal to $F(\eta^*)$).

We have thus reduced the proof of the optimality of the Hashin-Shtrikman-Walpole bound to the proof of Proposition 1.4. Unfortunately, the latter requires a blind and brute force computation, and we have no clear understanding of its success. (We also know another proof of Theorem 1.5, but it, too, depends on a tedious computation; see Sec. 2.)

The proof of Proposition 1.4 starts with another lemma connecting the critical k's for the functions g and $\overline{g}$ (which are defined as suprema over the unit sphere $|k| = 1$; see (1.5) and (1.7)).

LEMMA 1.6. Assume that the mixed reference material $\overline{\sigma}$ has $\overline{\lambda}$ positive, i.e., it satisfies (1.16). Let $\overline{\eta}$ be any symmetric second-order tensor.

($*$) Among all k extremal for $\overline{g}(\overline{\eta})$, the constant

$$c = \frac{\kappa_1 - \kappa_2}{2\mu_1 + \overline{\lambda}} \langle \overline{\eta} k, k\rangle$$

does not depend on k. Thus, we can define $\eta = \overline{\eta} + cI_2$.

($**$) Any k extremal for $\overline{g}(\overline{\eta})$ is also extremal for $g(\eta)$, and conversely.

($***$) For any extremal k, we have $f_{\overline{\sigma}}(k)\overline{\eta} = f_{\sigma_1}(k)\eta$.

REMARK 1.7. The constant c introduced in Lemma 1.6 corresponds exactly to that introduced in Proposition 1.4. Thus, in both lemmas, the correspondence between $\overline{\eta}$ and η is the same.

Proof of Lemma 1.6. In Propositions 7.2 and 7.4 of [AK], we proved that, assuming $\overline{\lambda}$ is positive and labelling the eigenvalues of $\overline{\eta}$ so that $\overline{\eta}_1 \leq \overline{\eta}_2 \leq \cdots \leq \overline{\eta}_N$, there are three regimes for $\langle \overline{\eta} k, k\rangle$, with k any extremal in $\overline{g}(\overline{\eta})$:

$$\overline{g}(\overline{\eta}) = \frac{\overline{\eta}_1^2}{2\mu_1 + \overline{\lambda}} \quad \text{and} \quad \langle \overline{\eta} k, k\rangle = \overline{\eta}_1 \quad \text{if } \overline{\eta}_1 \geq \frac{2\mu_1 + \overline{\lambda}}{2(\mu_1 + \overline{\lambda})}(\overline{\eta}_N + \overline{\eta}_1);$$

$$\overline{g}(\overline{\eta}) = \frac{\overline{\eta}_N^2}{2\mu_1 + \overline{\lambda}} \quad \text{and} \quad \langle \overline{\eta} k, k\rangle = \overline{\eta}_N \quad \text{if } \frac{2\mu_1 + \overline{\lambda}}{2(\mu_1 + \overline{\lambda})}(\overline{\eta}_N + \overline{\eta}_1) \geq \overline{\eta}_N;$$

$$\overline{g}(\overline{\eta}) = \frac{(\overline{\eta}_1 - \overline{\eta}_N)^2}{4\mu_1} + \frac{(\overline{\eta}_1 + \overline{\eta}_N)^2}{4(\mu_1 + \overline{\lambda})}$$

and

$$\langle \overline{\eta} k , k \rangle = \frac{2\mu_1 + \overline{\lambda}}{2(\mu_1 + \overline{\lambda})}(\overline{\eta}_1 + \overline{\eta}_N) \quad \text{in the remaining cases.}$$

Thus, the first point of the lemma is proved, and the definition of η is meaningful. The second and third points are straightforward computations. The optimality condition for k in the definition (1.5) of $\overline{g}(\overline{\eta})$ is

$$\frac{1}{\mu_1}\overline{\eta}^2 k + \left(\frac{1}{2\mu_1 + \overline{\lambda}} - \frac{1}{\mu_1}\right) 2\langle \overline{\eta} k , k \rangle \overline{\eta} k = Lk ,$$

where L is a Lagrange multiplier for the constraint $|k|^2 = 1$, and $f_{\overline{\sigma}}(k)\overline{\eta}$ is defined by

$$f_{\overline{\sigma}}(k)\overline{\eta} = \frac{1}{\mu_1}[(\overline{\eta} k) \otimes k - \langle \overline{\eta} k , k \rangle k \otimes k] + \frac{1}{2\mu_1 + \overline{\lambda}}\langle \overline{\eta} k , k \rangle k \otimes k .$$

Let us replace $\overline{\eta}$ by $\eta - cI_2$. Remarking that $c = (\kappa_1 - \kappa_2)\langle \overline{\eta} k , k \rangle/(2\mu_1 + \overline{\lambda})$ is also equal to $(\kappa_1 - \kappa_2)\langle \eta k , k \rangle/(2\mu_1 + \lambda_1)$, it is easily seen that k is a critical point in the definition of $g(\eta)$ and $f_{\overline{\sigma}}(k)\overline{\eta} = f_{\sigma_1}(k)\eta$. Q.E.D.

Proof of Proposition 1.4. Let $\overline{\eta}^*$ be the unique maximizer of $\overline{F}(\eta)$. It satisfies

$$0 \in \theta_2[2\Lambda_s \xi - 2(\sigma_2 - \sigma_1)^{-1}\Lambda_s \overline{\eta}^*] - \theta_1[2\Lambda_h \xi + 2(\sigma_1 - \sigma_2)^{-1}\Lambda_h \overline{\eta}^*] - \theta_1\theta_2 \partial \overline{g}(\overline{\eta}^*) . \quad (1.18)$$

Equivalently, there exists a family of positive reals m_i, with $\sum_{i=1}^n m_i = 1$, and vectors k_i, extremal for the definition of $\overline{g}(\overline{\eta}^*)$, such that

$$\theta_2[\Lambda_s \xi - (\sigma_2 - \sigma_1)^{-1}\Lambda_s \overline{\eta}^*] - \theta_1[\Lambda_h \xi + (\sigma_1 - \sigma_2)^{-1}\Lambda_h \overline{\eta}^*] = \theta_1\theta_2 \sum_{i=1}^n m_i f_{\overline{\sigma}}(k_i)\overline{\eta}^* . \quad (1.19)$$

Multiplying (1.19) successively by Λ_h and Λ_s leads to

$$\Lambda_h \xi + (\sigma_1 - \sigma_2)^{-1}\Lambda_h \overline{\eta}^* = -\theta_2 \Lambda_h \sum_{i=1}^n m_i f_{\overline{\sigma}}(k_i)\overline{\eta}^* ,$$

$$\Lambda_s \xi - (\sigma_2 - \sigma_1)^{-1}\Lambda_s \overline{\eta}^* = \theta_1 \Lambda_s \sum_{i=1}^n m_i f_{\overline{\sigma}}(k_i)\overline{\eta}^* .$$

Replacing $\overline{\eta}^*$ by $\eta^* - cI_2$, with $c = (\kappa_1 - \kappa_2)\langle \overline{\eta}^* k , k \rangle/(2\mu_1 + \overline{\lambda})$, and using Lemma 1.6 gives

$$\begin{aligned} \Lambda_h \xi - (\sigma_2 - \sigma_1)^{-1}\Lambda_h \eta^* &= -\theta_2 \Lambda_h \sum_{i=1}^n m_i f_{\sigma_1}(k_i)\eta^* + \frac{c}{N(\kappa_1 - \kappa_2)} , \\ \Lambda_s \xi - (\sigma_2 - \sigma_1)^{-1}\Lambda_s \eta^* &= \theta_1 \Lambda_s \sum_{i=1}^n m_i f_{\sigma_1}(k_i)\eta^* . \end{aligned} \quad (1.20)$$

Since $\Lambda_h f_{\sigma_1}(k_i)\eta^* = \langle \eta^* k_i , k_i \rangle / N(2\mu_1 + \lambda_1)$, the first equation in (1.20) becomes

$$\Lambda_h \xi - (\sigma_2 - \sigma_1)^{-1}\Lambda_h \eta^* = \theta_1 \Lambda_h \sum_{i=1}^n m_i f_{\sigma_1}(k_i)\eta^* .$$

Together with the second equation in (1.20) this implies that η^* is a critical point of $F(\eta)$. Moreover, η^* is unique, since the above calculation is reversible and $\overline{\eta}^*$ is unique too. It remains to check the equality

$$\langle \sigma_1 \xi , \xi \rangle + \theta_2 F(\eta^*) = \langle \overline{\sigma} \xi , \xi \rangle + \overline{F}(\overline{\eta}^*) .$$

We leave this easy calculation to the reader. Q.E.D.

2. The translation method. In this section we re-derive the optimal lower bound (1.17) by means of the translation method. The translation method was first introduced by Murat and Tartar [Ta2], and by Lurie and Cherkaev [LC]. Its link with the Hashin-Shtrikman method was found by Milton [Mi], who also introduced the name "translation method". For an introduction to that method and more references, see Sec. 4 of our paper [AK]. We begin by recalling two basic results.

PROPOSITION 2.1. Let τ be a constant fourth-order tensor (called a translation). By definition, a translation τ is said to be admissible if it satisfies

(i) $\sigma_1 - \tau \geq 0$ and $\sigma_2 - \tau \geq 0$,

(ii) τ is quasiconvex on strains, that is, for any Q-periodic function $\phi(y)$,

$$\int_Q \langle \tau e(\phi(y)) , e(\phi(y)) \rangle \, dy \geq 0 .$$

Then, to each admissible translation τ is associated the following translation bound:

$$\sigma^* \geq \left(\int_Q (\sigma - \tau)^{-1} \right)^{-1} + \tau . \tag{2.1}$$

Furthermore, for any symmetric second-order tensor ξ, the bound (2.1) admits the equivalent variational formulation

$$\langle \sigma^* \xi , \xi \rangle \geq \inf_{\nu} \int_Q [\langle \sigma(y)(\xi + \nu) , (\xi + \nu) \rangle - \langle \tau \nu , \nu \rangle] \, dy , \tag{2.2}$$

where the infimum is taken over all Q-periodic tensor fields ν with average value 0 on Q.

PROPOSITION 2.2. In the case of well-ordered components (i.e., $\sigma_1 \leq \sigma_2$), the particular translation

$$\tau_\eta = \sigma_1 - g(\eta)^{-1} \eta \otimes \eta , \tag{2.3}$$

where $g(\eta)$ is defined by (1.7), is admissible and yields the following bound:

$$\langle \sigma^* \xi , \xi \rangle \geq \langle \sigma_1 \xi , \xi \rangle + \frac{\theta_2 \langle \eta , \xi \rangle^2}{\theta_1 g(\eta) + \langle (\sigma_2 - \sigma_1)^{-1} \eta , \eta \rangle} . \tag{2.4}$$

REMARK 2.3. It is easily seen that by taking the envelope of the bounds (2.4) (i.e., maximizing over η) we recover the Hashin-Shtrikman bound (1.6) (for details see Proposition 4.9 in [AK]).

In the non-well-ordered case, the translation τ_η is not admissible in general, and the translation bound (2.4) does not hold, since there is no reason that $\sigma_2 - \tau_\eta$ should be positive. Our strategy for generalizing Proposition 2.2 is to decompose the "old" translation τ_η into a new *admissible* translation $\tilde{\tau}_\eta$ and a positive (convex) part ϕ_η:

$$\tau_\eta = \sigma_1 - g(\eta)^{-1}\eta \otimes \eta = \tilde{\tau}_\eta + \phi_\eta . \tag{2.5}$$

More precisely, we shall prove the following.

PROPOSITION 2.4. Assume that both materials have positive Lamé moduli, i.e.,

$$\lambda_i = \kappa_i - \frac{2}{N}\mu_i \geq 0, \qquad \text{for } i = 1, 2. \tag{2.6}$$

Then, for any tensor η, there exists an admissible (in the sense of Proposition 2.1) translation $\tilde{\tau}_\eta$ such that the difference $\tau_\eta - \tilde{\tau}_\eta$ is positive.

We postpone for a moment the proof of Proposition 2.4 in order to state the optimal translation bound. As in the first section, our starting point is the functional

$$F(\eta) = 2\langle \xi, \eta\rangle - \langle (\sigma_2 - \sigma_1)^{-1}\eta, \eta\rangle - \theta_1 g(\eta), \tag{2.7}$$

where $g(\eta)$ is defined by (1.7). Recall that the supremum of $F(\eta)$ is involved in the well-ordered Hashin-Shtrikman bound (1.6).

THEOREM 2.5. Assume the materials satisfy (2.6). Then

(*) there exists a critical point η^* of $F(\eta)$ and a unique critical value $F(\eta^*)$,

(**) the translation bound furnished by $\tilde{\tau}_{\eta^*}$ is optimal and is given by

$$\langle \sigma^*\xi, \xi\rangle \geq \inf_\nu \int_Q [\langle \sigma(y)(\xi+\nu), (\xi+\nu)\rangle - \langle \tilde{\tau}_{\eta^*}\nu, \nu\rangle]\, dy = \langle \sigma_1\xi, \xi\rangle + \theta_2 \operatorname{crit}_\eta F(\eta). \tag{2.8}$$

REMARK 2.6. Of course (2.8) coincides with the previous bound (1.17) obtained with the Hashin-Shtrikman-Walpole variational principle. However, the assumptions of Theorem 2.5 are stronger than those of Theorem 1.5 (both Lamé moduli λ_1 and λ_2 being positive implies that the "mixed" Lamé modulus $\bar{\lambda}$ is positive too). The critical point η^* of F is actually unique (see Theorem 1.5). We make only the weaker assertion that the critical *value* is unique in Theorem 2.5, because this is all that seems to follow from the methods of the present section.

The proof of Theorem 2.5 is based on several lemmas.

LEMMA 2.7. There exists at least one critical point η^* of $F(\eta)$.

Proof. This is clearly a consequence of Proposition 1.4, but we sketch here a more direct argument. Recall that Hooke's law for material $i = 1, 2$ is given by

$$\sigma_i = 2\mu_i\Lambda_s + N\kappa_i\Lambda_h .$$

Fixing a symmetric second-order tensor ξ, and any $t \in \mathbb{R}$, consider the function of η defined by

$$F(\eta, t) = 2\langle \xi + tI_2, \eta\rangle - |\Lambda_h\eta|^2 - (2\mu_2 - 2\mu_1)^{-1}|\Lambda_s\eta|^2 - \theta_1 g(\eta). \tag{2.9}$$

Recall that $g(\eta)$ is convex since it is the supremum of a family of quadratic functions. Therefore, there exists a unique maximizer $\eta(t)$ (which is easily seen to be a continuous function of t). It satisfies

$$0 \in \left(2\xi + 2tI_2 - 2\Lambda_h\eta(t) - 2(2\mu_2 - 2\mu_1)^{-1}\Lambda_s\eta(t) - \theta_1\partial g(\eta(t))\right), \tag{2.10}$$

where $\partial g(\eta(t))$ is the subdifferential of $g(\eta)$ at the point $\eta(t)$. On the other hand, a critical point η^* of (2.7), if any, would satisfy

$$0 \in \left(2\xi - 2(N\kappa_2 - N\kappa_1)^{-1}\Lambda_h\eta^* - 2(2\mu_2 - 2\mu_1)^{-1}\Lambda_s\eta^* - \theta_1\partial g(\eta^*)\right). \tag{2.11}$$

Comparing (2.10) and (2.11), we can choose $\eta^* = \eta(t)$ if

$$t - \frac{1}{N}\operatorname{tr}[\eta(t)] = \frac{1}{N^2(\kappa_1 - \kappa_2)}\operatorname{tr}[\eta(t)]. \tag{2.12}$$

Now, it is a relatively easy matter to show that there always exists a root of Eq. (2.12). This gives a critical point of $F(\eta)$. Q.E.D.

LEMMA 2.8. Let τ_η be the translation defined by $\sigma_1 - g(\eta)^{-1}\eta \otimes \eta$. Consider any decomposition $\tau_\eta = \tilde{\tau}_\eta + \phi_\eta$, with $\tilde{\tau}_\eta$ admissible and ϕ_η positive (by virtue of Proposition 2.4 there is at least one such decomposition). Then

$$\nu \in \partial g(\eta) \Rightarrow \phi_\eta \nu = 0, \tag{2.13}$$

where $\partial g(\eta)$ is the subdifferential of $g(\eta)$.

Proof. Recalling the definition (1.7), we have

$$g(\eta) = \sup_{|k|=1}\langle f_{\sigma_1}(k)\eta, \eta\rangle, \tag{2.14}$$

where the degenerate Hooke's law $f_{\sigma_1}(k)$ is defined in (1.1). For any tensor ν belonging to $\partial g(\eta)$, there exist an integer $n \geq 1$, unit vectors $(k_i)_{1\leq i\leq n}$ achieving the maximum in (2.14), and positive numbers $(m_i)_{1\leq i\leq n}$ with $\sum_{i=1}^n m_i = 1$ such that

$$\nu = \sum_{i=1}^n m_i f_{\sigma_1}(k_i)\eta.$$

By definition, each term $f_{\sigma_1}(k_i)\eta$ is a Fourier component of a periodic strain $e(\phi)$ (see the proof of Theorem 1.1 and the corresponding parts of [AK, Ko]). By quasiconvexity of the translations τ_η and $\tilde{\tau}_\eta$ (see Proposition 2.1), one has

$$\langle \tau_\eta[f_{\sigma_1}(k_i)\eta], [f_{\sigma_1}(k_i)\eta]\rangle \geq 0, \qquad \langle \tilde{\tau}_\eta[f_{\sigma_1}(k_i)\eta], [f_{\sigma_1}(k_i)\eta]\rangle \geq 0. \tag{2.15}$$

Furthermore, an easy calculation shows that the first term in (2.15) is actually equal to zero since k_i is extremal for $g(\eta)$. On the other hand ϕ_η is positive; thus each term $\langle \tilde{\tau}_\eta[f_{\sigma_1}(k_i)\eta], [f_{\sigma_1}(k_i)\eta]\rangle$ and $\langle \phi_\eta[f_{\sigma_1}(k_i)\eta], [f_{\sigma_1}(k_i)\eta]\rangle$ are equal to zero. Positivity of ϕ_η implies that

$$\phi_\eta[f_{\sigma_1}(k_i)\eta] = 0, \qquad \text{for } 1 \leq i \leq n. \tag{2.16}$$

Summing the equalities (2.16) gives the desired result. Q.E.D.

Proof of Theorem 2.5. Fix the tensor ξ. By Lemma 2.7, there exists a critical point η^* of F. So we have two tasks: (i) to verify that the resulting translation bound has the form (2.8), and (ii) to verify that this bound is optimal.

The proof of (i) is a bit circuitous. To motivate it, recall that in the well-ordered case the optimal bound is $\langle\sigma_1\xi, \xi\rangle + \theta_2 F(\eta^*)$, and this is the translation bound associated to τ_{η^*}. Now, it is a general fact about translations that if $\tau = \tilde{\tau} + \phi$, with τ, $\tilde{\tau}$ admissible and ϕ convex, then the translation bound associated to $\tilde{\tau}$ is stricter than that associated to τ (see Sec. 8 in [Mi]). Taking $\tau = \tau_{\eta^*}$, $\tilde{\tau} = \tilde{\tau}_{\eta^*}$, and remembering that no bound can be better than the optimal one, we deduce that in the well-ordered case τ_{η^*} and $\tilde{\tau}_{\eta^*}$ yield the *same* bound on $\langle \sigma^* \xi, \xi \rangle$. In other words, (2.8) holds in the well-ordered case. Our task is to find an alternative argument that proves (2.8) even when σ_1 and σ_2 are not well ordered.

Preparing to prove (i), let us explore the consequence of η^* being a critical point of F. It is easily seen that any such η^* is also a critical point of

$$G(\eta) = \frac{\langle \eta, \xi \rangle^2}{\theta_1 g(\eta) + \langle (\sigma_2 - \sigma_1)^{-1} \eta, \eta \rangle}, \tag{2.17}$$

and we have the equalities

$$G(\eta^*) = F(\eta^*) = \langle \xi, \eta^* \rangle. \tag{2.18}$$

When the materials are well ordered, we proved in Proposition 4.9 of [AK] that the translation bound (2.2) obtained with τ_η is exactly

$$\inf_\nu \int_Q [\langle \sigma(y)(\xi + \nu), (\xi + \nu) \rangle - \langle \tau_\eta \nu, \nu \rangle] \, dy = \langle \sigma_1 \xi, \xi \rangle + \theta_2 G(\eta). \tag{2.19}$$

In the non-well-ordered case, the infimum over ν in (2.19) may be infinite. Nevertheless, a careful examination of the proof of Proposition 4.9 in [AK] shows that there always exists a critical point $\nu(\eta)$ of the left-hand side of (2.19). Moreover, $\nu(\eta)$ is constant within each component material. Now consider the left-hand side of (2.19), evaluated at $\nu(\eta)$:

$$H(\eta) = \int_Q [\langle \sigma(y)(\xi + \nu(\eta)), (\xi + \nu(\eta)) \rangle - \langle \sigma_1 \nu(\eta), \nu(\eta) \rangle + g(\eta)^{-1} \langle \eta, \nu(\eta) \rangle^2] \, dy. \tag{2.20}$$

By virtue of Proposition 4.9 in [AK], (2.19) still holds when replacing the infimum by the critical value, i.e.,

$$H(\eta) = \langle \sigma_1 \xi, \xi \rangle + \theta_2 G(\eta).$$

Thus, η^* being a critical point of $G(\eta)$, it is also a critical point of $H(\eta)$. Let us compute the subdifferential of $H(\eta)$. Suppose briefly that "everything is smooth": then, if we differentiate $H(\eta)$, the chain-rule lemma allows us not to differentiate $\nu(\eta)$, because it is itself a critical point. This can be made rigorous in the framework of the subdifferential calculus by the "chain-rule" Theorem 2.3.9 of [Cl]. Thus

$$0 \in \partial H(\eta^*) \Rightarrow 0 \in \partial \left[g(\eta^*)^{-1} \int_Q \langle \eta^*, \nu(\eta^*) \rangle^2 \right]. \tag{2.21}$$

The critical point $\nu(\eta^*)$ takes a constant value ν_i in material σ_i $(i = 1, 2)$, and it has average zero on Q, i.e., $\theta_1 \nu_1 + \theta_2 \nu_2 = 0$. Therefore $\int_Q \langle \eta^*, \nu(\eta^*) \rangle^2 = \theta_1 / \theta_2 \langle \eta^*, \nu_1 \rangle^2$. The Leibniz rule for subdifferentials implies

$$\partial [g(\eta^*)^{-1} \langle \eta^*, \nu_1 \rangle^2] \subset [g(\eta^*)^{-1} 2 \langle \eta^*, \nu_1 \rangle \nu_1 - g(\eta^*)^{-2} \langle \eta^*, \nu_1 \rangle^2 \partial g(\eta^*)]. \tag{2.22}$$

From Eq. (4.28) in [AK] we know that

$$\langle \eta^*, \nu_1 \rangle = \frac{\theta_2 \langle \eta^*, \xi \rangle}{\theta_1 + g(\eta^*)^{-1} \langle (\sigma_2 - \sigma_1)^{-1} \eta^*, \eta^* \rangle}. \tag{2.23}$$

Together with (2.18), (2.23) yields $\langle \eta^*, \nu_1 \rangle = \theta_2 g(\eta^*)$. Therefore, we deduce from (2.21) and (2.23)

$$\nu_1 \in \frac{\theta_2}{2} \partial g(\eta^*). \tag{2.24}$$

Applying Lemma 2.8, (2.24) implies that $\phi_{\eta^*} \nu_1 = 0$; thus $\phi_{\eta^*} \nu(\eta^*) = 0$. Using the decomposition $\tau_{\eta^*} = \tilde{\tau}_{\eta^*} + \phi_{\eta^*}$, we obtain

$$H(\eta^*) = \int_Q [\langle \sigma(y)(\xi + \nu(\eta^*)), (\xi + \nu(\eta^*)) \rangle - \langle \tilde{\tau}_{\eta^*} \nu(\eta^*), \nu(\eta^*) \rangle] \, dy. \tag{2.25}$$

We are ready now to prove assertion (i), i.e., that (2.8) holds. The first inequality

$$\langle \sigma^* \xi, \xi \rangle \geq \inf_{\nu} \int_Q [\langle \sigma(y)(\xi + \nu), (\xi + \nu) \rangle - \langle \tilde{\tau}_{\eta^*} \nu, \nu \rangle] \, dy \tag{2.26}$$

is just the variational form of the translation bound associated to $\tilde{\tau}_{\eta^*}$. We assert that $\nu = \nu(\eta^*)$ (defined as a critical point for (2.19)) achieves the minimum in (2.26). Since $\sigma(y) - \tilde{\tau}_{\eta^*}$ is nonnegative, it suffices to check that $\nu = \nu(\eta^*)$ is a critical point for (2.26). By definition it satisfies

$$\int_Q [\langle \sigma(y)[\xi + \nu(\eta^*)], \nu' \rangle - \langle \tau_{\eta^*} \nu(\eta^*), \nu' \rangle] \, dy = 0 \quad \text{for any zero-average tensor } \nu'. \tag{2.27}$$

We know from (2.24) and Lemma 2.8 that

$$(\tau_{\eta^*} - \tilde{\tau}_{\eta^*}) \nu(\eta^*) = \phi_{\eta^*} \nu(\eta^*) = 0. \tag{2.28}$$

It follows easily that $\nu(\eta^*)$ achieves the minimum in (2.26). The value of the bound (2.26) is thus

$$H(\eta^*) = \langle \sigma_1 \xi, \xi \rangle + \theta_2 G(\eta^*) = \langle \sigma_1 \xi, \xi \rangle + \theta_2 F(\eta^*),$$

using (2.25) and (2.18). This proves (2.8).

The optimality of this bound—our assertion (ii)—is an immediate consequence of Proposition 1.3. Our argument does not show that the critical point η^* is unique. But it does show that the critical value $F(\eta^*)$ is unique. Indeed, the preceding applies for any critical point η^*, whereas the optimal bound is by definition unique. (In truth η^* is unique too; see Sec. 1.) Q.E.D.

The remainder of this section is devoted to the decomposition of τ_η.

Proof of Proposition 2.4. We shall make essential use of the assumption that both materials have positive Lamé moduli $\lambda_i = \kappa_i - 2\mu_i/N$. The main idea is to decompose the "old" translation $\tau_\eta = \sigma_1 - g(\eta)^{-1} \eta \otimes \eta$ into a positive part ϕ_η and a "new" translation $\tilde{\tau}_\eta$, which is a linear combination of the "elementary" translations τ_{ik} defined by

$$\langle \tau_{ik} \xi, \xi \rangle = \xi_{ik}^2 - \xi_{ii} \xi_{kk}. \tag{2.29}$$

(Here ξ is any symmetric second-order tensor, with entries denoted by ξ_{ik}.) The translations τ_{ik} are easily seen to be quasiconvex on strains (e.g., by Fourier analysis): for any Q-periodic function ϕ

$$\int_Q \langle \tau_{ik} e(\phi), e(\phi)\rangle \geq 0, \qquad \text{with } e(\phi) = \frac{\nabla\phi + {}^t\nabla\phi}{2}.$$

In Proposition 7.4 of [AK], we proved that, when $\lambda_1 \geq 0$, the nonlocal term $g(\eta)$ is defined by

$$g(\eta) = \begin{cases} \dfrac{1}{4\mu_1}(\eta_1 - \eta_N)^2 + \dfrac{1}{4(\mu_1+\lambda_1)}(\eta_1+\eta_N)^2 & \text{if } \eta_N \geq \dfrac{2\mu_1+\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1) \geq \eta_1, \\ \dfrac{\eta_1^2}{2\mu_1+\lambda_1} \quad \text{if } \eta_1 \geq \dfrac{2\mu_1+\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1), \\ \dfrac{\eta_N^2}{2\mu_1+\lambda_1} \quad \text{if } \dfrac{2\mu_1+\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1) \geq \eta_N, \end{cases} \tag{2.30}$$

where $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_N$ are the eigenvalues of η.

Corresponding to the different regimes of $g(\eta)$ in (2.30), we consider different cases in decomposing τ_η. With no loss of generality, we always work in a basis where η is diagonal.

Consider first the case

$$g(\eta) = \frac{\eta_N^2}{2\mu_1+\lambda_1} \quad \text{when } \frac{2\mu_1+\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1) \geq \eta_N. \tag{2.31}$$

For any symmetric second-order tensor ξ, replacing $g(\eta)$ by its value yields

$$\langle \tau_\eta\xi, \xi\rangle = \sum_{i,k=1}^N [(2\mu_1+\lambda_1)\frac{\eta_i\eta_k}{\eta_N^2} - \lambda_1](\xi_{ik}^2 - \xi_{ii}\xi_{kk}) + (2\mu_1+\lambda_1)\sum_{i,k=1}^N \left(1 - \frac{\eta_i\eta_k}{\eta_N^2}\right)\xi_{ik}^2. \tag{2.32}$$

Obviously all terms in the second sum in the right-hand side of (2.32) are positive. If the coefficients of the first sum were positive, the decomposition of τ_η would be straightforward: the first sum would be quasiconvex on strains, and the second positive. Unfortunately this is not the case. We therefore define the new translation $\tilde{\tau}_\eta$ by

$$\begin{aligned}\langle \tilde{\tau}_\eta\xi, \xi\rangle = \sum_{i,k=1}^N \Bigg[&(2\mu_1+\lambda_1)\frac{\eta_i\eta_k}{\eta_N^2} - \lambda_1 \\ &+ \frac{\lambda_1(2\mu_1+\lambda_1)}{2\mu_1}\left(1 - \frac{\eta_k}{\eta_N}\right)\left(1 - \frac{\eta_i}{\eta_N}\right)\Bigg](\xi_{ik}^2 - \xi_{ii}\xi_{kk}).\end{aligned} \tag{2.33}$$

Then the difference $\phi_\eta = \tau_\eta - \tilde{\tau}_\eta$ is given by

$$\begin{aligned}\langle \phi_\eta\xi, \xi\rangle = {}&(2\mu_1+\lambda_1)\sum_{i,k=1}^N \left(1 - \frac{\eta_k\eta_i}{\eta_N^2}\right)\xi_{ik}^2 \\ &- \frac{\lambda_1(2\mu_1+\lambda_1)}{2\mu_1}\sum_{i,k=1}^N \left(1 - \frac{\eta_k}{\eta_N}\right)\left(1 - \frac{\eta_i}{\eta_N}\right)(\xi_{ik}^2 - \xi_{ii}\xi_{kk}).\end{aligned} \tag{2.34}$$

Let us check that ϕ_η is positive. Equation (2.34) can be rewritten

$$\langle \phi_\eta \xi, \xi \rangle = \frac{\lambda_1(2\mu_1 + \lambda_1)}{2\mu_1} \left[\sum_{k=1}^{N} \left(1 - \frac{\eta_k}{\eta_N}\right) \xi_{kk} \right]^2 + (2\mu_1 + \lambda_1) \sum_{i,k=1}^{N} a_{ik} \xi_{ik}^2 ,$$

where the coefficient a_{ik} is defined by

$$a_{ik} = 1 - \frac{\eta_k \eta_i}{\eta_N^2} - \frac{\lambda_1}{2\mu_1} \left(1 - \frac{\eta_k}{\eta_N}\right) \left(1 - \frac{\eta_i}{\eta_N}\right) .$$

It is enough to prove that a_{ik} is positive. With no loss of generality we can assume that $\eta_k \geq \eta_i$. Remarking that (2.31) implies that η_N is positive, we infer

$$\begin{aligned} a_{ik} &= \frac{\eta_k - \eta_i}{\eta_N} + \left(1 - \frac{\eta_k}{\eta_N}\right) \left[1 + \frac{\eta_i}{\eta_N} - \frac{\lambda_1}{2\mu_1}\left(1 - \frac{\eta_i}{\eta_N}\right)\right] \\ &\geq \left(1 - \frac{\eta_k}{\eta_N}\right) \frac{(2\mu_1 + \lambda_1)\eta_i + (2\mu_1 - \lambda_1)\eta_N}{2\mu_1 \eta_N} \\ &\geq \left(1 - \frac{\eta_k}{\eta_N}\right) \frac{(2\mu_1 + \lambda_1)\eta_i - \lambda_1 \eta_N}{2\mu_1 \eta_N} \geq 0 \end{aligned}$$

as a consequence of (2.31).

To prove that $\tilde{\tau}_\eta$ is quasiconvex on strains, it is also enough to check that all coefficients in (2.33) are positive. The coefficient of τ_{ik} is

$$\begin{aligned} b_{ik} &= (2\mu_1 + \lambda_1)\frac{\eta_i \eta_k}{\eta_N^2} - \lambda_1 + \frac{\lambda_1(2\mu_1 + \lambda_1)}{2\mu_1} \left(1 - \frac{\eta_k}{\eta_N}\right) \left(1 - \frac{\eta_i}{\eta_N}\right) \\ &= \frac{[(2\mu_1 + \lambda_1)\eta_i - \lambda_1\eta_N][(2\mu_1 + \lambda_1)\eta_k - \lambda_1\eta_N]}{2\mu_1 \eta_N^2} . \end{aligned}$$

In view of (2.31), for any index i one has $(2\mu_1 + \lambda_1)\eta_i - \lambda_1 \eta_N \geq 0$. Thus b_{ik} is positive.

Finally, to prove that $\tilde{\tau}_\eta$ is admissible in the sense of Proposition 2.1, it remains to show that

$$\sigma_1 - \tilde{\tau}_\eta \geq 0, \quad \text{and} \quad \sigma_2 - \tilde{\tau}_\eta \geq 0. \tag{2.35}$$

The first inequality in (2.35) is obvious since $\tilde{\tau}_\eta = \sigma_1 - g(\eta)^{-1} \eta \otimes \eta - \phi_\eta$ and ϕ_η is positive. The second one is true if the quadratic form

$$\begin{aligned} P(\xi) &= \langle (\sigma_2 - \sigma_1)\xi, \xi \rangle + g(\eta)^{-1} \langle \eta, \xi \rangle^2 + \langle \phi_\eta \xi, \xi \rangle \\ &= 2(\mu_2 - \mu_1)|\xi|^2 + \lambda_2 (\operatorname{tr} \xi)^2 + \langle \phi_\eta \xi, \xi \rangle - \lambda_1 (\operatorname{tr} \xi)^2 + g(\eta)^{-1} \langle \eta, \xi \rangle^2 \end{aligned}$$

is positive. The two first terms are positive because $\mu_2 \geq \mu_1$ and $\lambda_2 \geq 0$. The last two terms yield

$$-\lambda_1(\operatorname{tr}\xi)^2 + g(\eta)^{-1}\langle\eta, \xi\rangle^2 = \sum_{i,k=1}^{N}\left(-\lambda_1 + (2\mu_1+\lambda_1)\frac{\eta_i\eta_k}{\eta_N^2}\right)\xi_{ii}\xi_{kk}$$

$$= \frac{1}{2\mu_1}\sum_{i,k=1}^{N}\left[-\lambda_1(2\mu_1+\lambda_1)\left(1-\frac{\eta_i}{\eta_N}\right)\left(1-\frac{\eta_k}{\eta_N}\right)\right.$$

$$\left. + \left(\lambda_1 - (2\mu_1+\lambda_1)\frac{\eta_i}{\eta_N}\right)\left(\lambda_1 - (2\mu_1+\lambda_1)\frac{\eta_k}{\eta_N}\right)\right]\xi_{ii}\xi_{kk}$$

$$= \frac{1}{2\mu_1}\left[\sum_{k=1}^{N}\left(\lambda_1 - (2\mu_1+\lambda_1)\frac{\eta_k}{\eta_N}\right)\xi_{kk}\right]^2 - \frac{\lambda_1(2\mu_1+\lambda_1)}{2\mu_1}\left[\sum_{k=1}^{N}\left(1-\frac{\eta_k}{\eta_N}\right)\xi_{kk}\right]^2.$$

Recalling that ϕ_η is

$$\langle\phi_\eta\xi, \xi\rangle = \frac{\lambda_1(2\mu_1+\lambda_1)}{2\mu_1}\left[\sum_{k=1}^{N}\left(1-\frac{\eta_k}{\eta_N}\right)\xi_{kk}\right]^2 + (2\mu_1+\lambda_1)\sum_{i,k=1}^{N}a_{ik}\xi_{ik}^2$$

where the coefficients a_{ik} are positive, we deduce that $P(\xi)$ is positive; thus $\sigma_2 - \tilde{\tau}_\eta \geq 0$.

The case $g(\eta) = \eta_1^2/(2\mu_1+\lambda_1)$, when $2(\mu_1+\lambda_1)\eta_1 \geq (2\mu_1+\lambda_1)(\eta_N+\eta_1)$ is completely symmetric to the first case (2.31). Thus, the second and last case is

$$g(\eta) = \frac{1}{4\mu_1}(\eta_1-\eta_N)^2 + \frac{1}{4(\mu_1+\lambda_1)}(\eta_1+\eta_N)^2 \quad \text{when } \eta_N \geq \frac{2\mu_1+\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1) \geq \eta_1. \tag{2.36}$$

Remark that the condition on η_1, η_N in (2.36) is equivalent to

$$\eta_N \geq \frac{\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1) \geq \eta_1. \tag{2.37}$$

We introduce an integer p $(1 \leq p \leq N-1)$ such that the following ordering of the eigenvalues holds:

$$\eta_1 \leq \cdots \leq \eta_p \leq \frac{\lambda_1}{2(\mu_1+\lambda_1)}(\eta_N+\eta_1) < \eta_{p+1} \leq \cdots \leq \eta_N. \tag{2.38}$$

Then, the new translation $\tilde{\tau}_\eta$ is defined by

$$\begin{aligned}\langle\tilde{\tau}_\eta\xi, \xi\rangle = 2\mu_1\sum_{i,k=1}^{p}&\frac{[(\lambda_1(\eta_1+\eta_N) - 2(\mu_1+\lambda_1)\eta_k][\lambda_1(\eta_1+\eta_N) - 2(\mu_1+\lambda_1)\eta_i]}{[\lambda_1\eta_N - (2\mu_1+\lambda_1)\eta_1]^2} \\ &\times(\xi_{ik}^2 - \xi_{ii}\xi_{kk}) \\ + 2\mu_1\sum_{i,k=p+1}^{N}&\frac{[\lambda_1(\eta_1+\eta_N) - 2(\mu_1+\lambda_1)\eta_k][\lambda_1(\eta_1+\eta_N) - 2(\mu_1+\lambda_1)\eta_i]}{[(2\mu_1+\lambda_1)\eta_N - \lambda_1\eta_1]^2} \\ &\times(\xi_{ik}^2 - \xi_{ii}\xi_{kk}).\end{aligned} \tag{2.39}$$

As before we denote by ϕ_η the difference $\tau_\eta - \tilde{\tau}_\eta$. A tedious (but easy) calculation shows that ϕ_η can be written as

$$\begin{aligned}
\langle \phi_\eta \xi, \xi \rangle = {} & 4\mu_1(\mu_1+\lambda_1) g(\eta)^{-1} \left[\sum_{k=1}^{p} \frac{\lambda_1 g(\eta) - \eta_N \eta_k}{\lambda_1 \eta_N - (2\mu_1 + \lambda_1)\eta_1} \xi_{kk} \right. \\
& \left. + \sum_{k=p+1}^{N} \frac{\lambda_1 g(\eta) - \eta_1 \eta_k}{(2\mu_1 + \lambda_1)\eta_N - \lambda_1 \eta_1} \xi_{kk} \right]^2 \\
& + 4\lambda_1 \mu_1 (\mu_1 + \lambda_1) \left[\sum_{k=1}^{p} \frac{\eta_k - \eta_1}{\lambda_1 \eta_N - (2\mu_1 + \lambda_1)\eta_1} \xi_{kk} \right. \\
& \left. + \sum_{k=p+1}^{N} \frac{\eta_N - \eta_k}{(2\mu_1 + \lambda_1)\eta_N - \lambda_1 \eta_1} \xi_{kk} \right]^2 \\
& + 4\mu_1(\mu_1 + \lambda_1) \sum_{i,k=1}^{p} \frac{2(\mu_1 + \lambda_1)(\eta_1^2 - \eta_i \eta_k) + \lambda_1(\eta_1 + \eta_N)(\eta_i + \eta_k - 2\eta_1)}{[\lambda_1 \eta_N - (2\mu_1 + \lambda_1)\eta_1]^2} \xi_{ik}^2 \\
& + 4\mu_1(\mu_1 + \lambda_1) \sum_{i,k=p+1}^{N} \frac{2(\mu_1 + \lambda_1)(\eta_N^2 - \eta_i \eta_k) + \lambda_1(\eta_1 + \eta_N)(\eta_i + \eta_k - 2\eta_N)}{[\lambda_1 \eta_N - (2\mu_1 + \lambda_1)\eta_1]^2} \xi_{ik}^2 \\
& + 4\mu_1 \sum_{i=1}^{p} \sum_{k=p+1}^{N} \xi_{ik}^2 .
\end{aligned} \tag{2.40}$$

The quadratic form ϕ_η is positive if the coefficients of ξ_{ik}^2 in the third and fourth lines of (2.40) are positive. For i, $k \in [1\,;p]$, their sign is the same as that of c_{ik} defined by

$$c_{ik} = 2(\mu_1 + \lambda_1)(\eta_1^2 - \eta_i \eta_k) + \lambda_1(\eta_1 + \eta_N)(\eta_i + \eta_k - 2\eta_1).$$

With no loss of generality we can assume that $\eta_i \geq \eta_k$; thus

$$\begin{aligned}
c_{ik} = {} & 4(\mu_1 + \lambda_1)(\eta_k - \eta_1) \left[\frac{\lambda_1}{2(\mu_1 + \lambda_1)} (\eta_1 + \eta_N) - \frac{(\eta_1 + \eta_k)}{2} \right] \\
& + 2(\mu_1 + \lambda_1)(\eta_i - \eta_k) \left[\frac{\lambda_1}{2(\mu_1 + \lambda_1)})(\eta_1 + \eta_N) - \eta_k \right] .
\end{aligned} \tag{2.41}$$

In view of the ordering condition (2.38) each term of (2.41) is positive since i, $k \in [1\,;p]$. A similar computation holds for i, $k \in [p+1\,;N]$; thus ϕ_η is a positive quadratic form.

The new translation $\tilde{\tau}_\eta$ is easily seen to be quasiconvex on strains, since the ordering condition (2.38) implies that all the coefficients of $\xi_{ik}^2 - \xi_{ii}\xi_{kk}$ are positive. To prove that $\tilde{\tau}_\eta$ is an admissible translation in the sense of Proposition 2.1, it remains to show that

$$\sigma_1 - \tilde{\tau}_\eta \geq 0, \quad \text{and} \quad \sigma_2 - \tilde{\tau}_\eta \geq 0. \tag{2.42}$$

The first inequality in (2.42) is obvious since $\tilde{\tau}_\eta = \sigma_1 - g(\eta)^{-1}\eta \otimes \eta - \phi_\eta$ and ϕ_η is positive. The second one is true if the quadratic form

$$
\begin{aligned}
P(\xi) &= \langle(\sigma_2 - \sigma_1)\xi, \xi\rangle + g(\eta)^{-1}\langle\eta, \xi\rangle^2 + \langle\phi_\eta\xi, \xi\rangle^2 \\
&= 2(\mu_2 - \mu_1)|\xi|^2 + \lambda_2(\operatorname{tr}\xi)^2 + \langle\phi_\eta\xi, \xi\rangle - \lambda_1(\operatorname{tr}\xi)^2 + g(\eta)^{-1}\langle\eta, \xi\rangle^2
\end{aligned}
$$

is positive. The two first terms are positive because $\mu_2 \geq \mu_1$ and $\lambda_2 \geq 0$. Another tedious computation yields

$$
\begin{aligned}
&-\lambda_1(\operatorname{tr}\xi)^2 + g(\eta)^{-1}\langle\eta, \xi\rangle^2 \\
&\quad= -4\mu_1(\mu_1+\lambda_1)g(\eta)^{-1}\left[\sum_{k=1}^{p}\frac{\lambda_1 g(\eta) - \eta_N\eta_k}{\lambda_1\eta_N - (2\mu_1+\lambda_1)\eta_1}\xi_{kk}\right. \\
&\qquad\qquad \left. + \sum_{k=p+1}^{N}\frac{\lambda_1 g(\eta) - \eta_1\eta_k}{(2\mu_1+\lambda_1)\eta_N - \lambda_1\eta_1}\xi_{kk}\right]^2 \\
&\qquad - 4\lambda_1\mu_1(\mu_1+\lambda_1)\left[\sum_{k=1}^{p}\frac{\eta_k - \eta_1}{\lambda_1\eta_N - (2\mu_1+\lambda_1)\eta_1}\xi_{kk}\right. \\
&\qquad\qquad \left. + \sum_{k=p+1}^{N}\frac{\eta_N - \eta_k}{(2\mu_1+\lambda_1)\eta_N - \lambda_1\eta_1}\xi_{kk}\right]^2 \\
&\qquad + 2\mu_1\left[\sum_{k=1}^{p}\frac{\lambda_1(\eta_1+\eta_N) - 2(\mu_1+\lambda_1)\eta_k}{\lambda_1\eta_N - (2\mu_1+\lambda_1)\eta_1}\xi_{kk}\right]^2 \\
&\qquad + 2\mu_1\left[\sum_{k=p+1}^{N}\frac{\lambda_1(\eta_1+\eta_N) - 2(\mu_1+\lambda_1)\eta_k}{(2\mu_1+\lambda_1)\eta_N - \lambda_1\eta_1}\xi_{kk}\right]^2.
\end{aligned}
\tag{2.43}
$$

Thus $\langle\phi_\eta\xi, \xi\rangle - \lambda_1(\operatorname{tr}\xi)^2 + g(\eta)^{-1}\langle\eta, \xi\rangle^2$ is positive, and so is $\sigma_2 - \tilde{\tau}_\eta$. Q.E.D.

REMARK 2.9. One can easily show that $\tilde{\tau}_\eta$ is "extremal", in the sense that no further convex quadratic form can be removed while maintaining quasiconvexity. In other words, if $\tilde{\tau}_\eta = \hat{\tau} + \hat{\phi}$, with $\hat{\tau}$ quasiconvex and $\hat{\phi}$ nonnegative, then $\tilde{\tau}_\eta = \hat{\tau}$ and $\hat{\phi} = 0$.

3. Related issues. In this section we discuss some potential generalizations of our previous result. By considering the 2-D case, where explicit bounds are known for any ordering of the materials, we explain why we are unable to obtain an optimal *upper* bound for non-well-ordered materials with our methods. For sums of energies, we establish optimal *trace* bounds (as introduced in [MK, Mi]) for non-well-ordered materials.

Explicit bounds in two space dimensions. In two space dimensions, we know explicit optimal upper and lower bounds on the elastic energy of a composite made of two isotropic components [GC, AK2]. Of course, the optimal lower bound in the non-well-ordered case coincides with that obtained here. Furthermore, it is valid without any assumption on the moduli of the materials.

Concerning the lower bound, we have seen that, whatever the ordering of the materials, it is always the material with the *lower* shear modulus that is the reference material, or equivalently the *matrix* material in the matrix-inclusion microstructure achieving equality in the bound. Thus, a naive guess would be that, for the upper bound too, it is always the material with stronger shear modulus that is the reference or matrix material. This is true if the materials are well ordered, but it can readily be checked in the explicit formulae of [AK2] that this is wrong in the non-well-ordered case. Rather, either material may be the reference material, depending on the value of ξ (see Remark 2.8 in [AK2]).

Optimal upper bounds and complementary energy. Although the form of the optimal lower bound was simple and concise

$$\langle \sigma^* \xi , \xi \rangle \geq \langle \sigma_1 \xi , \xi \rangle + \theta_2 \operatorname{crit}_\eta F(\eta) \tag{3.1}$$

(remark that material one is always the matrix material), it was obtained by two different, but equally complicated, methods involving a lot of tedious computations. For the optimal upper bound, we already know in the 2-D case that the correct formula for the bound is more complicated (it can involve either σ_1 or σ_2 as the matrix material). Thus our method evidently cannot work for upper bounds on elastic energy.

However, as a consequence of Theorem 8.2 in [AK], we easily obtain an *optimal upper* bound for *complementary* energy $\langle (\sigma^*)^{-1} \xi , \xi \rangle$ by simply taking the Fenchel (or Legendre) transform of the optimal lower bound (3.1) on primal energy. For details, we refer the interested reader to Sec. 8 of [AK].

Optimal trace bounds. Another possible generalization of the lower bound (3.1) would be a similar bound for a sum of energies rather than a single one. Unfortunately, our methods are very specific to the case of a single energy, since they rely strongly on the explicit formulae for $g(\eta)$ or $\overline{g}(\eta)$ defined in (1.5), (1.7). There is however one type of bound on sums of energies, so-called *trace* bounds, which can be established in the non-well-ordered case by using the mixed reference material introduced in Sec. 1. In the case of two well-ordered (possibly nonisotropic) materials, trace bounds have been introduced by Milton and Kohn [MK, Mi]. Let us recall the lower trace bound, as presented in Sec. 5 of [AK]. For a collection $(\eta_i)_{1 \leq i \leq p}$ of symmetric second-order tensors, writing $M = \sum_{i=1}^p \eta_i \otimes \eta_i$, and assuming $\sigma_1 \leq \sigma_2$, the lower trace bound is

$$\theta_2 \langle (\sigma^* - \sigma_1)^{-1} , M \rangle \leq \langle (\sigma_2 - \sigma_1)^{-1} , M \rangle + \theta_1 g(M) , \tag{3.2}$$

where the nonlocal term is

$$g(M) = \sup_{|k|=1} \langle f_{\sigma_1}(k) , M \rangle . \tag{3.3}$$

A similar upper trace bound holds for $\langle (\sigma_2 - \sigma^*)^{-1} , M \rangle$. An interesting feature of such trace bounds is that equality is achieved in (3.2) for a *single* layering of materials (recall that the Hashin-Shtrikman bound (1.6) is usually attained by a multi-layered microstructure, not by a single-layered microstructure; see Theorem 3.5 in [AK]). We now generalize these trace bounds to the case of two *isotropic and non-well-ordered* materials.

THEOREM 3.1. Consider two isotropic materials with Hooke's law

$$\sigma_i = 2\mu_i\Lambda_s + N\kappa_i\Lambda_h \quad \text{for } i = 1, 2$$

that are not well ordered, i.e.,

$$\mu_1 < \mu_2, \qquad \kappa_1 > \kappa_2. \tag{3.4}$$

Denote by $\overline{\sigma}$ the weak "mixed" reference material $\overline{\sigma} = 2\mu_1\Lambda_s + N\kappa_2\Lambda_h$. For any collection $(\eta_i)_{1\le i\le p}$ of symmetric second-order tensors, we write $M = \sum_{i=1}^{p} \eta_i \otimes \eta_i$, $M_s = \sum_{i=1}^{p} \Lambda_s\eta_i \otimes \Lambda_s\eta_i$, and $M_h = \sum_{i=1}^{p} \Lambda_h\eta_i \otimes \Lambda_h\eta_i$. Then, the following lower trace bound holds:

$$\langle(\sigma^* - \overline{\sigma})^{-1}, M\rangle \le \frac{1}{\theta_1}\langle(\sigma_1 - \overline{\sigma})^{-1}, M_h\rangle + \frac{1}{\theta_2}\langle(\sigma_2 - \overline{\sigma})^{-1}, M_s\rangle + \theta_1\theta_2\overline{g}\left(\frac{M_h}{\theta_1} - \frac{M_s}{\theta_2}\right). \tag{3.5}$$

Here, as usual, $\overline{g}$ is defined by $\overline{g}(P) = \sup_{|k|=1}\langle f_{\overline{\sigma}}(k), P\rangle$. Furthermore, the trace bound (3.5) is optimal, i.e., for any M there is a single-layered microstructure that achieves equality in (3.5).

Similarly, we establish an upper trace bound.

THEOREM 3.2. Let σ_1 and σ_2 be two isotropic, non-well-ordered materials, with $\mu_1 < \mu_2$ and $\kappa_1 > \kappa_2$. Denote by $\underline{\sigma}$ the strong "mixed" reference material $\underline{\sigma} = 2\mu_2\Lambda_s + N\kappa_1\Lambda_h$. For any collection $(\eta_i)_{1\le i\le p}$ of symmetric second-order tensors, writing $M = \sum_{i=1}^{p} \eta_i \otimes \eta_i$, the upper trace bound is

$$\langle(\underline{\sigma} - \sigma^*)^{-1}, M\rangle \le \frac{1}{\theta_1}\langle(\underline{\sigma} - \sigma_1)^{-1}, M_s\rangle + \frac{1}{\theta_2}\langle(\underline{\sigma} - \sigma_2)^{-1}, M_h\rangle - \theta_1\theta_2\underline{g}\left(\frac{M_s}{\theta_1} - \frac{M_h}{\theta_2}\right), \tag{3.6}$$

where M_s and M_h are defined as in Theorem 3.1, and $\underline{g}$ is given by

$$\underline{g}(P) = \inf_{|k|=1}\langle f_{\underline{\sigma}}(k), P\rangle. \tag{3.7}$$

Furthermore, the trace bound (3.6) is optimal, i.e., for any M there is a single-layered microstructure that achieves equality in (3.6).

Proof of Theorem 3.1. To simplify the exposition we shall establish the trace bound (3.5) for a single energy, i.e., $M = \eta \otimes \eta$. Since $\sigma^* - \overline{\sigma}$ is positive, by Fenchel transform we have

$$\langle(\sigma^* - \overline{\sigma})^{-1}\eta, \eta\rangle = \sup_{\xi} 2\langle\xi, \eta\rangle - \langle(\sigma^* - \overline{\sigma})\xi, \xi\rangle. \tag{3.8}$$

By definition (0.11) of σ^*, (3.8) becomes

$$\begin{aligned}\langle(\sigma^* - \overline{\sigma})^{-1}\eta, \eta\rangle = {} & \sup_{\xi} 2\langle\xi, \eta\rangle + \langle\overline{\sigma}\xi, \xi\rangle \\ & - \inf_{\phi}\int_Q [\langle(\sigma(y) - \overline{\sigma})[\xi + e(\phi)], [\xi + e(\phi)]\rangle \\ & + \langle\overline{\sigma}[\xi + e(\phi)], [\xi + e(\phi)]\rangle]\, dy.\end{aligned} \tag{3.9}$$

Since $\sigma(y) - \overline{\sigma}$ is positive, by Fenchel transform we have

$$\langle(\sigma(y)-\overline{\sigma})[\xi+e(\phi)]\,,\,[\xi+e(\phi)]\rangle = \sup_{\varepsilon(y)} 2\langle[\xi+e(\phi)]\,,\,\varepsilon(y)\rangle - \langle(\sigma(y)-\overline{\sigma})^{-1}\varepsilon(y)\,,\,\varepsilon(y)\rangle\,. \tag{3.10}$$

Plugging (3.10) in (3.9) and commuting the supremum in ε and the infimum in ϕ (this is licit by a standard min-max principle since the right-hand side of (3.11) is concave in ε and convex in ϕ) yield

$$\begin{aligned}\langle(\sigma^* - \overline{\sigma})^{-1}\eta\,,\,\eta\rangle = {}& \sup_{\xi}\inf_{\varepsilon} 2\langle\xi\,,\,\eta\rangle - 2\int_Q \langle\varepsilon\,,\,\xi\rangle + \int_Q \langle(\sigma(y)-\overline{\sigma})^{-1}\varepsilon\,,\,\varepsilon\rangle \\ & - \inf_{\phi}\int_Q [2\langle\varepsilon\,,\,e(\phi)\rangle + \langle\overline{\sigma}e(\phi)\,,\,e(\phi)\rangle]\,dy\,.\end{aligned} \tag{3.11}$$

Applying again a standard min-max principle, we commute the supremum in ξ and the infimum in ε. Since (3.11) is linear in ξ, the supremum in ξ is replaced by a constraint on ε. We obtain

$$\langle(\sigma^* - \overline{\sigma})^{-1}\eta\,,\,\eta\rangle = \inf_{\varepsilon}\left[\int_Q \langle(\sigma(y)-\overline{\sigma})^{-1}\varepsilon\,,\,\varepsilon\rangle - \inf_{\phi}\int_Q (2\langle\varepsilon\,,\,e(\phi)\rangle + \langle\overline{\sigma}e(\phi)\,,\,e(\phi)\rangle)\right] \tag{3.12}$$

with the constraint

$$\int_Q \varepsilon(y)\,dy = \eta\,. \tag{3.13}$$

Since $\sigma(y)-\overline{\sigma}$ is a multiple of Λ_h in material one, and a multiple of Λ_s in material two, this further restricts $\varepsilon(y)$ to be a multiple of I_2 in material one, and trace-free in material two. Furthermore, one can get an inequality in (3.12) by choosing $\varepsilon(y)$ constant in each material: this, added to the constraint (3.13), uniquely determines $\varepsilon(y)$ as

$$\varepsilon(y) = \frac{\Lambda_h\eta}{\theta_1}\chi_1(y) + \frac{\Lambda_s\eta}{\theta_2}\chi_2(y)\,.$$

Consequently, from (3.12) we deduce

$$\begin{aligned}\langle(\sigma^* - \overline{\sigma})^{-1}\eta\,,\,\eta\rangle \le {}& \frac{1}{\theta_1}\langle(\sigma_1 - \overline{\sigma})^{-1}\Lambda_h\eta\,,\,\Lambda_h\eta\rangle + \frac{1}{\theta_2}\langle(\sigma_2 - \overline{\sigma})^{-1}\Lambda_s\eta\,,\,\Lambda_s\eta\rangle \\ & - \inf_{\phi}\int_Q \left[2\left\langle\left(\frac{\Lambda_h\eta}{\theta_1} - \frac{\Lambda_s\eta}{\theta_2}\right)\chi_1\,,\,e(\phi)\right\rangle + \langle\overline{\sigma}e(\phi)\,,\,e(\phi)\rangle\right]dy\,.\end{aligned} \tag{3.14}$$

As in Theorem 1.1, the last term in (3.14) (the so-called nonlocal term) can be computed by Fourier analysis and is bounded above by $\theta_1\theta_2\overline{g}(\theta_1^{-1}\Lambda_h\eta-\theta_2^{-1}\Lambda_s\eta)$, with $\overline{g}$ defined by (1.5). Thus, (3.14) gives the desired trace bound. To assert its optimality, we shall show that (3.14) is actually an *equality* for a single-layered microstructure. Indeed, take any k extremal in the definition (1.5) of $\overline{g}(\theta_1^{-1}\Lambda_h\eta - \theta_2^{-1}\Lambda_s\eta)$, and consider a microstructure made of layers of material one and two, in proportions θ_1 and θ_2 respectively, orthogonal to direction k. For such a microstructure, the nonlocal term in (3.14) is exactly equal to $\theta_1\theta_2\overline{g}(\theta_1^{-1}\Lambda_h\eta - \theta_2^{-1}\Lambda_s\eta)$, and it is well known that the strain $e(\phi)$ is constant in each material. Thus, the field $\varepsilon(y)$ is also constant in each material, and there was no restriction in passing from (3.12) to (3.14). This proves equality in (3.14) for that single-layered (in direction k) microstructure. Q.E.D.

The proof of Theorem 3.2 is similar and can safely be left to the reader.

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