Discrete polytopal complexes for fluid mechanics, electromagnetism and solid mechanics

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The well-posedness of PDE models involving "incomplete" differential operators, such as curl and div, relies on fine calculus properties related to these operators. These properties are translated in the concept of Hilbert complexes and their cohomology, the most common of which being the de Rham complex. On a domain Ω, this complex is the following sequence of space and operators:

$$\mathbb{R} \longrightarrow H(\mathbf{grad};\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \longrightarrow \{0\},$$

where, for $\mathcal{D} = \mathbf{grad}, \mathbf{curl}, \mathrm{div}$, we denote by $H(\mathcal{D};\Omega)$ the space of $L^2(\Omega)$-integrable (scalar or vector-valued) functions v such that $\mathcal{D}v$ is also L^2-integrable. The standard calculus formulas $\mathbf{curl}\,\mathbf{grad} = 0$ and $\mathrm{div}\,\mathbf{curl} = 0$ translate into the fact that, in this sequence, the image of an operator is included in the kernel of the next one; the cohomology of the complex describes the "gap" at each level between these images and kernels.

The cohomology of the de Rham complex is related to the well-posedness of some models involving $\mathbf{grad}$, $\mathbf{curl}$ and/or div, and designing stable numerical discretisations of these models requires to create discrete versions of this complex. In the finite element setting, for example, the Lagrange–Nédélec–Raviart-Thomas spaces form a discrete complex with the same cohomology as the continuous one, which is therefore suitable to discretise the considered models. Finite element methods, however, have restricted flexibility in terms of admissible meshes, which limits their capacity for local refinement procedures or to capture detailed geometries at a reasonable computational cost.

In this presentation, I will talk about the recent theory of polytopal complexes, discrete versions of the de Rham complex that are applicable on generic polytopal meshes. I will focus on the Discrete De Rham (DDR) complex

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathbf{grad},h}} \underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\},$$

in which spaces are made of vectors of polynomials attached to mesh entities (vertices, edges, faces, elements), and discrete operators representing $\mathbf{grad}, \mathbf{curl}, \mathrm{div}$ are constructed mimicking integration-by-parts formulae. I will briefly present the key ideas involved in the design of this polytopal complex, as well as its homological and analytical properties. I will then show how it can be used to discretise various models, including the Stokes/Navier–Stokes equations, electromagnetism equations, and even plate problems. For each of these models, the properties of the DDR complex leads to significant outcomes, not only in terms of the flexibility of the admissible meshes, but also in terms of robustness with respect to physical parameters.