MASTER M2 NUMERICAL ANALYSIS AND P.D.E.s
UNIVERSITE PARIS 6 - ECOLE POLYTECHNIQUE
Course of G. Allaire, "Homogenization"
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The evaluation procedure will pay attention to the quality of the dissertation and most particularly to the clarity and readability in the proposed argumentation. As usual the subscript # denotes spaces of periodic functions. Throughout the problem C denotes a positive constant which does not depend on ϵ.

The goal of this problem is to study a model of reactive transport in porous media. We denote by Ω a smooth bounded open set in $\mathbb{R}^N$ which represents a porous medium. The domain Ω is tiled by a square periodic tiling of size ϵ. The cubes of this tiling $(Y_p^\epsilon)_{1\le p\le n(\epsilon)}$, with $n(\epsilon) \approx |\Omega|\epsilon^{-N}$, are all equal, up to a translation, to $[0,\epsilon]^N$. Thus, after translation each cube is homothetic of ratio ϵ to the unit cell $Y = [0,1]^N$ which is decomposed in a fluid part Y_f and a solid part Y_s, separated by an interface Γ, with $Y = Y_f \cup Y_s$. Using the same notation in each cube, $Y_p^\epsilon = Y_{f,p}^\epsilon \cup Y_{s,p}^\epsilon$, the fluid part of the porous medium Ω_ϵ (assumed to be smooth and connected) is defined by

$$\Omega_\epsilon = \Omega \setminus \left(\cup_{p=1}^{n(\epsilon)} Y_{s,p}^\epsilon \right).$$

The interface Γ_ϵ between the fluid and solid parts of the porous medium is defined by

$$\Gamma_\epsilon = \partial\Omega_\epsilon \setminus \partial\Omega.$$

The fluid part Ω_ϵ (as indicated by its name) is filled with an incompressible fluid with a given velocity $b\left(\frac{x}{\epsilon}\right)$ where $b(y) \in C^1_\#(Y_f)^N$ is a smooth periodic vector field satisfying

$$\mathrm{div}_y b(y) = 0 \text{ in } Y_f, \quad b(y) = 0 \text{ on } \Gamma.$$

The molecular diffusion tensor in the fluid is $A\left(\frac{x}{\epsilon}\right)$ where $A(y) \in L^\infty_\#(Y_f)^{N\times N}$ is a periodic coercive symmetric matrix satisfying, for $0 < \alpha \le \beta$,

$$\alpha|\xi|^2 \le A(y)\xi\cdot\xi \le \beta|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^N, y \in Y_f.$$

A chemical species is dissolved in the fluid and can react with the solid walls by absorption/desorption. Its concentration in the fluid is denoted by $u_\epsilon(t,x)$ while its concentration on the fluid/solid interface is denoted by $v_\epsilon(t,x)$. The initial concentrations are $u_{in}(x)$ and $v_{in}(x)$, respectively, which belong to $H^1_0(\Omega)$.

Denoting by $k > 0$ and $K > 0$ two positive chemical constants and by n the unit exterior normal to Ω_ϵ, the model is a system of evolution equations for these concentrations:

$$
\begin{cases}
\dfrac{\partial u_\epsilon}{\partial t} + b\left(\dfrac{x}{\epsilon}\right) \cdot \nabla u_\epsilon - \operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right) \nabla u_\epsilon\right) = 0 & \text{in } \Omega_\epsilon \times (0,T), \\
-A\left(\dfrac{x}{\epsilon}\right) \nabla u_\epsilon \cdot n = \epsilon \dfrac{\partial v_\epsilon}{\partial t} & \text{on } \Gamma_\epsilon \times (0,T), \\
\dfrac{\partial v_\epsilon}{\partial t} = \dfrac{k}{\epsilon^2}\left(u_\epsilon - \dfrac{v_\epsilon}{K}\right) & \text{on } \Gamma_\epsilon \times (0,T), \\
u_\epsilon = 0 & \text{on } \partial\Omega \times (0,T), \\
u_\epsilon(x,0) = u_{in}(x) & \text{in } \Omega_\epsilon, \\
v_\epsilon(x,0) = v_{in}(x) & \text{on } \Gamma_\epsilon.
\end{cases}
\tag{1}
$$

In system (1) the second line is a boundary condition expressing the conservation of mass of the species at the fluid/solid interface, while the third line is an ordinary differential equation governing the evolution of the concentration on the solid walls.

Part I

In this part the method of formal two-scale asymptotic expansions is applied in order to find the homogenized problem for (1). It is thus assumed that the solution (u_ϵ, v_ϵ) can be written as a series

$$
u_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i u_i(t,x,\frac{x}{\epsilon}), \quad v_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i v_i(t,x,\frac{x}{\epsilon})
$$

with $u_i(t,x,y)$ and $v_i(t,x,y)$ Y-periodic functions with respect to the variable $y \in Y$.

1. Write the equations in the cell Y_f and the boundary conditions on the solid wall Γ satisfied by u_0, v_0, u_1, v_1, and u_2, v_2. Show in particular that each v_i, for $0 \le i \le 2$, can be explicitly computed in terms of the u_j's with $0 \le j \le i$.

2. Let $g(y) \in L^2_\#(Y_f)$ and $h(y) \in L^2(\Gamma)$. Prove that the following problem admits a unique solution in $H^1_\#(Y_f)/\mathbb{R}$

$$
\begin{cases}
-\operatorname{div}_y \left(A(y)\nabla_y w\right) = g & \text{in } Y_f \\
A(y)\nabla_y w \cdot n = h & \text{on } \Gamma \\
y \to w(y) \ Y\text{-periodic}
\end{cases}
\tag{2}
$$

if and only if the data satisfy the compatibility condition

$$
\int_{Y_f} g(y)dy + \int_\Gamma h(y)ds = 0.
$$

3. Deduce that $u_0(t,x,y)$ does not depend on y, and that $u_1(t,x,y)$ can be written in terms of the gradient of u_0 multiplied by solutions of a family of cell problems which should precisely be defined.

4. Write the necessary and sufficient compatibility condition for solving for $u_2(t,x,y)$. Deduce from it the homogenized equation as well as the boundary condition on $\partial\Omega$ (it is not required to find the initial condition).

Part II

This part is devoted to the proof of a priori estimates for (1). We denote by V_ϵ the subspace of $H^1(\Omega_\epsilon)$ made of functions vanishing on $\partial\Omega$. In the sequel we shall assume that there exists a linear continuous extension operator from V_ϵ into $H^1_0(\Omega)$, denoted by X_ϵ such that, for any $\phi \in V_\epsilon$

$$X_\epsilon\phi = \phi \text{ in } \Omega_\epsilon, \text{ and } \|\nabla(X_\epsilon\phi)\|_{L^2(\Omega)^N} \leq C\|\nabla\phi\|_{L^2(\Omega_\epsilon)^N}.$$

By a slight abuse of notations any function ϕ will be identified with its extension $X_\epsilon\phi$.

1. Show that (1) admits the variational formulation

$$\int_{\Omega_\epsilon} \frac{\partial u_\epsilon}{\partial t}\phi\, dx + \frac{\epsilon}{K}\int_{\Gamma_\epsilon} \frac{\partial v_\epsilon}{\partial t}\psi\, ds + a\Big((u_\epsilon, v_\epsilon), (\phi, \psi)\Big) = 0 \tag{3}$$

   for any test function $(\phi,\psi) \in L^2((0,T);V_\epsilon) \times L^2((0,T)\times\Gamma_\epsilon)$, with the bilinear form

$$\begin{aligned} a\Big((u_\epsilon, v_\epsilon), (\phi, \psi)\Big) &= \int_{\Omega_\epsilon} \Big(b\left(\frac{x}{\epsilon}\right)\cdot\nabla u_\epsilon\phi + A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\cdot\nabla\phi\Big)dx \\ &+\frac{k}{\epsilon}\int_{\Gamma_\epsilon}\left(u_\epsilon - \frac{v_\epsilon}{K}\right)\left(\phi - \frac{\psi}{K}\right)ds. \end{aligned} \tag{4}$$

2. Prove that, for a fixed ϵ, the bilinear form (4), integrated in time from 0 to T, is coercive on the space $L^2((0,T);V_\epsilon) \times L^2((0,T)\times\Gamma_\epsilon)$. We shall assume that it is enough to prove that (1) admits a unique solution (u_ϵ, v_ϵ) in $\big\{L^2((0,T);V_\epsilon) \cap C([0,T];L^2(\Omega_\epsilon))\big\} \times C([0,T];L^2(\Gamma_\epsilon))$.

3. Prove the existence of $C > 0$ such that, for any $w \in V_\epsilon$,

$$\|w\|_{L^2(\Omega_\epsilon)} \leq C\big(\epsilon\|\nabla w\|_{L^2(\Omega_\epsilon)} + \sqrt{\epsilon}\|w\|_{L^2(\Gamma_\epsilon)}\big),$$

   and

$$\sqrt{\epsilon}\|w\|_{L^2(\Gamma_\epsilon)} \leq C\big(\|w\|_{L^2(\Omega_\epsilon)} + \epsilon\|\nabla w\|_{L^2(\Omega_\epsilon)}\big).$$

4. Integrating in time (3) with $(\phi, \psi) = (u_\epsilon, v_\epsilon)$, prove the existence of $C > 0$ such that

$$\begin{aligned} &\|u_\epsilon\|_{L^\infty((0,T);L^2(\Omega_\epsilon))} + \sqrt{\epsilon}\|v_\epsilon\|_{L^\infty((0,T);L^2(\Gamma_\epsilon))} + \|\nabla u_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} \\ &+\sqrt{\epsilon}\|\epsilon^{-1}(u_\epsilon - \tfrac{v_\epsilon}{K})\|_{L^2((0,T)\times\Gamma_\epsilon)} \le C\left(\|u_{in}\|_{L^2(\Omega)} + \|v_{in}\|_{H^1(\Omega)}\right). \end{aligned} \tag{5}$$

5. Assuming u_ϵ to be given, write explicitly the solution v_ϵ of the ordinary differential equation on the third line of (1). Deduce from this formula, since $v_{in} \in H_0^1(\Omega)$ and $u_\epsilon \in L^2((0,T);V_\epsilon)$, that v_ϵ has a natural extension in $L^2((0,T);V_\epsilon)$ which satisfies

$$\|v_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} \le C\left(\|u_\epsilon\|_{L^\infty((0,T);L^2(\Omega_\epsilon))} + \epsilon\|v_{in}\|_{L^2(\Omega)}\right),$$

and

$$\epsilon\|\nabla v_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} \le C\left(\|\nabla u_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} + \epsilon\|v_{in}\|_{H_0^1(\Omega)}\right).$$

Hint: rely on the fact that $\displaystyle\int_0^t \epsilon^{-2} e^{\frac{2k(s-t)}{K\epsilon^2}}\,ds \le C$.

Part III

In this part a rigorous convergence theorem is proved by using the method of two-scale convergence. In the present unsteady context we recall the main results of two-scale convergence: for any sequence $z_\epsilon(t,x)$, uniformly bounded in $L^2((0,T)\times\Omega_\epsilon)$, there exist a subsequence ϵ and a limit $z_0(t,x,y) \in L^2((0,T)\times\Omega\times Y_f)$ such that

$$\lim_{\epsilon\to 0}\int_0^T\int_{\Omega_\epsilon} z_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right)dt\,dx = \int_0^T\int_\Omega\int_{Y_f} z_0(t,x,y)\phi\left(t,x,y\right)dt\,dx\,dy$$

for any smooth test function $\phi(t,x,y)$ which is Y-periodic in y. We also recall the notion of two-scale convergence for sequences of functions defined on the boundary Γ_ϵ: let $\zeta_\epsilon(t,x)$ be a sequence satisfying

$$\sqrt{\epsilon}\|\zeta_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} \le C,$$

there exist a subsequence ϵ and a limit $\zeta_0(t,x,y) \in L^2((0,T)\times\Omega\times\Gamma)$ such that

$$\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{\Gamma_\epsilon}\zeta_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right)dt\,dx = \int_0^T\int_\Omega\int_\Gamma \zeta_0(t,x,y)\phi\left(t,x,y\right)dt\,dx\,dy$$

for any smooth test function $\phi(t,x,y)$ which is Y-periodic in y.

1. Recall the results of the course on the structure of the two-scale limits of a sequence z_ϵ and its gradient when z_ϵ is uniformly bounded in $L^2((0,T);V_\epsilon)$. Same question for a sequence ζ_ϵ such that

$$\|\zeta_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} + \epsilon\|\nabla\zeta_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} + \sqrt{\epsilon}\|\zeta_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} \leq C.$$

   In each case one should make the connection between the limits obtained by the two types of convergence defined above.

2. Deduce from (5) (and more precisely from the estimate on $\epsilon^{-1}(u_\epsilon - \frac{v_\epsilon}{K})$) that the two-scale limits of u_ϵ and v_ϵ coincide on Γ.

3. Multiplying the equation for u_ϵ in (1) by a test function $\phi(t,x)+\epsilon\phi_1(t,x,\frac{x}{\epsilon})$, find the two-scale homogenized problem (under its variational form).

4. Deduce from the previous question the cell problem and show that it admits a unique solution in a functional space independent of time. Deduce the homogenized problem as well, the initial condition of which has to be clearly recovered. Prove that the homogenized problem is well-posed, i.e., indicate why the homogenized diffusion tensor is coercive. What can be said on the convergence of the entire sequence ?