MASTER M2 NUMERICAL ANALYSIS AND P.D.E.s
UNIVERSITE PARIS 6 - ECOLE POLYTECHNIQUE
Course of G. Allaire and F. Alouges, "Homogenization"
January 10th, 2013 (3 hours)

Important: The evaluation procedure will pay attention to the quality of the dissertation and most particularly to the clarity and readability of the proposed argumentation. The subject is composed of 3 Parts. Questions II.5 to II.9 must ONLY be treated by students from mathematics and NOT by those from mechanics (M4S). We are aware of the length and difficulty of the subject. An indicative number of points is provided which simply reflects the relative difficulty of the different parts.

As usual the subscript # denotes spaces of periodic functions. Throughout the problem C denotes a positive constant which does not depend on ϵ.

The goal of this problem is to study the influence of a zero-order term in the homogenization process of a diffusion equation. Such a zero-order term models a reaction or absorption process. Three different scalings are studied.

Let Ω be a smooth bounded open set of $\mathbb{R}^N$ which represents a periodic porous medium. Let $\epsilon > 0$ be the small parameter which defines the periodicity of the coefficients. Let $Y = [0,1]^N$ be the unit cell. The diffusion tensor is $A\left(\frac{x}{\epsilon}\right)$ where $A(y) \in L^\infty_\#(Y)^{N\times N}$ is a periodic coercive symmetric matrix satisfying, for $0 < \alpha \leq \beta$,

$$\alpha|\xi|^2 \leq A(y)\xi\cdot\xi \leq \beta|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^N, y \in Y.$$

A chemical species is diffused in the domain and can react with the underlying medium by absorption/desorption. The reaction coefficient is $c\left(\frac{x}{\epsilon}\right)$ where $c(y) \in L^\infty_\#(Y)$ is a periodic bounded coefficient (with no specific sign). The species concentration is denoted by $u_\epsilon(t,x)$. The initial concentration is $u_{in}(x) \in H^1_0(\Omega)$. The model is an evolution equation for this concentration:

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + \dfrac{1}{\epsilon^\gamma} c\left(\dfrac{x}{\epsilon}\right) u_\epsilon - \operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon\right) = 0 & \text{in } \Omega\times(0,T), \\ u_\epsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ u_\epsilon(x,0) = u_{in}(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\gamma = 0, 1, 2$ is an integer which will change its values in the three different parts of the present exam.

Part I (4 points)

In this part, which is a straightforward application of what has been seen in class, we study the simplest case, $\gamma = 0$, of model (1). We apply the method of formal two-scale asymptotic expansions to find the homogenized problem for (1). It is thus assumed that the solution u_ϵ can be written as a series

$$u_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i u_i(t,x,\frac{x}{\epsilon}) \tag{2}$$

with Y-periodic functions $y \to u_i(t,x,y)$.

1. Write the equations in the cell Y satisfied by u_0, u_1, and u_2.
2. Recall the so-called Fredholm alternative or compatibility condition on the source term $g(y) \in L^2_\#(Y)$ for the existence and uniqueness of a solution $w \in H^1_\#(Y)/\mathbb{R}$ (up to an additive constant) of

$$\begin{cases} -\mathrm{div}_y\,(A(y)\nabla_y w) = g & \text{in } Y \\ y \to w(y)\ Y\text{-periodic.} \end{cases} \tag{3}$$

3. Deduce that $u_0(t,x,y)$ does not depend on y, and that $u_1(t,x,y)$ can be written in terms of the gradient of u_0 multiplied by solutions of a family of cell problems which should precisely be defined.
4. Write the necessary and sufficient compatibility condition for solving for $u_2(t,x,y)$. Deduce from it the homogenized equation. What is the boundary condition on $\partial\Omega$ and the initial condition ?

Part II
(II.1 to II.4:10 points
II.5 to II.9: 8 points)

In this part we consider the case, $\gamma = 1$, of model (1) with the additional assumption

$$\int_Y c(y)\,dy = 0. \tag{4}$$

We again use the formal method of two-scale asymptotic expansions, i.e., we assume that the solution u_ϵ can be written as the series (2).

1. Write the equations in the cell Y satisfied by u_0, u_1, and u_2.
2. Deduce that $u_0(t,x,y)$ does not depend on y, and that $u_1(t,x,y)$ can be written as

$$u_1(t,x,y) = w_0(y)u_0(t,x) + \sum_{k=1}^{N} w_k(y)\frac{\partial u_0}{\partial x_k}(t,x)$$

where the $(w_k)_{1\le k\le N}$ are the solutions of the usual cell problems and w_0 is the solution of a new cell problem which should precisely be defined. Show that condition (4) is necessary for solving in w_0.

3. Discuss the numerical approximation of w_0 with a P^1 finite element technique on the unit cell Y. In particular, assuming h is the space step of a family of meshes of Y, give the exact and approximate variational formulations, and the error estimate between the exact solution w_0 and approximate one $w_{0,h}$ with respect to h. For this question one can assume that $A \in \mathcal{C}^\infty(Y)$.

4. Write the necessary and sufficient compatibility condition for solving for $u_2(t,x,y)$. Show that the homogenized equation is of the type

$$\frac{\partial u_0}{\partial t} + c^* u_0 + b^* \cdot \nabla_x u_0 - \mathrm{div}_x \left(A^* \nabla_x u_0\right) = 0 \quad \text{in } \Omega \times (0,T),$$

with precise formulas for c^*, b^* and A^*. By using the cell equations, prove that $b^* = 0$ and $c^* \leq 0$ (it shows in particular, that there is no convective term in the homogenized equation).

The rest of Part II must NOT be treated by students from M4S (mechanics). They must skip directly to Part III.

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5. We now turn to the rigorous justification of the homogenization process. To simplify the analysis, we first apply the Laplace transform to (1). For a positive parameter $p > 0$ we define

$$\hat{u}_\epsilon(x) = \int_0^{+\infty} e^{-pt} u_\epsilon(t,x)\, dt,$$

and we assume that, for sufficiently large p, the limit as t goes to $+\infty$ of $e^{-pt}u_\epsilon(t,x)$ is zero in $H_0^1(\Omega)$. Prove that (1) yields

$$\left\{ \begin{array}{ll} p\,\hat{u}_\epsilon + \dfrac{1}{\epsilon} c\left(\dfrac{x}{\epsilon}\right) \hat{u}_\epsilon - \mathrm{div}\left(A\left(\dfrac{x}{\epsilon}\right) \nabla \hat{u}_\epsilon\right) = u_{in} & \text{in } \Omega, \\ \hat{u}_\epsilon = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{5}$$

We shall merely justify the homogenization of (5) and not that of (1).

6. Prove that there exists a vector field $b(y) \in L^\infty_\#(Y)^N$ such that

$$\left\{ \begin{array}{ll} -\mathrm{div}_y b(y) = c(y) & \text{in } Y \\ y \to b(y)\ Y\text{-periodic.} & \end{array} \right. \tag{6}$$

(Hint: look for $b = \nabla_y \phi$.)

7. Prove that, for $p > 0$ sufficiently large, there exists a unique solution of (5) in $H_0^1(\Omega)$ and that the sequence $\hat{u}_\epsilon(x)$ is uniformly bounded in $H_0^1(\Omega)$. (Hint: Use Lax-Milgram lemma.)

8. Apply the two-scale convergence method to (5) and prove that the sequence $\hat{u}_\epsilon$ converges (in a sense to be made precise) to a limit $\hat{u}_0$ which is the solution of the homogenized problem

$$\begin{cases} p\,\hat{u}_0 + c^*\hat{u}_0 - \operatorname{div}_x\left(A^*\nabla_x \hat{u}_0\right) = u_{in} & \text{in } \Omega, \\ \hat{u}_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

with the same coefficients c^* and A^* as in question II.3.

9. We now consider for $u \in H_0^1(\Omega)$

$$\mathcal{E}_\epsilon(u) = \frac{1}{2}\int_\Omega \left(pu^2(x) + \frac{1}{\epsilon}c\left(\frac{x}{\epsilon}\right)u^2(x)\right)dx + \frac{1}{2}\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u(x)\cdot\nabla(x)\,dx$$

and

$$\mathcal{E}_0(u) = \frac{1}{2}\int_\Omega (p+c^*)\,u^2(x)\,dx + \frac{1}{2}\int_\Omega A^*\nabla u(x)\cdot\nabla(x)\,dx.$$

We also consider the minimization problems

$$(P_\epsilon) \quad \min_{u\in H_0^1(\Omega)} \mathcal{E}_\epsilon(u)\,,$$

and

$$(P_0) \quad \min_{u\in H_0^1(\Omega)} \mathcal{E}_0(u)\,,$$

The aim of the next questions is to show that the family of minimization problems $(P_\epsilon)_\epsilon$ $\Gamma-$converges to P_0 in $H^1(\Omega)$ for the weak H^1 convergence.

(a) Show that a family $(u_\epsilon)_\epsilon$ of $H_0^1(\Omega)$ maps, which is such that $\mathcal{E}_\epsilon(u_\epsilon) < C$ is bounded in $H^1(\Omega)$. (Hint: use $c = -\operatorname{div} b$ and integrate by parts the corresponding term.)

(b) Let $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ and $(u_\epsilon)_\epsilon$ a family of $H_0^1(\Omega)$ maps which is such that

$$u_\epsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega)\,.$$

Show that

$$\liminf_{\epsilon\to 0} \mathcal{E}_\epsilon(u_\epsilon) \geq \mathcal{E}_0(u_0).$$

(c) Conversely, build a family $(v_\epsilon)_\epsilon$ a family of $H_0^1(\Omega)$ maps which is such that

$$v_\epsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega)$$

and which satisfies

$$\lim_{\epsilon\to 0} \mathcal{E}_\epsilon(u_\epsilon) = \mathcal{E}_0(u_0).$$

(d) Conclude.

Part III
(III.1 to III.3:6 points
III.4 to III.5: optional 4 points)

In this part we consider the case, $\gamma = 2$, of model (1) with no assumption on the coefficient $c(y)$. We again use the formal method of two-scale asymptotic expansions but with a different ansatz which is

$$u_\epsilon(t,x) = e^{-\epsilon^{-2}\lambda t} \sum_{i=0}^{+\infty} \epsilon^i u_i(t,x,\frac{x}{\epsilon}) \tag{7}$$

where $\lambda \in \mathbb{R}$ is a parameter to be determined and with Y-periodic functions $y \to u_i(t,x,y)$.

1. Show that λ and u_0 must satisfy

$$\left\{ \begin{array}{l} c(y)u_0 - \mathrm{div}_y\,(A(y)\nabla_y u_0) = \lambda\, u_0 \quad \text{in } Y \\ y \to u_0(t,x,y) \ Y\text{-periodic.} \end{array} \right. \tag{8}$$

System (8) is interpreted as a spectral problem: λ is an eigenvalue and u_0 is a (non-zero) corresponding eigenfunction (or eigenvector). As usual (t,x) are just parameters in (8) where the only meaningfull variable is $y \in Y$.

We now recall some basic results on spectral problems that we shall admit in the sequel. System (8) admits an infinite number of independent solutions, eigenvalues $\lambda_i \in \mathbb{R}$ and eigenfunctions $\psi_i(y) \in H^1_\#(Y)$ (as usual, eigenfunctions are defined up to a multiplicative constant), labeled in increasing order

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le ...$$

We normalize the eigenfunctions so that $\|\psi_i\|_{L^2(Y)} = 1$. Furthermore, the first eigenfunction $\psi_1(y)$ (corresponding to the smallest eigenvalue λ_1) is the only one which is positive in the unit cell Y (with a suitable multiplicative constant). From a physical point of view, if we interpret the eigenfunctions $\psi_i(y)$ as concentrations (which obviously take non-negative values), only the first one makes physical sense. Therefore, from now on, we admit that the solution of (8) is precisely

$$\lambda = \lambda_1, \quad u_0(t,x,y) = u(t,x)\,\psi_1(y),$$

where $u(t,x)$ is the (so far unknown) multiplicative constant of the eigenfunction (constant in y but not in the other variables).

2. For $g(y) \in L^2_\#(Y)$ we consider the following problem

$$\left\{ \begin{array}{l} c(y)w - \mathrm{div}_y\,(A(y)\nabla_y w) - \lambda_1\, w = g \quad \text{in } Y \\ y \to w(y) \ Y\text{-periodic.} \end{array} \right. \tag{9}$$

Show that if $w \in H^1_\#(Y)$ is a solution, then $w + C\psi_1$ is a solution too, whatever the constant C. Prove that a necessary condition for the existence of a solution of (9) is that

$$\int_Y g(y)\,\psi_1(y)\,dy = 0. \tag{10}$$

(Hint: multiply the equation by ψ_1 and integrate by parts.)

From now on, we admit that (10) is a sufficient and necessary condition for the existence of a solution $w \in H^1_\#(Y)$ of (9), which is unique up to the addition of a multiple of ψ_1 (this is again called Fredholm alternative).

3. Coming back to the ansatz (7) show that, for $i = 1$ and 2, u_i must satisfy

$$\begin{cases} c(y)u_i - \mathrm{div}_y\,(A(y)\nabla_y u_i) - \lambda_1\,u_i = g_i & \text{in } Y \\ y \to u_i(t,x,y)\ Y\text{-periodic,} \end{cases} \tag{11}$$

with source terms given by

$$g_1 = \mathrm{div}_y(A(y)\nabla_x u_0) + \mathrm{div}_x(A(y)\nabla_y u_0)$$

and

$$g_2 = -\frac{\partial u_0}{\partial t} + \mathrm{div}_x\,(A(y)(\nabla_x u_0 + \nabla_y u_1)) + \mathrm{div}_y(A(y)\nabla_x u_1).$$

The last two questions of this part are optional.

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4. Check that g_1 satisfies the compatibility condition (10) and show that

$$u_1(t,x,y) = \sum_{k=1}^{N} z_k(y)\frac{\partial u}{\partial x_k}(t,x)$$

where the $(z_k)_{1\le k\le N}$ are the solutions of some new cell problems which should precisely be defined.

5. Write the necessary and sufficient compatibility condition for g_2 and deduce that the homogenized equation is of the type

$$\frac{\partial u}{\partial t} - \mathrm{div}_x\,(A^*\nabla_x u) = 0 \qquad \text{in } \Omega\times(0,T),$$

with a precise formula for A^* depending on the (z_k).