A REMARK ON TRACE BOUNDS FOR ELASTIC COMPOSITE MATERIALS

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ABSTRACT

This paper is concerned with optimal bounds for effective properties of two-phase, linearly elastic, composite materials, at given volume fractions. By convex duality applied to the well-known Hashin-Shtrikman bounds on sums of energies, Milton and Kohn introduced the so-called trace bounds on the effective Hooke's law. A generalization of this trace bound is presented here, which is related to differences (instead of sums) of energies. This new trace bound is optimal : it is saturated for a single layering of the two constituents.

1 Introduction

In this paper we study the effective, or homogenized, properties of composite materials which arise by mixing two linearly elastic components in given proportions. We denote by σ_1 and σ_2 the fourth-order tensors of effective moduli (Hooke's law), and by θ_1 and θ_2 the prescribed volume fractions of the two phases. Throughout this paper, the only assumption placed on σ_1 and σ_2 is that they are well ordered, i.e. for any strain η (a symmetric second-order tensor) they satisfy

$$< \sigma_1\eta, \eta > \leq < \sigma_2\eta, \eta > . \tag{1}$$

The macroscopic properties of a composite material are described by its (possibly anisotropic) Hooke's law σ^* which depends on the microgeometry of the mixture as well as on the elastic properties of the components. Since the microstructure is usually unknown, we are interested in bounding these effective properties. For example, for any strain η, it is well-known that σ^* must satisfy the arithmetic and harmonic mean bounds :

$$< (\theta_1\sigma_1^{-1} + \theta_2\sigma_2^{-1})^{-1}\eta, \eta > \leq < \sigma^*\eta, \eta > \leq < (\theta_1\sigma_1 + \theta_2\sigma_2)\eta, \eta > . \tag{2}$$

However, these bounds are also known to be not optimal, i.e. for most choices of η it is impossible to find a microstructure which saturates either inequality.

Since the pioneering work of Hashin and Shtrikman [7], optimal bounds are known which improve 2. By *optimal bounds*, we mean a pair of functions f_+, f_- depending on $(\sigma_1, \sigma_2, \theta_1, \theta_2, \eta)$ such that

$$f_- \ \leq < \sigma^* \eta, \eta > \leq \ f_+, \tag{3}$$

and such that each inequality can be saturated by a microstructure which depends on the strain η (for details, or explicit expressions for $f_\pm$, see e.g. [1]). The bounds 3 have also been extended to *sums* of energies for different strains $\eta_1, \eta_2, ..., \eta_p$

$$f_-^p \ \leq \ \sum_{i=1}^{p} < \sigma^* \eta_i, \eta_i > \leq \ f_+^p. \tag{4}$$

As in the case of a single energy, they are known to be optimal for a special choice of the microstructure, namely sequential laminations of the two components [4]. By convex duality (i.e. by applying the Fenchel, or Legendre, transform to 4), Milton and Kohn [12] obtained the so-called trace bounds.

Let us present, in the notations of [1], the lower trace bound. For a collection $\xi_1, \xi_2, ..., \xi_p$ of stresses (symmetric second-order tensors too), writing $M = \sum_{i=1}^{p} \xi_i \otimes \xi_i$, it reads

$$\theta_2 < (\sigma^* - \sigma_1)^{-1} : M > \leq < (\sigma_2 - \sigma_1)^{-1} : M > + \theta_1 g_-(M), \tag{5}$$

where $g_-(M)$ is called the non-local term (a kind of two-point correlation function of the microstructure). It is defined by

$$g_-(M) \ = \ \sup_{|k|=1} < f_{\sigma_1}(k) : M >, \tag{6}$$

where $f_{\sigma_1}(k)$ is a degenerate Hooke's law depending on σ_1 and on a direction k (see 12, 14 below). An interesting feature of the lower trace bound 5 is that optimality is achieved for a very simple microstructure. Indeed, a single layering of the two components (in the direction of one of the vector k which is optimal in 6) is enough to saturate 5. This is in contrast with the bound 4 which requires an iterative procedure of layering for achieving optimality. Of course, a similar upper trace bound holds

$$\theta_1 < (\sigma_2 - \sigma^*)^{-1} : M > \leq < (\sigma_2 - \sigma_1)^{-1} : M > + \theta_2 g_+(M), \tag{7}$$

where $g_+(M)$ is another non-local term depending on σ_2.

The goal of this paper is to give a generalization of the above trace bounds. Denoting by $(\xi_1^-, \xi_2^-, ..., \xi_{p^-}^-)$, and $(\xi_1^+, \xi_2^+, ..., \xi_{p^+}^+)$, two collections of stresses, and writing $M^- = \sum_{i=1}^{p^-} \xi_i^- \otimes \xi_i^-$ and $M^+ = \sum_{i=1}^{p^+} \xi_i^+ \otimes \xi_i^+$, our main result reads

$$\begin{aligned} \theta_2^2 < (\sigma^* - \sigma_1)^{-1} : M^- > + \theta_1^2 < (\sigma_2 - \sigma^*)^{-1} : M^+ > \leq \\ < (\sigma_2 - \sigma_1)^{-1} : (\theta_1 M^+ + \theta_2 M^-) > \ + \theta_1 \theta_2 g(M^-, M^+), \end{aligned} \tag{8}$$

where $g(M^-, M^+)$ is the non-local term defined in terms of the degenerate Hooke's law $f_{\sigma_1}(k)$ and $f_{\sigma_2}(k)$ by

$$g(M^-, M^+) \;=\; \sup_{|k|=1} \Big(< f_{\sigma_1}(k) : M^- > - < f_{\sigma_2}(k) : M^+ > \Big) . \tag{9}$$

Optimality is also achieved in the trace bound 8 by a single layering of the two components. It is worth noticing that 8 degenerates in one of the previous trace bounds 5 or 7 when $M^- = 0$ or $M^+ = 0$. However, for most choices of M^-, M^+ it is not a trivial bound in the sense that it is not a linear combination of 5 and 7.

The proof of 8 is given in section 2, where, by convex duality, a bound, similar to 4, is derived for *differences* of energies. However, optimality of this latter bound is unknown (we believe it is usually not optimal). Finally in section 3, we discuss various motivation of the trace bounds, including the Hashin-Shtrikman bounds on the moduli of an isotropic composite material, the G-closure problem, and the relaxation of optimal design problems. Unfortunately, we must acknowledge that the new bound 8 yields little progress in these areas.

A final word of caution about our notations : we denote by $< .,. >$, and $< . : . >$, the inner product between second order tensors, and fourth order tensors, respectively. Let us remark that the inner product between M^1 and M^2 is also the trace of the product M^1M^2 :

$$< M^1 : M^2 > = \; Tr(M^1M^2) \;=\; \sum_{i,j,k,l=1}^{N} M^1_{ijkl} M^2_{ijkl}, \tag{10}$$

where the trace of a fourth-order tensor is defined by $Tr(M) = \sum_{i,j=1}^{N} M_{ijij}$. This explains why Milton and Kohn called the bounds 5 and 7 *trace bounds*.

2 Derivation of the trace bound

The new trace bound 8 is derived by means of the well-known Hashin-Shtrikman variational principle. Throughout this section, we follow the notations of [1] to which we refer for any further details. To make life easier, we establish it in the special case of single energies $M^+ = \xi^+ \otimes \xi^+$ and $M^- = \xi^- \otimes \xi^-$ (the generalization to sums of energies is straightforward albeit tedious).

We begin by introducing the so-called degenerate Hooke's laws $f_{\sigma_i}(k)$ which play an important role in the definition of the non-local term in trace bounds. Let $V(k)$ be the space of Fourier transform of strains at frequency k :

$$V(k) = \{ k \otimes v + v \otimes k \quad with v \in \Re^N \}. \tag{11}$$

For any tensor ξ, we denote by $\Pi_V \xi$ the orthogonal projection of ξ on a linear subspace V. Then, for a given frequency k and Hooke's law σ_i, the symmetric fourth-order tensor $f_{\sigma_i}(k)$ is defined by the quadratic form

$$< f_{\sigma_i}(k)\,\xi,\xi > = |\ \Pi_{\sigma_i^{1/2}V(k)}\sigma_i^{-1/2}\xi\ |^2\ . \qquad (12)$$

In the case of an isotropic Hooke's law σ_i, i.e.

$$\sigma_i\xi \ = \ 2\mu_i\xi + \lambda_i(tr\xi)I_2 \qquad (13)$$

where μ_i, λ_i are the Lamé coefficients of the material σ_i, and I_2 is the second-order identity tensor, the degenerate Hooke's laws $f_{\sigma_i}(k)$ can be explicitely computed (see e.g. Lemma 4.2 in [8]) :

$$f_{\sigma_i}(k)\,\xi \ = \ \frac{1}{\mu_i}((\xi k)\otimes k - < \xi k, k > k\otimes k) + \frac{1}{2\mu_i+\lambda_i} < \xi k, k > k\otimes k. \qquad (14)$$

We are now equiped to state a first bound on a difference of two energies.

Theorem 2.1 *Assume that $\sigma_1 \le \sigma_2$. Let σ^* be the effective Hooke's law of a composite made from σ_1 and σ_2, in volume fractions θ_1 and θ_2 respectively. Then, for any symetric second order tensors η^+ and η^-, we have*

$$\begin{aligned} < \sigma^*\eta^+,\eta^+ > - < \sigma^*\eta^-,\eta^- > \ge & \quad < \sigma_1\eta^+,\eta^+ > - < \sigma_2\eta^-,\eta^- > \\ + & \quad \sup_{\xi^+,\xi^-}\Big(2\theta_2 < \eta^+,\xi^+ > +2\theta_1 < \eta^-,\xi^- > \\ - & \quad \theta_2 < (\sigma_2-\sigma_1)^{-1}\xi^+,\xi^+ > -\theta_1 < (\sigma_2-\sigma_1)^{-1}\xi^-,\xi^- > \\ - & \quad \theta_1\theta_2 g(\xi^-,\xi^+)\Big), \end{aligned} \qquad (15)$$

where the non-local term $g(\xi^-,\xi^+)$ is defined by

$$g(\xi^-,\xi^+) \ = \ \sup_{|k|=1}\Big(< f_{\sigma_1}(k)\xi^-,\xi^- > - < f_{\sigma_2}(k)\xi^+,\xi^+ >\Big)\,. \qquad (16)$$

Remark 2.1 *The function of ξ^+ and ξ^- which is maximized in the right hand side of 15 is concave (and even strictly concave if $\sigma_1 < \sigma_2$). Thus, this maximum is always attained and the bound 15 makes sense. However, we do not know if it is optimal. Actually, we believe it is not for most choices of η^+ and η^-.*

Proof : As is well known, there is no loss of generality in considering only periodic composite materials. Let σ^* be obtained by mixing materials 1 and 2 with characteristic functions $\chi_1(y)$ and $\chi_2(y)$ in the unit period Q. By definition, it satisfies

$$< \sigma^*\eta^+,\eta^+ > = \inf_{\phi(y)}\int_Q < (\chi_1(y)\sigma_1 + \chi_2(y)\sigma_2)(\eta^+ + e(\phi)),(\eta^+ + e(\phi)) > \,dy. \quad (17)$$

By adding and substracting a reference energy in σ_1, we obtain

$$\begin{aligned} < \sigma^*\eta^+,\eta^+ > = \inf_{\phi(y)}\Big(& \quad \textstyle\int_Q < \chi_2(y)(\sigma_2-\sigma_1)(\eta^+ + e(\phi)),(\eta^+ + e(\phi)) > \,dy \\ + & \quad \textstyle\int_Q < \sigma_1(\eta^+ + e(\phi)),(\eta^+ + e(\phi)) > \,dy\Big). \end{aligned} \qquad (18)$$

Using the positivity of $\sigma_2 - \sigma_1$ and convex duality, the first integral in the right hand side of 18 is rewriten

$$\sup_{\xi^+(y)} \int_Q \Big(2 < \xi^+(y), \eta^+ + e(\phi) > - < (\sigma_2 - \sigma_1)^{-1} \xi^+(y), \xi^+(y) > \Big) \chi_2(y)\, dy. \tag{19}$$

One can get a lower bound of 19 by specializing to constant tensors ξ^+ :

$$\sup_{\xi^+(y)} \geq 2\theta_2 < \eta^+, \xi^+ > -\theta_2 < (\sigma_2 - \sigma_1)^{-1} \xi^+, \xi^+ > + \int_Q 2\chi_2(y) < \xi^+, e(\phi) > \, dy. \tag{20}$$

Substitution in 18 yields after some simplification

$$\begin{aligned} < \sigma^* \eta^+, \eta^+ > \geq < \sigma_1 \eta^+, \eta^+ > + 2\theta_2 < \eta^+, \xi^+ > -\theta_2 < (\sigma_2 - \sigma_1)^{-1} \xi^+, \xi^+ > \\ + \inf_{\phi(y)} \int_Q \Big(< \sigma_1 e(\phi), e(\phi) > + 2\chi_2 < \xi^+, e(\phi) > \Big)\, dy. \end{aligned} \tag{21}$$

The above infimum in ϕ (the last term in the right hand side of 21) is easily computed by Fourier analysis (see e.g. Proposition 2.1 in [1]). Denoting by $\hat{\chi_2}(k)$ the Fourier component at frequency k of the characteristic function $\chi_2(y)$, it is exactly equal to

$$- \sum_{k \neq 0} | \hat{\chi_2}(k) |^2 | \Pi_{\sigma_1^{1/2} V(k)} \sigma_1^{-1/2} \xi^+ |^2 . \tag{22}$$

Combined with 21, it gives

$$\begin{aligned} < \sigma^* \eta^+, \eta^+ > \geq < \sigma_1 \eta^+, \eta^+ > + 2\theta_2 < \eta^+, \xi^+ > -\theta_2 < (\sigma_2 - \sigma_1)^{-1} \xi^+, \xi^+ > \\ - \sum_{k \neq 0} | \hat{\chi_2}(k) |^2 | \Pi_{\sigma_1^{1/2} V(k)} \sigma_1^{-1/2} \xi^+ |^2 . \end{aligned} \tag{23}$$

By a similar argument (using material 2 as the reference material), for the same composite σ^*, but for a different tensor η^-, we obtain a converse inequality

$$\begin{aligned} < \sigma^* \eta^-, \eta^- > \leq < \sigma_2 \eta^-, \eta^- > - 2\theta_1 < \eta^-, \xi^- > + \theta_1 < (\sigma_2 - \sigma_1)^{-1} \xi^-, \xi^- > \\ - \sum_{k \neq 0} | \hat{\chi_2}(k) |^2 | \Pi_{\sigma_2^{1/2} V(k)} \sigma_2^{-1/2} \xi^- |^2 . \end{aligned} \tag{24}$$

Substracting 24 to 23, the desired result 15 is obtained by bounding from below the Fourier series

$$- \sum_{k \neq 0} | \hat{\chi_2}(k) |^2 \Big(| \Pi_{\sigma_1^{1/2} V(k)} \sigma_1^{-1/2} \xi^+ |^2 - | \Pi_{\sigma_2^{1/2} V(k)} \sigma_2^{-1/2} \xi^- |^2 \Big) \geq -\theta_1 \theta_2 g(\xi^-, \xi^+), \tag{25}$$

where we have used the identity $\sum_{k \neq 0} | \hat{\chi_2}(k) |^2 = \theta_1 \theta_2$. Let us remark that the right hand side of 24 is convex in ξ^- since its last term is always larger than $-\theta_1 \theta_2 < \sigma_2 \xi^-, \xi^- >$. Thus, the right hand side of the difference between 23 and 24 is concave in ξ^-, ξ^-, as claimed in Remark 2.1.

Theorem 2.2 *Under the same assumptions as for Theorem 2.1, and for any symmetric second order tensors* ξ^+ *and* ξ^-*, we have the following optimal trace bound*

$$\begin{aligned} \theta_2^2 < (\sigma^* - \sigma_1)^{-1}\xi^-, \xi^- > +\theta_1^2 < (\sigma_2 - \sigma^*)^{-1}\xi^+, \xi^+ > \leq \\ \theta_1 < (\sigma_2 - \sigma_1)^{-1}\xi^+, \xi^+ > +\theta_2 < (\sigma_2 - \sigma_1)^{-1}\xi^-, \xi^- > +\theta_1\theta_2 g(\xi^-, \xi^+). \end{aligned} \tag{26}$$

Remark 2.2 *The above trace bound 26 is optimal in the sense that, for any couple* (ξ^+, ξ^-)*, there exists a least one composite material* σ^* *which achieves equality in the bound. An example of such an optimal composite is obtained by a single lamination of materials 1 and 2 (in proportions* θ_1 *and* θ_2*) in a direction* k_0 *which achieves the maximum in the definition 16 of the non-local term* $g(\xi^-, \xi^+)$*.*

Proof : By convex duality, the first line of 25 is exactly

$$\begin{aligned} \sup_{\zeta^+, \zeta^-} \Big(2 < \zeta^+, \xi^+ > +2 < \zeta^-, \xi^- > \\ -\theta_2^{-2} < (\sigma^* - \sigma_1)\zeta^+, \zeta^+ > -\theta_1^{-2} < (\sigma_2 - \sigma^*)\zeta^-, \zeta^- > \Big). \end{aligned} \tag{27}$$

Replacing $\theta_2^{-1}\zeta^+$ by η^+, and $\theta_1^{-1}\zeta^-$ by η^-, we can use Theorem 2.1 to get an upper bound of 27

$$\begin{aligned} \sup_{\eta^+, \eta^-} \inf_{\xi'^+, \xi'^-} \Big(2\theta_2 < \eta^+, (\xi^+ - \xi'^+) > +2\theta_1 < \eta^-, (\xi^- - \xi'^-) > \\ +\theta_1 < (\sigma_2 - \sigma_1)^{-1}\xi'^+, \xi'^+ > +\theta_2 < (\sigma_2 - \sigma_1)^{-1}\xi'^-, \xi'^- > +\theta_1\theta_2 g(\xi'^-, \xi'^+) \Big). \end{aligned} \tag{28}$$

The above functional is linear in (η^+, η^-) and convex in (ξ'^+, ξ'^-), thus we can interchange the order of minimization and maximization. The computation of this saddle point is now obvious : ξ' must be equal ξ, and 28 coincides with the upper bound in 26.

To prove the optimality of the trace bound 26, we re-do the proof of Theorem 2.1 for a single lamination of materials 1 and 2 in a given direction k_0, i.e. for a periodic composite material whose phases have characteristic functions $\chi_1(y.k_0)$ and $\chi_2(y.k_0)$ in the unit period Q. The computation proceeds exactly as above, and we point out the only two differences. First, when specializing to constant tensors ξ^+ in 20, we don't obtain an inequality, but an *equality*, since the true field $e(\phi)(y)$ is known to be constant in each phase for this special microstructure. Second, the lower bound 25 can be improved in an *equality*, since all frequencies, but k_0, contribute to nothing :

$$\begin{aligned} -\sum_{k\neq 0} \mid \hat{\chi}_2(k) \mid^2 \Big(\mid \Pi_{\sigma_1^{1/2}V(k)} \sigma_1^{-1/2}\xi^- \mid^2 - \mid \Pi_{\sigma_2^{1/2}V(k)} \sigma_2^{-1/2}\xi^+ \mid^2 \Big) = \\ -\theta_1\theta_2 \Big(\mid \Pi_{\sigma_1^{1/2}V(k_0)} \sigma_1^{-1/2}\xi^+ \mid^2 - \mid \Pi_{\sigma_2^{1/2}V(k_0)} \sigma_2^{-1/2}\xi^- \mid^2 \Big). \end{aligned} \tag{29}$$

Thus, for this simple laminated microstructure we obtain the *equality*

$$
\begin{aligned}
< \sigma^* \eta^+, \eta^+ > - < \sigma^* \eta^-, \eta^- > = & \quad < \sigma_1 \eta^+, \eta^+ > - < \sigma_2 \eta^-, \eta^- > \\
+ & \quad \sup_{\xi^+, \xi^-} \Big(2\theta_2 < \eta^+, \xi^+ > + 2\theta_1 < \eta^-, \xi^- > \\
- & \quad \theta_2 < (\sigma_2 - \sigma_1)^{-1} \xi^+, \xi^+ > - \theta_1 < (\sigma_2 - \sigma_1)^{-1} \xi^-, \xi^- > \\
- & \quad \theta_1 \theta_2 (\mid \Pi_{\sigma_1^{1/2} V(k_0)} \sigma_1^{-1/2} \xi^- \mid^2 - \mid \Pi_{\sigma_2^{1/2} V(k_0)} \sigma_2^{-1/2} \xi^+ \mid^2) \Big).
\end{aligned}
\tag{30}
$$

Now, taking the Legendre transform of equation 30 we obtain an equality similar to the trace bound 26, except that the non-local term is specified at frequency k_0. If we choose k_0 as one of the maximizer of the non-local term, this implies that this single lamination in direction k_0 saturates the trace bound.

Remark 2.3 *The optimality of the trace bound 26 can also be checked very easily by comparison with the explicit formula for a layered composite materials (see e.g. Theorem 4.1 in [6]).*

Remark 2.4 *Our new trace bound 26 is not implied by the previous upper and lower trace bounds 5 and 7. Indeed by summing 5 and 7, with weights θ_2 and θ_1 respectively, we obtain an upper bound which is always worse than 26 since we can easily check that*

$$
\theta_1 \theta_2 g(M^-, M^+) \ \leq \ \theta_1 \theta_2 \Big(g_-(M^-) + g_+(M^+) \Big). \tag{31}
$$

For some choices of M^-, M^+ (for example $M^- = M^+ = I_2 \otimes I_2$), 31 could be an equality, but usually it is a strict inequality, as one can be convinced by inspection of the explicit formula for g_- and g_+ in the case of one energy and isotropic constituents (see section 7 in [1]).

Remark 2.5 *Throughout this paper we assume that the two component materials σ_1 and σ_2 are well-ordered. If they were isotropic, but non well-ordered, the usual trace bounds would still hold in a slightly different form involving a mixed reference material (see [2] for details). Of course, Theorem 2.2 could also be generalized to this case.*

3 Motivation and discussion of the trace bounds

This section is concerned with various motivations and applications of the trace bounds. A first important application is the derivation of optimal bounds on effective moduli of isotropic composites. Before discussing the potential applications of our new trace bound, and for the sake of completeness, we recall how Milton and Kohn [12] obtained, from the trace bounds, the celebrated Hashin-Shtrikman bounds on the bulk and shear moduli κ^*, μ^*.

We consider an isotropic elastic composite made of two isotropic constituents with moduli $(\kappa_i, \mu_i)_{i=1,2}$ (thus $\lambda_i = \kappa_i - \frac{2\mu_i}{N}$), i.e. their Hooke's law σ_i is given by

$$\sigma_i \;=\; 2\mu_i \Lambda_s + N\kappa_i \Lambda_h \tag{32}$$

where Λ_s and Λ_h are the orthogonal projections on shear and hydrostatic tensors, respectively, defined by

$$\Lambda_s \;=\; I_4 - \frac{1}{N} I_2 \otimes I_2, \quad \Lambda_h \;=\; \frac{1}{N} I_2 \otimes I_2 \tag{33}$$

where N is the spatial dimension, and I_4, I_2 the fourth, and second, order identity tensors. Since Λ_s and Λ_h are projections on rotationnaly invariant subspaces, one can easily check that, for any vector k

$$< f_{\sigma_i}(k) : \Lambda_s > = \frac{N(N-1)(2\mu_i + \kappa_i)}{2\mu_i(2(N-1)\mu_i + N\kappa_i)}, \tag{34}$$

and

$$< f_{\sigma_i}(k) : \Lambda_h > = \frac{1}{2(N-1)\mu_i + N\kappa_i}. \tag{35}$$

Thus, by choosing M equal to Λ_h or Λ_s, the computation of the non-local term $g_{\pm}(M)$ is obvious since there is no need to optimize in k, thanks to the relations 34 and 35. This special choice of M in the usual lower and upper trace bounds 5 and 7 yields the Hashin-Shtrikman lower and upper bounds on κ^*, μ^*. For example, the lower bounds read as

$$\theta_2 \frac{1}{2(\mu^* - \mu_1)} \;\leq\; \frac{(N-1)(N+2)}{4(\mu_2 - \mu_1)} + \theta_1 \frac{N(N-1)(2\mu_1 + \kappa_1)}{2\mu_1(2(N-1)\mu_1 + N\kappa_1)}, \tag{36}$$

and

$$\theta_2 \frac{1}{N(\kappa^* - \kappa_1)} \;\leq\; \frac{1}{N(\kappa_2 - \kappa_1)} + \theta_1 \frac{1}{2(N-1)\mu_1 + N\kappa_1}. \tag{37}$$

Furthermore, these bounds are known to be optimal, since equality is attained in 36 and 37 for sequentially laminated composites (see [6]).

The Hashin-Shtrikman bounds defines a rectangle of "admissible" isotropic effective Hooke's law in the (κ, μ) plane. However, it is known that not all points of that rectangle are attained. To sharpen these bounds, one can try to obtain *coupled* bounds on κ^*, μ^* which cut a smaller "admissible" domain in this rectangle (see, e.g. [5], [13]). Theoretically, our new trace bound can furnish such coupled estimates since it can be viewed as, both, an upper and a lower bound on σ^*. However in practice, obvious choices of M^+ and M^-, as Λ_h or Λ_s, yield linear combinations of the previous Hashin-Shtrikman bounds. This is due to the fact that the non-local term, being constant for all k, doesn't couple the two energies $M^{\pm}$. Of course, one can hope that this is not the case for more general choices of $M^{\pm}$, but then the computation of the non-local term is tedious, or even out of reach. We must acknowledge that we have not succeeded in our quest of new bounds for effective moduli of isotropic composites starting from the trace bound established in Theorem 2.2.

Another motivation for the study of trace bounds is the so-called G-closure problem, i.e. the determination of the set of all possible effective Hooke's law obtained by mixing two materials in prescribed proportions. It is well-known that for the conductivity problem, the trace bounds are enough to characterize this set (see the original papers of Murat and Tartar [14]-[15], and [12] for an interpretation of their result in terms of trace bounds). It is also the case for incompressible elasticity in 2-D [9]. However, the picture is not so bright for the general elasticity where we only have a partial knowledge of the G-closure. Milton has given a geometrical interpretation of the trace bounds as tangent planes to some transformation of the G-closure set in the space of fourth order tensors (see section 14 in [11]). The convex hull of this G-closure set would be completly characterized if we could use any tensor M in the usual trace bounds 5 and 7. However, this is not the case since we are restricted to positive tensors M. Our new trace bound 8 doesn't remove this obstacle, but it improves a little the situation. Let us explain briefly why. Our new trace bound is indeed a coupled lower and upper bound on σ^*. Thus, we can "turn around" the G-closure set, passing continuously from a view from below to a view from above. However, we have not been able to obtain any quantitative results from this qualitative picture !

Finally, let us briefly discuss the relationship between optimal bounds and optimal design. We consider an example taken from our previous work [3] : the problem is to find the most rigid shape of an elastic body with prescribed weight. In other words, we seek the best arrangement of a given elastic material which minimizes its compliance (which plays the role of the cost function). This problem is known to be not well-posed, and needs to be relaxed. Actually, minimizing sequences are created by very fine perforations of the original material, and the effective behavior of the resulting design is that of a composite material. Consequently, the problem is relaxed by allowing perforated composite materials as admissible designs. This new relaxed formulation involves the minimization of the compliance over all possible effective Hooke's law. A priori, this requires the knowledge of the G-closure, which, unfortunately, is unknown. However, this problem can be reduced to a much simpler one by remarking that the compliance is equal to the complementary energy. Then, the minimization of the relaxed cost function is just the computation of an optimal lower bound on complementary energy (which has been studied in great details, see e.g. [1]).

In some sense, the crucial, and fortunate, point in the analysis of [3] is the choice of the compliance as cost function. A different cost function may require the knowledge of the entire G-closure, which is out of reach right now. As a simple example, let us consider the L^2-norm of the displacement. By introducing an adjoint state equation, one can check that this cost function is equal to the *difference* of two elastic energies. (Such problems are called *non self-adjoint* optimization problems by Lurie who studied them in the conductivity case [10].) Thus, to relax this problem requires an optimal lower bound on a difference of energies. Theorem 2.1 provides a bound of this type, but unfortunately it is not optimal. Again, this was a motivation for our new trace bound 8, but we haven't quite succeeded ! To conclude this paper, we hope that this little discussion of potential applications will motivate new research in the field of optimal bounds for effective properties of composite materials.

References

1. G. Allaire and R.V. Kohn, *Optimal bounds on the effective behavior of a mixture of two well-ordered elastic materials*, Quaterly of Applied Mathematics, in press (1993).

2. G. Allaire and R.V. Kohn, *Optimal lower bounds on the elastic energy of a composite made from two non well-ordered isotropic materials*, to appear in Quaterly of Applied Mathematics.

3. G. Allaire and R.V. Kohn, *Optimal design for minimum weight and compliance in plane stress using extremal microstructures*, Eur. J. Mech., A Solids, 12 (1993).

4. M. Avellaneda, *Optimal bounds and microgeometries for elastic two-phase composites*, SIAM J. Appl. Math. 47, 6, pp.1216-1228 (1987).

5. A. Cherkaev and L. Gibiansky, *Coupled estimates for the bulk and shear moduli of a two-dimensional isotropic elastic composite*, J. Mech. Phys. Solids 41, 5, pp.937-980 (1993).

6. G. Francfort and F. Murat, *Homogenization and optimal bounds in linear elasticity*, Arch. Rat. Mech. Anal. 94, pp.307-334 (1986).

7. Z. Hashin and S. Shtrikman, *A variational approach to the theory of the elastic behavior of multiphase materials*, J. Mech. Phys. Solids 11, pp.127-140 (1963).

8. R.V. Kohn, *The relaxation of a double-well energy*, Continuum Mech. Thermodyn. 3, pp.193-236 (1991).

9. R. Lipton, *On the effective elasticity of a two-dimensional homogenized incompressible elastic composite*, Proc. Roy. Soc. Edinburgh, 104A, pp.45-61 (1988).

10. K. Lurie, *The extension of optimisation problems containing controls in the coefficients*, Proc. Roy. Soc. Edinburgh, 114A, pp.81-97 (1990).

11. G. Milton, *On characterizing the set of possible effective tensors of composites*, Comm. Pure Appl. Math., XLIII, pp.63-125 (1990).

12. G. Milton and R.V. Kohn, *Variational bounds on the effective moduli of anisotropic composites*, J. Mech. Phys. Solids 36, 6, pp.597-629 (1988).

13. G. Milton and N. Phan-Thien, *New bounds on the elastic moduli of two-component materials*, Proc. Roy. Soc. Lond., A380, pp.305-331 (1982).

14. F. Murat and L. Tartar, *Calcul des Variations et Homogénéisation, in "Les Méthodes de l'Homogénéisation: Théorie et Applications en Physique"*, Eyrolles, pp.319-369 (1985).

15. L. Tartar, *Estimations fines des coefficients homogénéisés*, Ennio de Giorgi colloquium, P. Krée ed., Pitman Research Notes in Math. 125, pp.168-187 (1985).