1 Parametric optimization: 12 points

1. Because of the Dirichlet boundary conditions we choose the Sobolev space $H^1_0(\Omega)$. The variational formulation is: find $u^n \in H^1_0(\Omega)$ such that, for any test function $q^n \in H^1_0(\Omega)$,

$$
\int_\Omega (u^n q^n + h\Delta t \nabla u^n \cdot \nabla q^n) \, dx = \int_\Omega \left( u^{n-1} q^n + \Delta t f^n q^n \right) \, dx.
$$

2. The Lagrangian is the sum of the objective function and of the variational formulations for each $u^n$ (with, of course, different test functions). As usual, $u^n$ denotes the solution of the state equation and, in the Lagrangian, we replace it by the dummy function $v^n$. Thus it reads

$$
\mathcal{L}(h, \{v^n\}, \{q^n\}) = \sum_{n=1}^N \Delta t \int_\Omega j_1(v^n(x)) \, dx + \int_\Omega j_2(v^N(x)) \, dx + \sum_{n=1}^N \left( \int_\Omega (v^n q^n + h\Delta t \nabla v^n \cdot \nabla q^n) \, dx - \int_\Omega \left( v^{n-1} q^n + \Delta t f^n q^n \right) \, dx \right).
$$

3. The partial derivative of the Lagrangian with respect to $v^n$ is, for $1 \leq n \leq N - 1$,

$$
\langle \frac{\partial \mathcal{L}}{\partial v^n}, \psi \rangle = \Delta t \int_\Omega j'_1(v^n) \psi \, dx + \int_\Omega \left( \psi q^n + h\Delta t \nabla \psi \cdot \nabla q^n - \psi q^{n+1} \right) \, dx
$$

and for $n = N$

$$
\langle \frac{\partial \mathcal{L}}{\partial v^N}, \psi \rangle = \Delta t \int_\Omega j'_1(v^N) \psi \, dx + \int_\Omega j'_2(v^N) \psi \, dx + \int_\Omega \left( \psi q^N + h\Delta t \nabla \psi \cdot \nabla q^N \right) \, dx.
$$

Equating it to 0 and taking the value $v^n = u^n$ yields the variational formulation for the adjoint $p^n \in H^1_0(\Omega)$ such that, for any test function $\psi \in H^1_0(\Omega)$,

$$
\int_\Omega \left( \psi p^n + h\Delta t \nabla \psi \cdot \nabla p^n \right) \, dx = \int_\Omega \left( \psi p^{n+1} - \Delta t j'_1(u^n) \psi \right) \, dx
$$

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when $1 \leq n \leq N - 1$, while for $n = N$ the variational formulation is:

$$\text{find } p^N \in H^1_0(\Omega) \text{ such that, for any test function } \psi \in H^1_0(\Omega),$$

$$\int_{\Omega} (\psi p^N + h \Delta t \nabla \psi \cdot \nabla p^N) \, dx = - \int_{\Omega} (j'_2(u^N) + \Delta t j'_1(u^N) \psi) \, dx.$$ 

Disintegrating by parts yields the boundary value problem satisfied by $p^n$, for $1 \leq n \leq N - 1$,

$$\begin{cases}
P^n - \frac{p^{n+1}}{\Delta t} - \text{div} (h \nabla p^n) = -j'_1(u^n) & \text{in } \Omega, \\
p^n = 0 & \text{on } \partial \Omega,
\end{cases}$$

and for $p^N$

$$\begin{cases}
P^N - \Delta t \text{div} (h \nabla p^N) = -\Delta t j'_1(u^N) - j'_2(u^N) & \text{in } \Omega, \\
p^N = 0 & \text{on } \partial \Omega.
\end{cases}$$

To compute $p^n$, for $1 \leq n \leq N - 1$, we need to know $p^{n+1}$ and, on the other hand, $p^N$ depends solely on $u^N$. Thus, the adjoints $p^n$ have to be computed backward in time, namely in decreasing order from $n = N$ up to $n = 1$.

4. The formal derivative of $J_{\text{ds}}(h)$ is given by the formula

$$\langle J'_{\text{ds}}(h), k \rangle = \langle \frac{\partial L}{\partial h}(h, \{u^n\}, \{p^n\}), k \rangle.$$ 

Thus a simple computation (because the Lagrangian depends linearly on $h$ !) yields

$$\int_{\Omega} J'_{\text{ds}}(h) \, k \, dx = \sum_{n=1}^{N} \int_{\Omega} k \Delta t \nabla u^n \cdot \nabla p^n \, dx$$

or equivalently

$$J'_{\text{ds}}(h) = \sum_{n=1}^{N} \Delta t \nabla u^n \cdot \nabla p^n.$$ 

5. The boundary value problem for $p^n$, $1 \leq n \leq N - 1$, is obviously a time discretization of the evolution equation

$$\begin{cases}
- \frac{\partial p}{\partial t} - \text{div} (h \nabla p) = -j'_1(u) & \text{in } (0, T) \times \Omega, \\
p = 0 & \text{on } (0, T) \times \partial \Omega.
\end{cases}$$

Note the minus sign in front of the time derivative ! This parabolic equation must be complemented by an “initial” condition. However, in
condition attime


final
the present case it is a
with \( a_x(y) = \alpha_1\chi_1(y_1) + \alpha_2\chi_2(y_1) + \alpha_3\chi_3(y_1) \). Since the coefficient \( a_x \) depends only on the first component of the space variable \( y_1 \), the solutions are simply \( w_i \equiv 0 \), for \( 2 \leq i \leq N \), and \( w_1(y) \equiv w(y_1) \), the 1-d solution for \( i = 1 \). Then, using the following formula for the homogenized tensor \( A^* \)

\[
A^*_{ij} = \int_Y a_x(y) \left( e_i + \nabla_y w_i(y) \right) \cdot \left( e_i + \nabla_y w_i(y) \right) dy,
\]

a simple computation (see again Lemma 7.9 in the lecture notes) yields that

\[
A^* = \begin{pmatrix}
\lambda^-_\theta & 0 \\
0 & \lambda^+_\theta & \ldots & 0 \\
0 & \ldots & \lambda^+_\theta
\end{pmatrix},
\]

where \( \lambda^-_\theta = \left( \sum_{i=1}^3 \frac{\theta_i}{\alpha_i} \right)^{-1} \) is the harmonic mean and \( \lambda^+_\theta = \sum_{i=1}^3 \theta_i\alpha_i \) is the arithmetic mean of the phases conductivities.

2. Allowing only rotations of the previous simple laminate, i.e.,

\[
A^*(x) = R(x) A^* (\theta_1(x), \theta_2(x), \theta_3(x)) R^T(x),
\]

the relaxed state equation is just the homogenized equation

\[
\begin{aligned}
-\text{div} \left( A^* \nabla u \right) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

and the relaxed objective function does not change its expression

\[
\tilde{J}(\theta, R) = -\int_\Omega f(x) u(x) \, dx.
\]

3. By the energy minimization principle, the relaxed objective function can be written

\[
\tilde{J}(\theta, R) = \min_{v \in H^1_0(\Omega)} \int_\Omega (A^*(x) \nabla v \cdot \nabla v - 2f v) \, dx.
\]

Taking into account that

\[
A^*(x) \nabla v \cdot \nabla v = A^* (\theta_1(x), \theta_2(x), \theta_3(x)) \left( R^T(x) \nabla v(x) \right) \cdot \left( R^T(x) \nabla v(x) \right),
\]

the minimization with respect to the rotation matrix \( R(x) \) must align (pointwise) the lamination direction and the gradient of \( v \) so that only the smallest eigenvalue of \( A^* (\theta_1, \theta_2, \theta_3) \) plays a role. In other words

\[
\min_{R(x)} A^*(x) \nabla v \cdot \nabla v = \lambda^-_{\theta(x)} \left| \nabla v \right|^2.
\]
Thus the relaxed formulation is equivalent to

$$\inf_{\theta \in \mathcal{U}_{ad}, v \in H^1_0(\Omega)} \left\{ J^*(\theta, v) = \int_{\Omega} \left( \lambda_{ij} - \theta |\nabla v|^2 - 2 f v \right) dx \right\},$$

where the set of admissible densities is

$$\mathcal{U}_{ad}^* = \left\{ \theta = (\theta_1, \theta_2, \theta_3), 0 \leq \theta_i \leq 1, \sum_{i=1}^{3} \theta_i = 1, \int_{\Omega} \theta_i(x) dx = c_i|\Omega| \right\}.$$

4. By Lemma 5.8 in the lecture notes the function

$$(h, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N \rightarrow F(h, \xi) = h^{-1}|\xi|^2$$

is convex. By composition with a linear function, we deduce that the function

$$(\theta, \xi) \in (\mathbb{R}^+)^3 \times \mathbb{R}^N \rightarrow G(\theta, \xi) = \left( \sum_{i=1}^{3} \frac{\theta_i}{\alpha_i} \right)^{-1} |\xi|^2$$

is convex too. Indeed, an easy but tedious computation shows that the Hessian matrices satisfy

$$\nabla^2 G(\theta, \xi) \lambda \cdot \lambda = \nabla^2 F(h, \xi) \mu \cdot \mu \geq 0$$

for any $\lambda \in \mathbb{R}^{3+N}$ and $\mu \in \mathbb{R}^{1+N}$ such that $\mu_1 = \sum_{i=1}^{3} \lambda_i \alpha_i$ and $\mu_i = \lambda_{i+2}$ for $i \geq 2$. Furthermore $G(\theta, \xi)$ is infinite at infinity on the admissible set $\mathcal{U}_{ad}^*$ which features only linear equality and inequality constraints (which are clearly qualified). Thus, by Theorem 3.7 of the lecture notes, the relaxed formulation admits at least one optimal solution.