ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Answers to the written exam of March 16th, 2011.

1 Parametric optimization: 12 points

1. Because of the Dirichlet boundary conditions we choose the Sobolev space $H_0^1(\Omega)$. The variational formulation is: find $u^n \in H_0^1(\Omega)$ such that, for any test function $q^n \in H_0^1(\Omega)$,

$$\int_\Omega (u^n q^n + h\Delta t \nabla u^n \cdot \nabla q^n)\, dx = \int_\Omega \left(u^{n-1} q^n + \Delta t f^n q^n\right) dx.$$

2. The Lagrangian is the sum of the objective function and of the variational formulations for each u^n (with, of course, different test functions). As usual, u^n denotes the solution of the state equation and, in the Lagrangian, we replace it by the dummy function v^n. Thus it reads

$$\mathcal{L}(h, \{v^n\}, \{q^n\}) = \sum_{n=1}^{N} \Delta t \int_\Omega j_1(v^n(x))\, dx + \int_\Omega j_2(v^N(x))\, dx$$

$$+\sum_{n=1}^{N} \left(\int_\Omega (v^n q^n + h\Delta t \nabla v^n \cdot \nabla q^n)\, dx - \int_\Omega \left(v^{n-1} q^n + \Delta t f^n q^n\right) dx \right).$$

3. The partial derivative of the Lagrangian with respect to v^n is, for $1 \le n \le N-1$,

$$\langle \frac{\partial \mathcal{L}}{\partial v^n}, \psi \rangle = \Delta t \int_\Omega j_1'(v^n)\psi dx + \int_\Omega \left(\psi q^n + h\Delta t \nabla \psi \cdot \nabla q^n - \psi q^{n+1}\right) dx$$

and for $n = N$

$$\langle \frac{\partial \mathcal{L}}{\partial v^N}, \psi \rangle = \Delta t \int_\Omega j_1'(v^N)\psi dx + \int_\Omega j_2'(v^N)\psi dx + \int_\Omega \left(\psi q^N + h\Delta t \nabla \psi \cdot \nabla q^N\right) dx.$$

Equating it to 0 and taking the value $v^n = u^n$ yields the variational formulation for the adjoint $p^n \in H_0^1(\Omega)$ such that, for any test function $\psi \in H_0^1(\Omega)$,

$$\int_\Omega (\psi p^n + h\Delta t \nabla \psi \cdot \nabla p^n)\, dx = \int_\Omega \left(\psi p^{n+1} - \Delta t j_1'(u^n)\psi\right) dx$$

when $1 \le n \le N-1$, while for $n = N$ the variational formulation is: find $p^N \in H_0^1(\Omega)$ such that, for any test function $\psi \in H_0^1(\Omega)$,

$$\int_\Omega \left(\psi p^N + h\Delta t \nabla\psi \cdot \nabla p^N\right) dx = -\int_\Omega \left(j_2'(u^N) + \Delta t j_1'(u^N)\psi\right) dx.$$

Disintegrating by parts yields the boundary value problem satisfied by p^n, for $1 \le n \le N-1$,

$$\begin{cases} \frac{p^n - p^{n+1}}{\Delta t} - \operatorname{div}(h\nabla p^n) = -j_1'(u^n) & \text{in } \Omega, \\ p^n = 0 & \text{on } \partial\Omega, \end{cases}$$

and for p^N

$$\begin{cases} p^N - \Delta t \operatorname{div}\left(h\nabla p^N\right) = -\Delta t j_1'(u^N) - j_2'(u^N) & \text{in } \Omega, \\ p^N = 0 & \text{on } \partial\Omega. \end{cases}$$

To compute p^n, for $1 \le n \le N-1$, we need to know p^{n+1} and, on the other hand, p^N depends solely on u^N. Thus, the adjoints p^n have to be computed **backward in time**, namely in decreasing order from $n = N$ up to $n = 1$.

4. The formal derivative of $J_{\Delta t}(h)$ is given by the formula

$$\langle J'_{\Delta t}(h), k\rangle = \langle \frac{\partial \mathcal{L}}{\partial h}(h, \{u^n\}, \{p^n\}), k\rangle.$$

Thus a simple computation (because the Lagrangian depends linearly on h !) yields

$$\int_\Omega J'_{\Delta t}(h)\, k\, dx = \sum_{n=1}^{N} \int_\Omega k \Delta t \nabla u^n \cdot \nabla p^n dx$$

or equivalently

$$J'_{\Delta t}(h) = \sum_{n=1}^{N} \Delta t \nabla u^n \cdot \nabla p^n.$$

5. The boundary value problem for p^n, $1 \le n \le N-1$, is obviously a time discretization of the evolution equation

$$\begin{cases} -\frac{\partial p}{\partial t} - \operatorname{div}(h\nabla p) = -j_1'(u) & \text{in } (0,T)\times\Omega, \\ p = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

Note the minus sign in front of the time derivative ! This parabolic equation must be complemented by an “initial” condition. However, in the present case it is a **final** condition at time $t = T$. Indeed, formally when Δt goes to 0, the limit of the equation for p^N is just

$$p(T, x) = -j_2'\Big(u(T, x)\Big) \quad \text{in } \Omega.$$

Thus the evolution equation for p has to be solved **backward in time**. It is a well-posed problem because by changing the time variable and introducing $\tilde{p}(t, x) = p(T - t, x)$ we obtain the standard (and well-posed) parabolic equation

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t} - \operatorname{div}(h \nabla \tilde{p}) = -j_1'(u(T - t)) & \text{in } (0, T) \times \Omega, \\ \tilde{p} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \tilde{p}(0, x) = -j_2'\Big(u(T, x)\Big) & \text{in } \Omega, \end{cases}$$

where the sign in front of the time derivative is the "right" one.

Clearly, the previous derivative of $J_{\Delta t}(h)$ is a discretization of the following time integral

$$\int_0^T \nabla u(t, x) \cdot \nabla p(t, x)\, dt.$$

Remark. The statement of the present question was very cautious by saying that "p^n has possibly to be multiplied by a suitable coefficient". No such coefficient was necessary for the above definition of the Lagragian but remember that the variational formulation of u^n could have been multiplied by any coefficient (typically by $1/\Delta t$) without changing the definition of u^n but, of course, implying a change in the Lagrangian and in the definition of p^n...

6. The state u appears in the right hand side of the equation for the adjoint p. In the present time-dependent case, the difficulty is that p has to be computed backward, i.e., starting from the final time T and going back to the initial time 0. This is not a serious problem since, by the above change of variables $\tilde{p}(t, x) = p(T - t, x)$, the equation for p is well-posed, **except** for the fact that the state u has to be stored on the entire time interval $(0, T)$ before it can be put (backward) in the right hand side of the equation for p. If the number of time steps N is large, this storage process requires an enormous memory capacity and is the main computational bottle-neck for large applications.

2 Topology optimization: 8 points

1. Following a computation of the course (see Lemma 7.9 in the lecture notes) we compute the solutions of the cell problem

$$\begin{cases} -\operatorname{div}_{\mathrm{y}}\Big(a_\chi(y)\,(e_i + \nabla_y w_i(y))\Big) = 0 & \text{in } Y = (0,1)^N \\ y \to w_i(y) & Y\text{-periodic} \end{cases}$$

with $a_\chi(y) = \alpha_1\chi_1(y_1) + \alpha_2\chi_2(y_1) + \alpha_3\chi_3(y_1)$. Since the coefficient a_χ depends only on the first component of the space variable y_1, the solutions are simply $w_i \equiv 0$, for $2 \le i \le N$, and $w_1(y) \equiv w(y_1)$, the 1-d solution for $i = 1$. Then, using the following formula for the homogenized tensor A^*

$$A^*_{ij} = \int_Y a_\chi(y)\,(e_i + \nabla_y w_i(y)) \cdot (e_i + \nabla_y w_i(y))\,dy,$$

a simple computation (see again Lemma 7.9 in the lecture notes) yields that

$$A^* = \begin{pmatrix} \lambda_\theta^- & & & 0 \\ & \lambda_\theta^+ & & \\ & & \ddots & \\ 0 & & & \lambda_\theta^+ \end{pmatrix},$$

where $\lambda_\theta^- = \left(\sum_{i=1}^3 \frac{\theta_i}{\alpha_i}\right)^{-1}$ is the harmonic mean and $\lambda_\theta^+ = \sum_{i=1}^3 \theta_i\alpha_i$ is the arithmetic mean of the phases conductivities.

2. Allowing only rotations of the previous simple laminate, i.e.,

$$A^*(x) = R(x)\,A^*\left(\theta_1(x), \theta_2(x), \theta_3(x)\right)\,R^T(x),$$

the relaxed state equation is just the homogenized equation

$$\begin{cases} -\mathrm{div}\,(A^*\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the relaxed objective function does not change its expression

$$\tilde{J}(\theta, R) = -\int_\Omega f(x)\,u(x)\,dx.$$

3. By the energy minimization principle, the relaxed objective function can be written

$$\tilde{J}(\theta, R) = \min_{v \in H_0^1(\Omega)} \int_\Omega \left(A^*(x)\nabla v \cdot \nabla v - 2fv\right)dx.$$

Taking into account that

$$A^*(x)\nabla v \cdot \nabla v = A^*\left(\theta_1(x), \theta_2(x), \theta_3(x)\right)\left(R^T(x)\nabla v(x)\right) \cdot \left(R^T(x)\nabla v(x)\right),$$

the minimization with respect to the rotation matrix $R(x)$ must align (pointwise) the lamination direction and the gradient of v so that only the **smallest** eigenvalue of $A^*\left(\theta_1, \theta_2, \theta_3\right)$ plays a role. In other words

$$\min_{R(x)} A^*(x)\nabla v \cdot \nabla v = \lambda_{\theta(x)}^- |\nabla v|^2.$$

Thus the relaxed formulation is equivalent to

$$\inf_{\theta \in \mathcal{U}^*_{ad}, v \in H^1_0(\Omega)} \left\{ J^*(\theta, v) = \int_\Omega \left(\lambda_\theta^- |\nabla v|^2 - 2fv \right) dx \right\},$$

where the set of admissible densities is

$$\mathcal{U}^*_{ad} = \left\{ \theta = (\theta_1, \theta_2, \theta_3), 0 \le \theta_i \le 1,\ \sum_{i=1}^{3} \theta_i = 1, \int_\Omega \theta_i(x) dx = c_i |\Omega| \right\}.$$

4. By Lemma 5.8 in the lecture notes the function

$$(h, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N \longrightarrow F(h, \xi) = h^{-1} |\xi|^2$$

is convex. By composition with a linear function, we deduce that the function

$$(\theta, \xi) \in (\mathbb{R}^+)^3 \times \mathbb{R}^N \longrightarrow G(\theta, \xi) = \left(\sum_{i=1}^{3} \frac{\theta_i}{\alpha_i} \right)^{-1} |\xi|^2$$

is convex too. Indeed, an easy but tedious computation shows that the Hessian matrices satisfy

$$\nabla\nabla G(\theta, \xi) \lambda \cdot \lambda = \nabla\nabla F(h, \xi) \mu \cdot \mu \ge 0$$

for any $\lambda \in \mathbb{R}^{3+N}$ and $\mu \in \mathbb{R}^{1+N}$ such that $\mu_1 = \sum_{i=1}^3 \lambda_i \alpha_i$ and $\mu_i = \lambda_{i+2}$ for $i \ge 2$. Furthermore $G(\theta, \xi)$ is infinite at infinity on the admissible set $\mathcal{U}^*_{ad}$ which features only linear equality and inequality constraints (which are clearly qualified). Thus, by Theorem 3.7 of the lecture notes, the relaxed formulation admits at least one optimal solution.