ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Answers to the written exam of March 14th, 2012.

1 Parametric optimization: 10 points

1. The variational formulation is: find $u \in H^1(\Omega)$ such that, for any test function $q \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla q \, dx + \int_{\partial\Omega} kuq \, ds = \int_\Omega fq \, dx.$$

The Lagrangian is the sum of the objective function and of the variational formulation

$$\mathcal{L}(h, v, q) = \int_\Omega j(v(x)) \, dx + \int_\Omega (\nabla v \cdot \nabla q - fq) \, dx + \int_{\partial\Omega} kvq \, ds.$$

2. The partial derivative of the Lagrangian with respect to v is

$$\langle \frac{\partial \mathcal{L}}{\partial v}, \psi \rangle = \int_\Omega j'(v)\psi dx + \int_\Omega \nabla \psi \cdot \nabla q \, dx + \int_{\partial\Omega} k\psi q \, ds.$$

Equating it to 0 and taking the value $v = u$ yields the variational formulation for the adjoint $p \in H^1(\Omega)$ where $\psi \in H^1(\Omega)$ is any test function. Disintegrating by parts yields the boundary value problem satisfied by p

$$\begin{cases} -\Delta p = -j'(u) & \text{in } \Omega, \\ \frac{\partial p}{\partial n} + k\, p = 0 & \text{on } \partial\Omega. \end{cases}$$

3. The formal derivative of $J(k)$ is given by the formula

$$\langle J'(k), \theta \rangle = \langle \frac{\partial \mathcal{L}}{\partial k}(k, u, p), \theta \rangle.$$

Thus a simple computation (because the Lagrangian depends linearly on k !) yields

$$\int_\Omega J'(k) \, \theta \, dx = \int_{\partial\Omega} \theta up \, ds,$$

or equivalently

$$J'(k) = u\, p \quad \text{on } \partial\Omega.$$

4. When $j(v) = -fv$, we find $p = u$ so $J'(k) = u^2 \geq 0$. Since the derivative is always non-negative, the optimality condition is satisfied for the minimal value of k, namely $k(x) = k_{min}$ on $\partial\Omega$. Therefore, we expect this value to be the minimum of the objective function $J(k) = -\int_\Omega f(x)\,u(x)\,dx$.

   To make the proof rigorous, we rewrite $J(k)$ as the minimum of the (primal) energy

$$-\int_\Omega f(x)\,u(x)\,dx = \min_{v\in H^1(\Omega)} \int_\Omega |\nabla v|^2 dx + \int_{\partial\Omega} kv^2\,ds - 2\int_\Omega fv\,dx.$$

   The optimal design problem is thus equivalent to a double minimization

$$\min_{(k,v)\in\mathcal{U}_{ad}\times H^1(\Omega)} \int_\Omega |\nabla v|^2 dx + \int_{\partial\Omega} kv^2\,ds - 2\int_\Omega fv\,dx.$$

   For any fixed v, the minimal value is clearly attained by $k(x) = k_{min}$. Thus $k(x) = k_{min}$ is a global minimizer of the optimal design problem. (It may be not unique at those points $x \in \partial\Omega$ where $u(x) = 0$.)

2 Geometric optimization: 7 points

1. By the chain rule lemma, the shape derivative of $\mathcal{M}_\Omega(f)$ is, for any vector field $\theta \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$,

$$\mathcal{M}_\Omega(f)'(\theta) = \frac{1}{|\Omega|}\int_{\partial\Omega} f\,\theta\cdot n\,ds - \frac{1}{|\Omega|^2}\int_{\partial\Omega}\theta\cdot n\,ds\int_\Omega f(x)\,dx,$$

   which simplifies as

$$\mathcal{M}_\Omega(f)'(\theta) = \frac{1}{|\Omega|}\int_{\partial\Omega}(f - \mathcal{M}_\Omega(f))\,\theta\cdot n\,ds.$$

   Clearly, the derivative is zero (for any θ) if and only if $f = \mathcal{M}_\Omega(f)$ on $\partial\Omega$.

2. We rewrite the function $J(\Omega)$ as

$$J(\Omega) = \frac{1}{|\Omega|}\int_\Omega f^2(x)\,dx - (\mathcal{M}_\Omega(f))^2.$$

   We deduce from the previous question that

$$J'(\Omega)(\theta) = \frac{1}{|\Omega|}\int_{\partial\Omega}\left(f^2 - \mathcal{M}_\Omega(f^2)\right)\theta\cdot n\,ds - 2\mathcal{M}_\Omega(f)\frac{1}{|\Omega|}\int_{\partial\Omega}(f - \mathcal{M}_\Omega(f))\,\theta\cdot n\,ds.$$

   Recombining terms yields

$$J'(\Omega)(\theta) = \frac{1}{|\Omega|}\int_{\partial\Omega}\left((f - \mathcal{M}_\Omega(f))^2 + (\mathcal{M}_\Omega(f))^2 - \mathcal{M}_\Omega(f^2)\right)\theta\cdot n\,ds.$$

By Cauchy-Schwartz inequality we have $(\mathcal{M}_\Omega(f))^2 \leq \mathcal{M}_\Omega(f^2)$ and the inequality is strict if f is not constant on Ω. Therefore, if f is not constant and $f = \mathcal{M}_\Omega(f)$ on $\partial\Omega$, we deduce that $J'(\Omega)(\theta) < 0$ if the domain increases, namely when $\theta \cdot n > 0$ on $\partial\Omega$. Thus, if Ω is such that $J(\Omega) \leq \epsilon$, for a small enough θ satisfying $\theta \cdot n > 0$ on $\partial\Omega$, we still have $J((\mathrm{Id} + \theta)\Omega) \leq \epsilon$ while the volume increases, $|(\mathrm{Id} + \theta)\Omega| > |\Omega|$. In other words, $(\mathrm{Id} + \theta)\Omega$ is a better admissible design.

3. If the constraint is inactive, i.e., $J(\Omega) < \epsilon$, for a maximizer Ω with finite volume, then we can slightly increase its volume while keeping the constraint satisfied, therefore contradicting the assumption that Ω was a maximizer. Thus, for a finite-volume maximizer, the constraint must be active, i.e., $J(\Omega) = \epsilon$. In such a case, there existe a non-negative Lagrange multiplier $\lambda \geq 0$ such that, for any $\theta \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$,

$$\frac{\lambda}{|\Omega|} \int_{\partial\Omega} \left((f - \mathcal{M}_\Omega(f))^2 + (\mathcal{M}_\Omega(f))^2 - \mathcal{M}_\Omega(f^2)\right) \theta \cdot n \, ds + \int_{\partial\Omega} \theta \cdot n \, ds = 0.$$

In other words, the optimality condition is

$$\frac{\lambda}{|\Omega|} \Big((f - \mathcal{M}_\Omega(f))^2 + (\mathcal{M}_\Omega(f))^2 - \mathcal{M}_\Omega(f^2)\Big) + 1 = 0 \quad \text{on } \partial\Omega.$$

3 Homogenization: 3 points

In space dimension $N = 2$, for an isotropic homogenized tensor $A^* = a^*\mathrm{Id}$, the Hashin-Shtrikman upper bound reduces to

$$\frac{2}{\beta - a^*} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{1}{\beta - \lambda_\theta^+}$$

with $\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$ and $\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta$. Taking $\alpha = 0$ yields

$$\frac{2}{\beta - a^*} \leq \frac{1}{\beta} + \frac{1}{\theta\beta}.$$

A simple calculation gives the result $a^* \leq \frac{1-\theta}{1+\theta}\beta$.