ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Answers to the written exam of March 20th, 2013.

1 Parametric optimization: 12 points

1. Writing $v = u - u_0$ the problem becomes

$$\begin{cases} -\mathrm{div}\,(h\nabla v) = f + \mathrm{div}\,(h\nabla u_0) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The corresponding variational formulation is: find $v \in H_0^1(\Omega)$ such that, for any test function $q \in H_0^1(\Omega)$,

$$\int_\Omega h\nabla v \cdot \nabla q \, dx = \int_\Omega f q \, dx - \int_\Omega h\nabla u_0 \cdot \nabla q \, dx.$$

Replacing v by $u - u_0$ we get: find $u \in \mathcal{A}$ such that

$$\int_\Omega h\nabla u \cdot \nabla\phi \, dx = \int_\Omega f\phi \, dx \quad \forall\, \phi \in H_0^1(\Omega),$$

where

$$\mathcal{A} = \{\psi \in H^1(\Omega) \text{ such that } \psi = u_0 + \phi \text{ with } \phi \in H_0^1(\Omega)\}.$$

2. The Lagrangian is the sum of the objective function and of the variational formulation

$$\mathcal{L}(h, w, q) = \int_\Omega j(w(x))\, dx + \int_\Omega (h\nabla w \cdot \nabla q - f q)\, dx,$$

where $w \in \mathcal{A}$ and $q \in H_0^1(\Omega)$. For any $\psi \in H_0^1(\Omega)$ the sum $w + \psi$ belongs to $\mathcal{A}$, so the partial derivative of the Lagrangian with respect to w is, for any $\psi \in H_0^1(\Omega)$,

$$\langle \frac{\partial\mathcal{L}}{\partial w}, \psi\rangle = \int_\Omega j'(w)\psi \, dx + \int_\Omega h\nabla\psi \cdot \nabla q \, dx.$$

Equating it to 0 and taking the value $w = u$ yields the variational formulation for the adjoint $p \in H_0^1(\Omega)$ where $\psi \in H_0^1(\Omega)$ is any test function. Disintegrating by parts yields the boundary value problem satisfied by p

$$\begin{cases} -\mathrm{div}\,(h\nabla p) = -j'(u) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

3. The formal derivative of $J(h)$ is given by the formula

$$\langle J'(h), \theta \rangle = \langle \frac{\partial \mathcal{L}}{\partial h}(h, u, p), \theta \rangle.$$

   Thus a simple computation (because the Lagrangian depends linearly on h !) yields

$$\int_\Omega J'(h)\, \theta \, dx = \int_{\partial\Omega} \theta \nabla u \cdot \nabla p \, ds,$$

   or equivalently

$$J'(h) = \nabla u \cdot \nabla p \quad \text{ in } \Omega.$$

4. When $j(v) = fv$, we find that $p \neq \pm u$ because they don't satisfy the same boundary condition since $u_0 \neq 0$.

5. For the new objective function

$$J(h) = \int_\Omega j(h, u, \nabla u)\, dx,$$

   the Lagrangian is

$$\mathcal{L}(h, w, q) = \int_\Omega j(h, w, \nabla w)\, dx + \int_\Omega (h \nabla w \cdot \nabla q - fq)\, dx,$$

   where $w \in \mathcal{A}$ and $q \in H_0^1(\Omega)$. The new adjoint p is thus a solution of the variational formulation in $H_0^1(\Omega)$

$$\int_\Omega \left(\frac{\partial j}{\partial w}(h, u, \nabla u)\psi + \frac{\partial j}{\partial \zeta}(h, u, \nabla u) \cdot \nabla \psi \right) dx + \int_\Omega h \nabla \psi \cdot \nabla p \, dx = 0$$

   for any test function $\psi \in H_0^1(\Omega)$. Disintegrating by parts yields the boundary value problem satisfied by p

$$\begin{cases} -\mathrm{div}\,(h \nabla p) = -\frac{\partial j}{\partial w}(h, u, \nabla u) + \mathrm{div}\left(\frac{\partial j}{\partial \zeta}(h, u, \nabla u) \right) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

6. For the specific example

$$J(h) = \int_\Omega \left(f\, u - \frac{1}{2} h \nabla u \cdot \nabla u \right) dx$$

   we have

$$\frac{\partial j}{\partial w}(h, u, \nabla u) = f \quad \text{ and } \quad \frac{\partial j}{\partial \zeta}(h, u, \nabla u) = h \nabla u,$$

   so that the right hand side of the state equation is zero which implies $p = 0$.

As usual we have

$$\langle J'(h), \theta \rangle = \langle \frac{\partial \mathcal{L}}{\partial h}(h, u, p), \theta \rangle,$$

which implies that

$$\langle J'(h), \theta \rangle = \int_\Omega \frac{\partial j}{\partial h}(h, u, \nabla u)\, \theta \, dx + \int_\Omega \theta \nabla u \cdot \nabla p \, dx.$$

Since $p = 0$ we deduce

$$\langle J'(h), \theta \rangle = -\frac{1}{2} \int_\Omega \theta \nabla u \cdot \nabla u \, dx,$$

which is the "usual" formula for the derivative of compliance minimization problem.

To minimize the objective function we should choose to increase the thickness since $-\frac{1}{2}\nabla u \cdot \nabla u \leq 0$.

Eventually, when $u_0 = 0$, we recover the standard case of compliance minimization with homogeneous Dirichlet boundary condition. We simply defined in an equivalent way the compliance

$$J(h) = \int_\Omega \left(f\, u - \frac{1}{2} h \nabla u \cdot \nabla u \right) dx = \frac{1}{2} \int_\Omega f\, u \, dx.$$

2 Geometric optimization: 8 points

1. For Dirichlet boundary conditions we introduce two Lagrange multipliers: q for the p.d.e. and λ for the boundary condition. For any functions $v, q, \lambda \in H^1(\mathbb{R}^N)$ we define the Lagrangian

$$\mathcal{L}(\Omega, v, q, \lambda) = \frac{1}{2} \int_\Omega |v - u_0|^2 \, dx + \int_\Omega (V \cdot \nabla v - \nu \Delta v - f)\, q \, dx + \int_{\partial\Omega} v \, \lambda \, ds.$$

Clearly we have

$$\max_{q,\lambda} \mathcal{L}(\Omega, v, q, \lambda) = \begin{cases} \frac{1}{2} \int_\Omega |u - u_0|^2 \, dx & \text{if } v = u \text{ is the state,} \\ +\infty & \text{otherwise.} \end{cases}$$

2. Two successive integrations by parts yield

$$\mathcal{L}(\Omega, v, q, \lambda) = \frac{1}{2} \int_\Omega |v - u_0|^2 \, dx + \int_\Omega (-V \cdot \nabla q - \nu \Delta q)\, v \, dx - \int_\Omega f\, q \, dx$$
$$+ \int_{\partial\Omega} (V \cdot n\, v - \nu \partial_n v) q \, ds + \int_{\partial\Omega} \nu \partial_n q \, v \, ds + \int_{\partial\Omega} v \, \lambda \, ds.$$

To get the adjoint problem we differentiate the Lagrangian with respect to v and set this partial derivative equal to 0

$$\langle\frac{\partial\mathcal{L}}{\partial v}(\Omega,v,q,\lambda),\phi\rangle = \int_\Omega (v-u_0)\phi\,dx + \int_\Omega (-V\cdot\nabla q - \nu\Delta q)\,\phi\,dx$$
$$+\int_{\partial\Omega}(V\cdot n\,\phi - \nu\partial_n\phi)q\,ds + \int_{\partial\Omega}\nu\partial_n q\,\phi\,ds + \int_{\partial\Omega}\phi\,\lambda\,ds.$$

We first take a test function ϕ with compact support in Ω, so we deduce that the optimal value of q, the adjoint p, satisfies

$$-V\cdot\nabla p - \nu\Delta p = -(u-u_0) \quad \text{in } \Omega.$$

Next, we take $\phi = 0$ on $\partial\Omega$ but $\partial_n\phi$ can be any function on $\partial\Omega$. It yields $p = 0$ on $\partial\Omega$. Finally, taking a general test function such that its trace ϕ on $\partial\Omega$ is any function, we get the optimal value of λ

$$\lambda = -\nu\partial_n p \quad \text{on } \partial\Omega.$$

The adjoint problem is thus

$$\begin{cases} -V\cdot\nabla p - \nu\Delta p = -(u-u_0) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

The differential operator of the adjoint equation is different from the one of the state equation: the sign of the velocity is changed in the convective term.

3. Formally we know that the shape derivative is given by

$$J'(\Omega)(\theta) = \frac{\partial\mathcal{L}}{\partial\Omega}(\Omega,u,p,\lambda)(\theta).$$

We compute

$$\frac{\partial\mathcal{L}}{\partial\Omega}(\Omega,v,q,\lambda)(\theta) = \frac{1}{2}\int_{\partial\Omega}|v-u_0|^2\,\theta\cdot n\,ds$$
$$+\int_{\partial\Omega}(V\cdot\nabla v - \nu\Delta v - f)\,q\,\theta\cdot n\,ds + \int_{\partial\Omega}\left(H\,v\,\lambda + \frac{\partial}{\partial n}(v\lambda)\right)\,ds,$$

where H is the mean curvature. Replacing v by u, q by p and λ by its optimal value $(-V\cdot n\,p - \nu\partial_n p)$, and noticing that $u = p = 0$ on $\partial\Omega$, we deduce

$$J'(\Omega)(\theta) = \frac{1}{2}\int_{\partial\Omega}|u_0|^2\,\theta\cdot n\,ds - \nu\int_{\partial\Omega}\frac{\partial u}{\partial n}\frac{\partial p}{\partial n}\,\theta\cdot n\,ds.$$