1 Parametric optimization: 12 points

1. Writing $v = u - u_0$ the problem becomes
\[
\begin{align*}
- \text{div} (h \nabla v) &= f + \text{div} (h \nabla u_0) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

The corresponding variational formulation is: find $v \in H^1_0(\Omega)$ such that, for any test function $q \in H^1_0(\Omega)$,
\[
\int_\Omega h \nabla v \cdot \nabla q \, dx = \int_\Omega f q \, dx - \int_\Omega h \nabla u_0 \cdot \nabla q \, dx.
\]

Replacing $v$ by $u - u_0$ we get: find $u \in A$ such that
\[
\int_\Omega h \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in H^1_0(\Omega),
\]

where
\[
A = \{ \psi \in H^1(\Omega) \text{ such that } \psi = u_0 + \phi \text{ with } \phi \in H^1_0(\Omega) \}.
\]

2. The Lagrangian is the sum of the objective function and of the variational formulation
\[
\mathcal{L}(h, w, q) = \int_\Omega j(w(x)) \, dx + \int_\Omega (h \nabla w \cdot \nabla q - f q) \, dx,
\]

where $w \in A$ and $q \in H^1_0(\Omega)$. For any $\psi \in H^1_0(\Omega)$ the sum $w + \psi$ belongs to $A$, so the partial derivative of the Lagrangian with respect to $w$ is, for any $\psi \in H^1_0(\Omega)$,
\[
\left( \frac{\partial \mathcal{L}}{\partial w}, \psi \right) = \int_\Omega j'(w) \psi \, dx + \int_\Omega h \nabla \psi \cdot \nabla q \, dx.
\]

Equating it to 0 and taking the value $w = u$ yields the variational formulation for the adjoint $p \in H^1_0(\Omega)$ where $\psi \in H^1_0(\Omega)$ is any test function. Disintegrating by parts yields the boundary value problem satisfied by $p$
\[
\begin{align*}
- \text{div} (h \nabla p) &= -j'(u) \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
3. The formal derivative of $J(h)$ is given by the formula

$$\langle J'(h), \theta \rangle = \langle \frac{\partial L}{\partial h}(h, u, p), \theta \rangle.$$

Thus a simple computation (because the Lagrangian depends linearly on $h$!) yields

$$\int_{\Omega} J'(h) \theta \, dx = \int_{\partial \Omega} \theta \nabla u \cdot \nabla p \, ds,$$

or equivalently

$$J'(h) = \nabla u \cdot \nabla p \quad \text{in } \Omega.$$

4. When $j(v) = f v$, we find that $p \neq \pm u$ because they don’t satisfy the same boundary condition since $u_0 \neq 0$.

5. For the new objective function

$$J(h) = \int_{\Omega} j(h, u, \nabla u) \, dx,$$

the Lagrangian is

$$L(h, w, q) = \int_{\Omega} j(h, w, \nabla w) \, dx + \int_{\Omega} (h \nabla w \cdot \nabla q - f q) \, dx,$$

where $w \in \mathcal{A}$ and $q \in H^1_0(\Omega)$. The new adjoint $p$ is thus a solution of the variational formulation in $H^1_0(\Omega)$

$$\int_{\Omega} \left( \frac{\partial j}{\partial w}(h, u, \nabla u) \psi + \frac{\partial j}{\partial \zeta}(h, u, \nabla u) \cdot \nabla \psi \right) \, dx + \int_{\Omega} h \nabla \psi \cdot \nabla p \, dx = 0$$

for any test function $\psi \in H^1_0(\Omega)$. Disintegrating by parts yields the boundary value problem satisfied by $p$

$$\begin{cases}
-\text{div}(h \nabla p) = -\frac{\partial j}{\partial w}(h, u, \nabla u) + \text{div} \left( \frac{\partial j}{\partial \zeta}(h, u, \nabla u) \right) & \text{in } \Omega, \\
p = 0 & \text{on } \partial \Omega.
\end{cases}$$

6. For the specific example

$$J(h) = \int_{\Omega} \left( f u - \frac{1}{2} h \nabla u \cdot \nabla u \right) \, dx$$

we have

$$\frac{\partial j}{\partial w}(h, u, \nabla u) = f \quad \text{and} \quad \frac{\partial j}{\partial \zeta}(h, u, \nabla u) = h \nabla u,$$

so that the right hand side of the state equation is zero which implies $p = 0$. 

As usual we have
\[ \langle J'(h), \theta \rangle = \langle \frac{\partial L}{\partial h}(h, u, p), \theta \rangle, \]
which implies that
\[ \langle J'(h), \theta \rangle = \int_\Omega \frac{\partial j}{\partial h}(h, u, \nabla u) \theta \, dx + \int_\Omega \theta \nabla u \cdot \nabla p \, dx. \]
Since \( p = 0 \) we deduce
\[ \langle J'(h), \theta \rangle = -\frac{1}{2} \int_\Omega \theta \nabla u \cdot \nabla u \, dx, \]
which is the "usual" formula for the derivative of compliance minimization problem.

To minimize the objective function we should choose to increase the thickness since \( -\frac{1}{2} \nabla u \cdot \nabla u \leq 0 \).

Eventually, when \( u_0 = 0 \), we recover the standard case of compliance minimization with homogeneous Dirichlet boundary condition. We simply defined in an equivalent way the compliance
\[ J(h) = \int_\Omega \left( f u - \frac{1}{2} h \nabla u \cdot \nabla u \right) \, dx = \frac{1}{2} \int_\Omega f u \, dx. \]

2 Geometric optimization: 8 points

1. For Dirichlet boundary conditions we introduce two Lagrange multipliers: \( q \) for the p.d.e. and \( \lambda \) for the boundary condition. For any functions \( v, q, \lambda \in H^1(\mathbb{R}^N) \) we define the Lagrangian
\[ L(\Omega, v, q, \lambda) = \frac{1}{2} \int_\Omega |v-u_0|^2 \, dx + \int_\Omega (V \cdot \nabla v - \nu \Delta v - f) q \, dx + \int_{\partial \Omega} v \lambda \, ds. \]

Clearly we have
\[ \max_{q, \lambda} L(\Omega, v, q, \lambda) = \begin{cases} \frac{1}{2} \int_\Omega |u-u_0|^2 \, dx & \text{if } v = u \text{ is the state,} \\ +\infty & \text{otherwise.} \end{cases} \]

2. Two successive integrations by parts yield
\[ L(\Omega, v, q, \lambda) = \frac{1}{2} \int_\Omega |v-u_0|^2 \, dx + \int_\Omega (V \cdot \nabla q - \nu \Delta q) v \, dx - \int_\Omega f q \, dx \]
\[ + \int_{\partial \Omega} (V \cdot n v - \nu \partial_n v) q \, ds + \int_{\partial \Omega} \nu \partial_n q \, v \, ds + \int_{\partial \Omega} v \lambda \, ds. \]
To get the adjoint problem we differentiate the Lagrangian with respect to \( v \) and set this partial derivative equal to 0

\[
\left\langle \frac{\partial L}{\partial v}(\Omega, v, q, \lambda), \phi \right\rangle = \int_{\Omega} (v - u_0) \phi \, dx + \int_{\Omega} (-V \cdot \nabla q - \nu \Delta q) \phi \, dx + \int_{\partial\Omega} (V \cdot n \phi - \nu \partial_n \phi) q \, ds + \int_{\partial\Omega} \nu \partial_n q \phi \, ds + \int_{\partial\Omega} \phi \lambda \, ds.
\]

We first take a test function \( \phi \) with compact support in \( \Omega \), so we deduce that the optimal value of \( q \), the adjoint \( p \), satisfies

\[
-V \cdot \nabla p - \nu \Delta p = -(u - u_0) \quad \text{in } \Omega.
\]

Next, we take \( \phi = 0 \) on \( \partial\Omega \) but \( \partial_n \phi \) can be any function on \( \partial\Omega \). It yields \( p = 0 \) on \( \partial\Omega \). Finally, taking a general test function such that its trace \( \phi \) on \( \partial\Omega \) is any function, we get the optimal value of \( \lambda \)

\[
\lambda = -\nu \partial_n p \quad \text{on } \partial\Omega.
\]

The adjoint problem is thus

\[
\begin{cases}
-V \cdot \nabla p - \nu \Delta p = -(u - u_0) & \text{in } \Omega, \\
p = 0 & \text{on } \partial\Omega.
\end{cases}
\]

The differential operator of the adjoint equation is different from the one of the state equation: the sign of the velocity is changed in the convective term.

3. Formally we know that the shape derivative is given by

\[
J'(\Omega)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega, u, p, \lambda)(\theta).
\]

We compute

\[
\frac{\partial L}{\partial \Omega}(\Omega, v, q, \lambda)(\theta) = \frac{1}{2} \int_{\partial\Omega} |v - u_0|^2 \theta \cdot n \, ds + \int_{\partial\Omega} (V \cdot \nabla v - \nu \Delta v - f) q \theta \cdot n \, ds + \int_{\partial\Omega} \left( H v \lambda + \frac{\partial}{\partial n} (v \lambda) \right) \, ds,
\]

where \( H \) is the mean curvature. Replacing \( v \) by \( u \), \( q \) by \( p \) and \( \lambda \) by its optimal value \((-V \cdot np - \nu \partial_n p)\), and noticing that \( u = p = 0 \) on \( \partial\Omega \), we deduce

\[
J'(\Omega)(\theta) = \frac{1}{2} \int_{\partial\Omega} |u_0|^2 \theta \cdot n \, ds - \nu \int_{\partial\Omega} \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \theta \cdot n \, ds.
\]