ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Answers to the written exam of March 19th, 2014.

1 Parametric optimization: 10 points

1. Writing $v = \langle u'(h), k\rangle$ and differentiating problem (1) yields

$$\begin{cases} -\mathrm{div}\,(h\nabla v) = \mathrm{div}\,(k\nabla u) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

2. Writing $w = \langle v'(h), \tilde{k}\rangle$ and differentiating the previous problem leads to

$$\begin{cases} -\mathrm{div}\,(h\nabla w) = \mathrm{div}\left(\tilde{k}\nabla v\right) + \mathrm{div}\,(k\nabla \tilde{v}) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{v} = \langle u'(h), \tilde{k}\rangle$. Since v depends linearly on k and $\tilde{v}$ depends linearly on $\tilde{k}$ throught the same linear operator, the right hand side of the above equation is symmetric in $(k, \tilde{k})$, and so is w.

3. The first order derivative is

$$\langle J'(h), k\rangle = \int_\Omega f\, v\, dx$$

and the second order derivative is

$$\langle J''(h), (k, \tilde{k})\rangle = \int_\Omega f\, w\, dx.$$

4. To eliminate w in the formula for J'' we first multiply the equation for u by w and integrate by parts

$$\int_\Omega h\nabla u \cdot \nabla w\, dx = \int_\Omega f\, w\, dx.$$

Second, we multiply the equation for w by u and integrate by parts

$$\int_\Omega h\nabla w \cdot \nabla u\, dx = -\int_\Omega k\nabla \tilde{v} \cdot \nabla u\, dx - \int_\Omega \tilde{k}\nabla v \cdot \nabla u\, dx.$$

Then, by comparison we get

$$\langle J''(h), (k, \tilde{k})\rangle = \int_\Omega h\nabla u \cdot \nabla w\, dx = -\int_\Omega k\nabla \tilde{v} \cdot \nabla u\, dx - \int_\Omega \tilde{k}\nabla v \cdot \nabla u\, dx.$$

This last formula is symmetric in $(k, \tilde{k})$ by the same reasons as in question 2.

5. We multiply the equation for v by v to get

$$\int_\Omega h\nabla v\cdot\nabla v\,dx = -\int_\Omega k\nabla v\cdot\nabla u\,dx.$$

When $\tilde{k}=k$, we deduce from the above that

$$\langle J''(h),(k,k)\rangle = -2\int_\Omega k\nabla v\cdot\nabla u\,dx = 2\int_\Omega h\nabla v\cdot\nabla v\,dx \geq 0.$$

It implies that the objective function $h\to J(h)$ is convex. In particular, it implies that any local minimizer is actually a global minimizer.

2 Geometric optimization: 10 points

1. Since the boundary conditions are fixed (only the subdomain Ω is varying), we can use the standard variational formulation of problem (3) to build the Lagrangian. Introduce the space

$$V=\{\phi\in H^1(D) \text{ such that } \phi=0 \text{ on } \Gamma_D\}.$$

Thus, for any functions $v,q\in V$ and any subset $\Omega\subset D$, we define the Lagrangian

$$\mathcal{L}(\Omega,v,q)=\frac{1}{2}\int_{\Gamma_N}|v-u_0|^2\,ds+\int_D(\nabla v\cdot\nabla q+\chi_\Omega vq)\,dx-\int_{\Gamma_N}gq\,ds.$$

2. To get the adjoint problem we differentiate the Lagrangian with respect to v and set this partial derivative equal to 0. For any $\phi\in V$ we have

$$\langle\frac{\partial\mathcal{L}}{\partial v}(\Omega,v,q),\phi\rangle=\int_{\Gamma_N}(v-u_0)\phi\,ds+\int_D(\nabla\phi\cdot\nabla q+\chi_\Omega\phi q)\,dx.$$

By integration by parts, since $\phi=0$ on Γ_D, we deduce

$$\langle\frac{\partial\mathcal{L}}{\partial v}(\Omega,v,q),\phi\rangle=\int_{\Gamma_N}(v-u_0+\frac{\partial q}{\partial n})\phi\,ds+\int_D(-\Delta q+\chi_\Omega q)\phi\,dx.$$

We first take a test function ϕ with compact support in Ω, so we deduce that the optimal value of q, the adjoint p, satisfies

$$-\Delta p+\chi_\Omega p=0 \quad \text{in } D.$$

Then we take $\phi\neq 0$ on Γ_N so that

$$\frac{\partial p}{\partial n}=-(u-u_0) \quad \text{on } \Gamma_N.$$

Eventually, since $p \in V$ we have $p = 0$ on Γ_D. The adjoint problem is thus

$$\begin{cases} -\Delta p + \chi_\Omega p = 0 & \text{in } D, \\ p = 0 & \text{on } \Gamma_D \\ \frac{\partial p}{\partial n} = -(u - u_0) & \text{on } \Gamma_N. \end{cases}$$

3. Formally we know that the shape derivative is given by

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u, p)(\theta).$$

The only term which depends on Ω in $\mathcal{L}(\Omega, v, q)$ is

$$\int_D \chi_\Omega v q \, dx = \int_\Omega vq \, dx.$$

We compute its derivative

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, v, q)(\theta) = \int_{\partial\Omega} vq \, \theta \cdot n \, ds.$$

Therefore

$$J'(\Omega)(\theta) = \int_{\partial\Omega} up \, \theta \cdot n \, ds.$$

4. We multiply the equation for u by $u^- = \min(u, 0)$ to get

$$\int_D (\nabla u \cdot \nabla u^- + \chi_\Omega u u^-) \, dx = \int_{\Gamma_N} g u^- \, ds.$$

Since $u^- = 0$ if $u > 0$, we deduce

$$\int_D (|\nabla u^-|^2 + \chi_\Omega |u^-|^2) \, dx = \int_{\Gamma_N} g u^- \, ds.$$

The left hand side is non negative while the right hand side is non positive because $g \geq 0$ and $u^- \leq 0$. Therefore all terms are zero which implies that $u^- = 0$ everywhere in D. In other words, we have proved that $u \geq 0$ in D.

5. If $g \geq 0$, then $u \geq 0$. If $u \geq u_0$, then a similar argument as in the previous question shows that $p \leq 0$. Therefore, if $\theta \cdot n \geq 0$, we deduce

$$J'(\Omega)(\theta) = \int_{\partial\Omega} up \, \theta \cdot n \, ds \leq 0.$$

In other words the objective function decreases if the subdomain Ω is enlarged.

Of course, if $u \leq u_0$, then $p \geq 0$ and $J'(\Omega)(\theta) \leq 0$ for $\theta \cdot n \leq 0$ which means that the objective function decreases if the subdomain Ω is reduced.