1 Parametric optimization: 10 points

1. Writing \( v = h'(h), k_i \) and differentiating problem (1) yields
   \[
   \begin{cases}
   -\text{div} (h \nabla v) = \text{div} (k \nabla u) & \text{in } \Omega, \\
   v = 0 & \text{on } \partial \Omega.
   \end{cases}
   \]

2. Writing \( w = h'(h), \tilde{k}_i \) and differentiating the previous problem leads to
   \[
   \begin{cases}
   -\text{div} (h \nabla w) = \text{div} (\tilde{k} \nabla \tilde{v}) + \text{div} (k \nabla \tilde{v}) & \text{in } \Omega, \\
   w = 0 & \text{on } \partial \Omega,
   \end{cases}
   \]
   where \( \tilde{v} = h'(h), \tilde{k}_i \). Since \( v \) depends linearly on \( k \) and \( \tilde{v} \) depends linearly on \( \tilde{k} \) through the same linear operator, the right hand side of the above equation is symmetric in \((k, k_i)\), and so is \( w \).

3. The first order derivative is
   \[
   \mathcal{H}'(h), k_i = \int_{\Omega} f v \, dx
   \]
   and the second order derivative is
   \[
   \mathcal{H}''(h), (k, \tilde{k})_i = \int_{\Omega} f w \, dx.
   \]

4. To eliminate \( w \) in the formula for \( \mathcal{H}'' \) we first multiply the equation for \( u \) by \( w \) and integrate by parts
   \[
   \int_{\Omega} h \nabla u \cdot \nabla w \, dx = \int_{\Omega} f w \, dx.
   \]
   Second, we multiply the equation for \( w \) by \( u \) and integrate by parts
   \[
   \int_{\Omega} h \nabla w \cdot \nabla u \, dx = -\int_{\Omega} k \nabla \tilde{v} \cdot \nabla u \, dx - \int_{\Omega} \tilde{k} \nabla v \cdot \nabla u \, dx.
   \]
   Then, by comparison we get
   \[
   \mathcal{H}''(h), (k, \tilde{k})_i = \int_{\Omega} h \nabla u \cdot \nabla w \, dx = -\int_{\Omega} k \nabla \tilde{v} \cdot \nabla u \, dx - \int_{\Omega} \tilde{k} \nabla v \cdot \nabla u \, dx.
   \]
   This last formula is symmetric in \((k, \tilde{k})\) by the same reasons as in question 2.
5. We multiply the equation for \( v \) by \( v \) to get
\[
\int_{\Omega} h \nabla v \cdot \nabla v \, dx = - \int_{\Omega} k \nabla v \cdot \nabla u \, dx.
\]

When \( \tilde{k} = k \), we deduce from the above that
\[
\langle J''(h), (k,k) \rangle = -2 \int_{\Omega} k \nabla v \cdot \nabla u \, dx = 2 \int_{\Omega} h \nabla v \cdot \nabla v \, dx \geq 0.
\]

It implies that the objective function \( h \to J(h) \) is convex. In particular, it implies that any local minimizer is actually a global minimizer.

2 Geometric optimization: 10 points

1. Since the boundary conditions are fixed (only the subdomain \( \Omega \) is varying), we can use the standard variational formulation of problem (3) to build the Lagrangian. Introduce the space
\[
V = \{ \varphi \in H^1(D) \text{ such that } \varphi = 0 \text{ on } \Gamma_D \}.
\]

Thus, for any functions \( v, q \in V \) and any subset \( \Omega \subset D \), we define the Lagrangian
\[
L(\Omega, v, q) = \frac{1}{2} \int_{\Gamma_N} |v - u_0|^2 \, ds + \int_{D} (\nabla v \cdot \nabla q + \chi_{\Omega} v q) \, dx - \int_{\Gamma_N} g q \, ds.
\]

2. To get the adjoint problem we differentiate the Lagrangian with respect to \( v \) and set this partial derivative equal to 0. For any \( \varphi \in V \) we have
\[
\frac{\partial L}{\partial v}(\Omega, v, q), \varphi = \int_{\Gamma_N} (v - u_0) \varphi \, ds + \int_{D} (\nabla \varphi \cdot \nabla q + \chi_{\Omega} \varphi q) \, dx.
\]

By integration by parts, since \( \varphi = 0 \) on \( \Gamma_D \), we deduce
\[
\frac{\partial L}{\partial v}(\Omega, v, q), \varphi = \int_{\Gamma_N} (v - u_0) \varphi \, ds + \int_{D} (-\Delta q + \chi_{\Omega} q) \varphi \, dx.
\]

We first take a test function \( \varphi \) with compact support in \( \Omega \), so we deduce that the optimal value of \( q \), the adjoint \( p \), satisfies
\[
-\Delta p + \chi_{\Omega} p = 0 \quad \text{in } D.
\]

Then we take \( \varphi \equiv 0 \) on \( \Gamma_N \) so that
\[
\frac{\partial p}{\partial n} = -(u - u_0) \quad \text{on } \Gamma_N.
\]
Eventually, since \( p \in V \) we have \( p = 0 \) on \( \Gamma_D \). The adjoint problem is thus
\[
\begin{cases}
-\Delta p + \chi_\Omega p = 0 & \text{in } D, \\
p = 0 & \text{on } \Gamma_D, \\
\frac{\partial p}{\partial n} = -(u - u_0) & \text{on } \Gamma_N.
\end{cases}
\]

3. Formally we know that the shape derivative is given by
\[
J'(\Omega)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega, u, p)(\theta).
\]
The only term which depends on \( \Omega \) in \( L(\Omega, v, q) \) is
\[
\int_D \chi_\Omega v q \, dx = \int_\Omega v q \, dx.
\]
We compute its derivative
\[
\frac{\partial L}{\partial \Omega}(\Omega, v, q)(\theta) = \int_{\partial \Omega} v q \theta \cdot n \, ds.
\]
Therefore
\[
J'(\Omega)(\theta) = \int_{\partial \Omega} u p \theta \cdot n \, ds.
\]

4. We multiply the equation for \( u \) by \( u^- = \min(u, 0) \) to get
\[
\int_D (\nabla u \cdot \nabla u^- + \chi_\Omega u u^-) \, dx = \int_{\Gamma_N} g u^- \, ds.
\]
Since \( u^- = 0 \) if \( u > 0 \), we deduce
\[
\int_D (|\nabla u^-|^2 + \chi_\Omega |u^-|^2) \, dx = \int_{\Gamma_N} g u^- \, ds.
\]
The left hand side is non negative while the right hand side is non positive because \( g \geq 0 \) and \( u^- \leq 0 \). Therefore all terms are zero which implies that \( u^- = 0 \) everywhere in \( D \). In other words, we have proved that \( u \geq 0 \) in \( D \).

5. If \( g \geq 0 \), then \( u \geq 0 \). If \( u \geq u_0 \), then a similar argument as in the previous question shows that \( p \leq 0 \). Therefore, if \( \theta \cdot n \geq 0 \), we deduce
\[
J'(\Omega)(\theta) = \int_{\partial \Omega} u p \theta \cdot n \, ds \leq 0.
\]
In other words the objective function decreases if the subdomain \( \Omega \) is enlarged.

Of course, if \( u \leq u_0 \), then \( p \geq 0 \) and \( J'(\Omega)(\theta) \leq 0 \) for \( \theta \cdot n \leq 0 \) which means that the objective function decreases if the subdomain \( \Omega \) is reduced.