OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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CHAPTER VI

GEOMETRIC OPTIMIZATION (Second Part)

"Strategy" of the course

Computing the shape derivative of the solution of a p.d.e. is not easy !

- We explain **once** the rigorous method for computing a shape derivative.
- It is a bit involved and quite calculus-intensive...
- At the end we shall introduce a formal simpler method which is the one to be used **in practice**.
- This formal method is called the Lagrangian method and you should learn how to use it !

6.3.3. Derivation of a function depending on the shape

Let $u(\Omega, x)$ be a function defined on the domain Ω.

There exist two notions of derivative:

1) Eulerian (or shape) derivative U

$$u((\mathrm{Id}+\theta)\Omega_0, x) = u(\Omega_0, x) + U(\theta, x) + o(\theta) \quad , \quad \text{with} \quad \lim_{\theta\to 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0$$

OK if $x \in \Omega_0 \cap (\mathrm{Id}+\theta)\Omega_0$ (local definition, makes no sense on the boundary).

2) Lagrangian (or material) derivative Y

We define the **transported** function $\overline{u}(\theta)$ on Ω_0 by

$$\overline{u}(\theta, x) = u \circ (\mathrm{Id}+\theta) = u\Big((\mathrm{Id}+\theta)\Omega_0, x + \theta(x)\Big) \quad \forall\, x \in \Omega_0.$$

The Lagrangian derivative Y is obtained by differentiating $\overline{u}(\theta, x)$

$$\overline{u}(\theta, x) = \overline{u}(0, x) + Y(\theta, x) + o(\theta) \quad , \quad \text{with} \quad \lim_{\theta\to 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0\ ,$$

Differentiating $\overline{u} = u \circ (\text{Id} + \theta)$, one can check that

$$Y(\theta, x) = U(\theta, x) + \theta(x) \cdot \nabla u(\Omega_0, x).$$

The Eulerian derivative, although being simpler, is very delicate to use and often not rigorous. For example, if $u \in H_0^1(\Omega)$, the space of definition varies with Ω... Equivalently what boundary condition should the derivative satisfy ?

We recommend to use the Lagrangian derivative to avoid mistakes.

Remark. Computations will be made with Y but the final result is stated with U (which is simpler).

Composed shape derivative

Proposition 6.28. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$, and $u(\Omega) \in L^1(\mathbb{R}^N)$. We assume that the transported function $\overline{u}$ is diffrentiable at 0 from $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$, with derivative Y. Then

$$J(\Omega) = \int_\Omega u(\Omega)\, dx$$

is differentiable at Ω_0 and $\forall \theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \big(u(\Omega_0)\, \mathrm{div}\theta + Y(\theta)\big)\, dx.$$

In other words, using the Eulerian derivative U,

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \big(U(\theta) + \mathrm{div}(u(\Omega_0)\theta)\big)\, dx.$$

Similarly, if $\overline{u}(\theta)$ is differentiable at 0 as a function from $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ into $L^1(\partial\Omega_0)$, then

$$J(\Omega) = \int_{\partial\Omega} u(\Omega)\, dx$$

is differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \Big(u(\Omega_0)\,(\,\mathrm{div}\theta - \nabla\theta n\cdot n) + Y(\theta)\Big) ds.$$

In other words, using the Eulerian derivative U,

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(U(\theta) + \theta\cdot n \left(\frac{\partial u(\Omega_0)}{\partial n} + H u(\Omega_0) \right) \right) dx.$$

6.3.4 Shape derivation of an equation

From now on, $u(\Omega)$ is the solution of a p.d.e. in the domain Ω.

Recall that

$$Y(\theta, x) = U(\theta, x) + \theta(x) \cdot \nabla u(\Omega_0, x).$$

The Eulerian derivative, although being simpler, is very delicate to use and often not rigorous. For example, if $u \in H_0^1(\Omega)$, the space of definition varies with Ω... Equivalently what boundary condition should the derivative satisfy ?

We recommend to use the Lagrangian derivative: after getting back to the fixed reference domain Ω_0 we differentiate with respect to θ. This is the safest and most rigorous way for computing the shape derivative of u, but the details can be tricky.

We shall later introduce a heuristic method which is simpler.

The results depend on the type of boundary conditions.

Dirichlet boundary conditions

For $f \in L^2(\mathbb{R}^N)$ we consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits a unique solution $u(\Omega) \in H_0^1(\Omega)$.

Its variational formulation is: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \, \phi \in H_0^1(\Omega).$$

(Simplification with respect to the textbook since here $g = 0$.)

For $\Omega = (\,\mathrm{Id} + \theta)(\Omega_0)$ we define the change of variables

$$x = y + \theta(y) \quad y \in \Omega_0 \quad x \in \Omega.$$

Proposition 6.30. Let $u(\Omega) \in H_0^1(\Omega)$ be the solution and $\overline{u}(\theta) \in H_0^1(\Omega_0)$ be its transported function

$$\overline{u}(\theta)(y) = u(\Omega)(x) = u\Big((\,\mathrm{Id} + \theta)(\Omega_0)\Big) \circ (\,\mathrm{Id} + \theta)(y).$$

The functional $\theta \to \overline{u}(\theta)$, from $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ into $H^1(\Omega_0)$, is differentiable at 0, and its derivative in the direction θ, called Lagrangian derivative is

$$Y = \langle \overline{u}'(0), \theta \rangle$$

where $Y \in H_0^1(\Omega_0)$ is the unique solution of

$$\begin{cases} -\Delta Y = -\Delta\big(\theta \cdot \nabla u(\Omega_0)\big) & \text{in } \Omega_0 \\ Y = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Proof. We perform the change of variables $x = y + \theta(y)$ with $y \in \Omega_0$ in the variational formulation

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \, \phi \in H_0^1(\Omega).$$

Take a test function $\phi = \psi \circ (\,\mathrm{Id} + \theta)^{-1}$, i.e., $\psi(y) = \phi(x)$. Recall that

$$\left(\nabla \phi\right) \circ (\,\mathrm{Id} + \theta) = \left((I + \nabla\theta)^{-1}\right)^t \nabla \Big(\phi \circ (\,\mathrm{Id} + \theta)\Big).$$

We obtain: find $\overline{u} \in H_0^1(\Omega_0)$ such that, for any $\psi \in H_0^1(\Omega_0)$,

$$\int_{\Omega_0} A(\theta) \nabla \overline{u} \cdot \nabla \psi \, dy = \int_{\Omega_0} f \circ (\,\mathrm{Id} + \theta) \, \psi \, |\det(\,\mathrm{Id} + \nabla\theta)| dy$$

with $A(\theta) = |\det(I + \nabla\theta)| (I + \nabla\theta)^{-1} \left((I + \nabla\theta)^{-1}\right)^t$.

We differentiate with respect to θ at 0 the variational formulation

$$\int_{\Omega_0} A(\theta)\nabla\overline{u}\cdot\nabla\psi\,dy = \int_{\Omega_0} f\circ(\,\mathrm{Id}+\theta)\,\psi\,|\det(\,\mathrm{Id}+\nabla\theta)|dy$$

where ψ is a function which does not depend on θ.

We already checked in the proof of Proposition 6.22 that the righ hand side is differentiable. Furthermore, the map $\theta \to A(\theta)$ is differentiable too because

$$A(\theta) = (1+\,\mathrm{div}\theta)I - \nabla\theta - (\nabla\theta)^t + o(\theta) \quad \text{with} \quad \lim_{\theta\to 0}\frac{\|o(\theta)\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^{N^2})}}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)}} = 0.$$

Since $\overline{u}(\theta=0)=u(\Omega_0)$, we get

$$\int_{\Omega_0} \nabla Y \cdot \nabla \psi \, dy + \int_{\Omega_0} \Big(\mathrm{div}\theta \, I - \nabla\theta - (\nabla\theta)^t \Big) \nabla u(\Omega_0) \cdot \nabla \psi \, dy = \int_{\Omega_0} \mathrm{div}\Big(f\theta \Big) \psi \, dy$$

Since $\overline{u}(\theta) \in H^1_0(\Omega_0)$, its derivative Y belongs to $H^1_0(\Omega_0)$ too. Thus Y is a solution of

$$\begin{cases} -\Delta Y = \mathrm{div}\left[\Big(\mathrm{div}\theta \, I - \nabla\theta - (\nabla\theta)^t \Big) \nabla u(\Omega_0) \right] + \mathrm{div}\Big(f\theta \Big) & \text{in } \Omega_0 \\ Y = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Recalling that $\Delta u(\Omega_0) = -f$ in Ω_0, and using the identity (true for any $v \in H^1(\Omega_0)$ such that $\Delta v \in L^2(\Omega_0)$)

$$\Delta\left(\nabla v \cdot \theta\right) = \mathrm{div}\left((\Delta v)\theta - (\mathrm{div}\theta)\nabla v + \Big(\nabla\theta + (\nabla\theta)^t \Big) \nabla v \right),$$

leads to the final result. (gotcha !)

Shape derivative U

Corollary 6.32. The Eulerian derivative U of the solution $u(\Omega)$, defined by formula

$$U = Y - \nabla u(\Omega_0) \cdot \theta,$$

is the solution in $H^1(\Omega_0)$ of

$$\begin{cases} -\Delta U = 0 & \text{in } \Omega_0 \\ U = -(\theta \cdot n) \frac{\partial u(\Omega_0)}{\partial n} & \text{on } \partial\Omega_0. \end{cases}$$

(Obvious proof starting from Y.)

We are going to recover **formally** this p.d.e. for U without using the knowledge of Y.

Let ϕ be a compactly supported test function in $\omega \subset \Omega$ for the variational formulation

$$\int_\omega \nabla u \cdot \nabla \phi \, dx = \int_\omega f \phi \, dx.$$

Differentiating with respect to Ω, neither the test function, nor the domain of integration depend on Ω. Thus it yields

$$\int_\omega \nabla U \cdot \nabla \phi \, dx = 0 \quad \Leftrightarrow \quad -\Delta U = 0.$$

To find the boundary condition we formally differentiate

$$\int_{\partial\Omega} u(\Omega) \psi \, ds = 0 \quad \forall \, \psi \in C^\infty(\mathbb{R}^N)$$

$$\Rightarrow \int_{\partial\Omega_0} U \psi \, ds + \int_{\partial\Omega_0} \left(\frac{\partial (u\psi)}{\partial n} + H u \psi \right) \theta \cdot n \, ds = 0$$

which leads to the correct result since $u = 0$ on $\partial\Omega_0$.

Remark. The direct computation of U is not always that easy !

Neumann boundary conditions

For $f \in H^1(\mathbb{R}^N)$ and $g \in H^2(\mathbb{R}^N)$ we consider the boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega \end{cases}$$

which admits a unique solution $u(\Omega) \in H^1(\Omega)$.

Its variational formulation is: find $u \in H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla \phi + u\phi)\, dx = \int_\Omega f\phi\, dx + \int_{\partial\Omega} g\phi\, ds \quad \forall\, \phi \in H^1(\Omega).$$

Proposition 6.34. For $\Omega = (\,\mathrm{Id} + \theta)(\Omega_0)$ we define the change of variables

$$x = y + \theta(y) \quad y \in \Omega_0 \quad x \in \Omega.$$

Let $u(\Omega) \in H^1(\Omega)$ be the solution and $\overline{u}(\theta) \in H^1(\Omega_0)$ be its transported function

$$\overline{u}(\theta)(y) = u(\Omega)(x) = u\Big((\,\mathrm{Id} + \theta)(\Omega_0)\Big) \circ (\,\mathrm{Id} + \theta)(y).$$

The functional $\theta \to \overline{u}(\theta)$, from $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ into $H^1(\Omega_0)$, is differentiable at 0, and its derivative in the direction θ, called Lagrangian derivative is

$$Y = \langle \overline{u}'(0), \theta \rangle$$

where $Y \in H^1(\Omega_0)$ is the unique solution of

$$\begin{cases} -\Delta Y + Y = -\Delta(\nabla u(\Omega_0) \cdot \theta) + \nabla u(\Omega_0) \cdot \theta & \text{in } \Omega_0 \\ \dfrac{\partial Y}{\partial n} = (\nabla \theta + (\nabla \theta)^t) \nabla u(\Omega_0) \cdot n + \nabla g \cdot \theta - g(\nabla \theta n \cdot n) & \text{on } \partial \Omega_0. \end{cases}$$

Proof. We perform the change of variables $x = y + \theta(y)$ with $y \in \Omega_0$ in the variational formulation. Take a test function $\phi = \psi \circ (\,\mathrm{Id} + \theta)^{-1}$, i.e., $\psi(y) = \phi(x)$. We get

$$\begin{aligned}\int_{\Omega_0} A(\theta)\nabla\overline{u}\cdot\nabla\psi\,dy \quad & + \int_{\Omega_0} \overline{u}\psi|\det(I+\nabla\theta)|dy \\ & = \int_{\Omega_0} f\circ(\,\mathrm{Id}+\theta)\,\psi|\det(I+\nabla\theta)|dy \\ & + \int_{\partial\Omega_0} g\circ(\,\mathrm{Id}+\theta)\,\psi|\det(I+\nabla\theta)|\,|\,(I+\nabla\theta)^{-t}n\,|\,ds\end{aligned}$$

with $A(\theta) = |\det(I+\nabla\theta)|(I+\nabla\theta)^{-1}\left((I+\nabla\theta)^{-1}\right)^t$.

We differentiate with respect to θ at 0.

The only new term is the boundary integral which can be differentiated like in Proposition 6.24.

Defining $Y = \langle \overline{u}'(0), \theta \rangle$ we deduce

$$\begin{aligned}\int_{\Omega_0} (\nabla Y \cdot \nabla \psi + Y\psi)\, dy + & \int_{\Omega_0} \big(\,\mathrm{div}\theta\, I - \nabla\theta - (\nabla\theta)^t\big)\, \nabla\overline{u} \cdot \nabla\psi\, dy \\ & + \int_{\Omega_0} \overline{u}\psi\, \mathrm{div}\theta\, dy = \int_{\Omega_0} \mathrm{div}(f\theta)\psi\, dy \\ & + \int_{\partial\Omega_0} \big(\nabla g \cdot \theta + g\big(\,\mathrm{div}\theta - \nabla\theta n \cdot n\big)\big)\, \psi ds\end{aligned}$$

Then we recall that $\overline{u}(0) = u(\Omega_0) = u$, $\Delta u = u - f$ in Ω_0 and $\frac{\partial u}{\partial n} = g$ on $\partial\Omega_0$, and the identity

$$\Delta\,(\nabla v \cdot \theta) = \,\mathrm{div}\,\big((\Delta v)\theta - (\,\mathrm{div}\theta)\nabla v + (\nabla\theta + (\nabla\theta)^t)\nabla v\big)\,,$$

to get the result. Simple in principle but computationally intensive...

Corollary 6.36. The Eulerian derivative U of the solution $u(\Omega)$, defined by

$$U = Y - \nabla u(\Omega_0) \cdot \theta,$$

is a solution in $H^1(\Omega_0)$ of

$$-\Delta U + U = 0 \quad \text{in } \Omega_0.$$

and satisfies the boundary condition

$$\frac{\partial U}{\partial n} = \theta \cdot n \left(\frac{\partial g}{\partial n} - \frac{\partial^2 u(\Omega_0)}{\partial n^2} \right) + \nabla_t(\theta \cdot n) \cdot \nabla_t u(\Omega_0) \quad \text{on} \quad \partial\Omega_0,$$

where $\nabla_t \phi = \nabla \phi - (\nabla \phi \cdot n) n$ denotes the tangential gradient on the boundary.

Proof. Easy but tedious computation.

6.4 Gradient and optimality condition

We consider the shape optimization problem

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega),$$

with $\mathcal{U}_{ad} = \left\{ \Omega = (\,\mathrm{Id} + \theta)(\Omega_0) \text{ and } \int_\Omega dx = V_0 \right\}$. The cost function $J(\Omega)$ is either the compliance, or a least square criterion for a target displacement $u_0(x) \in L^2(\mathbb{R}^N)$

$$J(\Omega) = \int_\Omega f u \, dx + \int_{\partial\Omega} g u \, ds \quad \text{or} \quad J(\Omega) = \int_\Omega |u - u_0|^2 dx.$$

The function $u(\Omega)$ is the solution in $H^1(\Omega)$ of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega, \end{cases}$$

with $f \in H^1(\mathbb{R}^N)$ and $g \in H^2(\mathbb{R}^N)$.

Gradient and optimality condition

Theorem 6.38. The functional $J(\Omega) = \int_\Omega |u - u_0|^2 dx$ is shape differentiable

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(|u - u_0|^2 + \nabla u \cdot \nabla p + p(u - f) - \frac{\partial(gp)}{\partial n} - Hgp \right) ds,$$

where p is the adjoint state, unique solution in $H^1(\Omega_0)$ of

$$\begin{cases} -\Delta p + p = -2\,(u - u_0) & \text{in } \Omega_0 \\ \frac{\partial p}{\partial n} = 0 & \text{on } \partial\Omega_0, \end{cases}$$

We recover the fact that the shape derivative depends only on the normal trace of θ on the boundary.

Proof. Applying Proposition 6.28 to the cost function yields

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \left(|u(\Omega_0) - u_0|^2 \operatorname{div}\theta + 2(u(\Omega_0) - u_0)(Y - \nabla u_0 \cdot \theta)\right) dx,$$

or equivalently, with $U = Y - \nabla u(\Omega_0) \cdot \theta$,

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \left[\operatorname{div} \left(\theta |u(\Omega_0) - u_0|^2\right) + 2(u(\Omega_0) - u_0)U \right] dx.$$

Multiplying the adjoint equation by U

$$\int_{\Omega_0} (\nabla p \cdot \nabla U + pU)\, dy = -2 \int_{\Omega_0} (u(\Omega_0) - u_0)\, U\, dy,$$

then the equation for U by p

$$\begin{aligned} \int_{\Omega_0} (\nabla p \cdot \nabla U + pU)\, dy \;&= \\ \int_{\partial\Omega_0} \theta \cdot n \left(-\nabla u(\Omega_0) \cdot \nabla p - p\Delta u(\Omega_0) + \frac{\partial (gp)}{\partial n} + Hgp \right) ds,& \end{aligned}$$

we deduce the result by comparison of the two equalities.

The compliance case (self-adjoint)

Theorem 6.40. The functional $J(\Omega) = \int_\Omega f u \, dx + \int_{\partial\Omega} g u \, ds$ is shape-differentiable

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(-|\nabla u(\Omega_0)|^2 - |u(\Omega_0)|^2 + 2u(\Omega_0) f \right) ds$$

$$+ \int_{\partial\Omega_0} \theta \cdot n \left(2 \frac{\partial (g u(\Omega_0))}{\partial n} + 2 H g u(\Omega_0) \right) ds,$$

Interpretation: assume $f = 0$ and $g = 0$ where $\theta \cdot n \neq 0$. The formula simplifies in

$$J'(\Omega_0)(\theta) = - \int_{\partial\Omega_0} \theta \cdot n \left(|\nabla u|^2 + u^2 \right) ds \leq 0$$

It is always advantageous to increase the domain (i.e., $\theta \cdot n > 0$) for decreasing the compliance.

Proof. Applying Proposition 6.28 to the cost function yields

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \left(fu \,\mathrm{div}\theta + u\theta \cdot \nabla f + fY \right) dx + \int_{\partial\Omega_0} \left(gu \left(\mathrm{div}\theta - \nabla\theta n \cdot n \right) + u\theta \cdot \nabla g + gY \right) ds,$$

or equivalently, with $U = Y - \nabla u \cdot \theta$,

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \left(\mathrm{div}(fu\theta) + fU \right) dx + \int_{\partial\Omega_0} \left(\theta \cdot n \left(\frac{\partial(gu)}{\partial n} + Hgu \right) + gU \right) ds.$$

Multiplying the equation for u by U and that for U by u, then comparing, leads to the result.

Remark. Same type of result for a Dirichlet boundary condition (but different formulas).

6.4.3 Fast derivation: the Lagrangian method

- The previous computations are quite tedious... but there is a simpler and faster (albeit formal) method, called the Lagrangian method (proposed in this context by J. Céa).
- The Lagrangian allows us to find the correct definition of the adjoint state too.
- It is easy for Neumann boundary conditions, a little more involved for Dirichlet ones.
- That is the method to be known !

Fast derivation for Neumann boundary conditions

If the objective function is

$$J(\Omega) = \int_\Omega j(u(\Omega))\, dx,$$

the Lagrangian is defined as the sum of J and of the variational formulation of the state equation

$$\mathcal{L}(\Omega, v, q) = \int_\Omega j(v)\, dx + \int_\Omega \Big(\nabla v \cdot \nabla q + vq - fq \Big) dx - \int_{\partial\Omega} gq\, ds,$$

with v and $q \in H^1(\mathbb{R}^N)$. It is important to notice that the space $H^1(\mathbb{R}^N)$ does not depend on Ω and thus the three variables in $\mathcal{L}$ are clearly independent.

The partial derivative of $\mathcal{L}$ with respect to q in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial q}(\Omega, v, q), \phi \rangle = \int_\Omega \Big(\nabla v \cdot \nabla \phi + v\phi - f\phi \Big) dx - \int_{\partial\Omega} g\phi \, ds,$$

which, upon equating to 0, gives the variational formulation of the state.

The partial derivative of $\mathcal{L}$ with respect to v in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \rangle = \int_\Omega j'(v)\phi \, dx + \int_\Omega \Big(\nabla \phi \cdot \nabla q + \phi q \Big) dx,$$

which, upon equating to 0, gives the variational formulation of the adjoint.

The partial derivative of $\mathcal{L}$ with respect to Ω in the direction θ is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, v, q)(\theta) = \int_{\partial\Omega} \theta \cdot n \left(j(v) + \nabla v \cdot \nabla q + vq - fq - \frac{\partial (gq)}{\partial n} - Hgq \right) ds.$$

When evaluating this derivative with the state $u(\Omega_0)$ and the adjoint $p(\Omega_0)$, we precisely find the derivative of the objective function

$$\boxed{\frac{\partial \mathcal{L}}{\partial \Omega}\Big(\Omega_0, u(\Omega_0), p(\Omega_0)\Big)(\theta) = J'(\Omega_0)(\theta)}$$

Indeed, if we differentiate the equality

$$\mathcal{L}(\Omega, u(\Omega), q) = J(\Omega) \quad \forall\, q \in H^1(\mathbb{R}^N),$$

the chain rule lemma yields

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u(\Omega_0), q)(\theta) + \langle \frac{\partial \mathcal{L}}{\partial v}(\Omega_0, u(\Omega_0), q), u'(\Omega_0)(\theta)\rangle$$

Taking $q = p(\Omega_0)$, the last term cancels since $p(\Omega_0)$ is the solution of the adjoint equation.

Thanks to this computation, the “correct” result can be guessed for $J'(\Omega_0)$ without using the notions of shape or material derivatives.

Nevertheless, in full rigor, this “fast” computation of the shape derivative $J'(\Omega_0)$ is valid only if we know that u is shape differentiable.

Fast derivation for Dirichlet boundary conditions

It is more involved ! Let $u \in H_0^1(\Omega)$ be the solution of

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

The "usual" Lagrangian is

$$\mathcal{L}(\Omega, v, q) = \int_\Omega j(v) \, dx + \int_\Omega \Big(\nabla v \cdot \nabla q - f q \Big) dx,$$

for $v, q \in H_0^1(\Omega)$. The variables (Ω, v, q) are not independent !

Indeed, the functions v and q satisfy

$$v = q = 0 \quad \text{on } \partial\Omega.$$

Another Lagrangian has to be introduced.

Lagrangian for Dirichlet boundary conditions

The Dirichlet boundary condition is penalized

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_\Omega j(v)\,dx - \int_\Omega (\Delta v + f)q\,dx + \int_{\partial\Omega} \lambda v\,ds$$

where λ is the Lagrange multiplier for the boundary condition. It is now possible to differentiate since the 4 variables $v, q, \lambda \in H^1(\mathbb{R}^N)$ are independent.

Of course, we recover

$$\sup_{q,\lambda} \mathcal{L}(\Omega, v, q, \lambda) = \begin{cases} \int_\Omega j(u)\,dx = J(\Omega) & \text{if } v \equiv u, \\ +\infty & \text{otherwise.} \end{cases}$$

By definition of the Lagrangian:

the partial derivative of $\mathcal{L}$ with respect to q in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial q}(\Omega, v, q, \lambda), \phi \rangle = - \int_\Omega \phi \Big(\Delta v + f \Big) dx,$$

which, upon equating to 0, gives the state equation,

the partial derivative of $\mathcal{L}$ with respect to λ in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial \lambda}(\Omega, v, q, \lambda), \phi \rangle = \int_{\partial\Omega} \phi v \, dx,$$

which, upon equating to 0, gives the Dirichlet boundary condition for the state equation.

To compute the partial derivative of $\mathcal{L}$ with respect to v, we perform a first integration by parts

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_\Omega j(v)\,dx + \int_\Omega (\nabla v \cdot \nabla q - fq)\,dx + \int_{\partial\Omega} \left(\lambda v - \frac{\partial v}{\partial n} q\right) ds,$$

then a second integration by parts

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_\Omega j(v)\,dx - \int_\Omega (v\Delta q - fq)\,dx + \int_{\partial\Omega} \left(\lambda v - \frac{\partial v}{\partial n} q + \frac{\partial q}{\partial n} v\right) ds.$$

We now can differentiate in the direction $\phi \in H^1(\mathbb{R}^N)$

$$\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \rangle = \int_\Omega j'(v)\phi\,dx - \int_\Omega \phi \Delta q\,dx + \int_{\partial\Omega} \left(-q\frac{\partial \phi}{\partial n} + \phi\left(\lambda + \frac{\partial q}{\partial n}\right)\right) ds$$

which, upon equating to 0, gives three relationships, the two first ones being the adjoint problem.

1. If ϕ has compact support in Ω_0, we get

$$-\Delta p = -j'(u) \quad \text{dans} \quad \Omega_0.$$

2. If $\phi = 0$ on $\partial\Omega_0$ with any value of $\frac{\partial \phi}{\partial n}$ in $L^2(\partial\Omega_0)$, we deduce

$$p = 0 \quad \text{sur} \quad \partial\Omega_0.$$

3. If ϕ is now varying in the full $H^1(\Omega_0)$, we find

$$\frac{\partial p}{\partial n} + \lambda = 0 \quad \text{sur} \quad \partial\Omega_0.$$

The adjoint problem has actually been recovered but furthermore the optimal Lagrange multiplier λ has been characterized.

Eventually, the shape partial derivative is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u, p, \lambda)(\theta) = \int_{\partial \Omega_0} \theta \cdot n \Big(j(u) - (\Delta u + f)p + \frac{\partial (u\lambda)}{\partial n} + Hu\lambda \Big) ds$$

Knowing that $u = p = 0$ on $\partial \Omega_0$ and $\lambda = -\frac{\partial p}{\partial n}$ we deduce

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u, p, \lambda)(\theta) = \int_{\partial \Omega_0} \theta \cdot n \Big(j(0) - \frac{\partial u}{\partial n}\frac{\partial p}{\partial n} \Big) ds = J'(\Omega_0)(\theta)$$

$$\boxed{J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}\Big(\Omega_0, u(\Omega_0), p(\Omega_0)\Big)(\theta)}$$

This formula is not a surprise because differentiating

$$\mathcal{L}(\Omega, u(\Omega), q, \lambda) = J(\Omega) \quad \forall q, \lambda$$

yields

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u(\Omega_0), q, \lambda)(\theta) + \langle \frac{\partial \mathcal{L}}{\partial v}(\Omega_0, u(\Omega_0), q, \lambda), u'(\Omega_0)(\theta)\rangle.$$

Then, taking $q = p(\Omega_0)$ (the adjoint state) and $\lambda = -\frac{\partial p}{\partial n}(\Omega_0)$, the last term cancels and we obtain the desired formula.

Application to compliance minimization

We minimize $J(\Omega) = \int_\Omega f u \, dx$ with $u \in H^1_0(\Omega)$ solution of

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \, \phi \in H^1_0(\Omega).$$

The adjoint state is just $p = -u$. The shape derivative is

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \Big(f u - \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \Big) ds = \int_{\partial\Omega_0} \theta \cdot n \left(\frac{\partial u}{\partial n} \right)^2 ds \leq 0$$

It is always advantageous to shrink the domain (i.e., $\theta \cdot n < 0$) to decrease the compliance.

This is the opposite conclusion compared to Neumann b.c., but it is logical !

Another example: the drum

We optimize the shape of a drum (an elastic membrane) in order it produces the lowest possible tune. Let $\lambda(\Omega)$ be the eigenvalue (the square of the eigenfrequency) and $u(x)$ be the eigenmode

$$\begin{cases} -\Delta u = \lambda(\Omega) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The fundamental mode is the smallest eigenvalue which is also characterized by

$$\lambda(\Omega) = \min_{u \in H^1_0(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

Thus we study

$$\inf_{\Omega \subset \mathbb{R}^2} \left(\lambda(\Omega) + \ell \int_\Omega dx \right),$$

where $\ell \geq 0$ is a given Lagrange multiplier for a constraint on the membrane area.

Eulerian derivation

For a test function ϕ with compact support $\omega \subset \Omega$ we derive

$$\int_\omega \nabla u \cdot \nabla \phi \, dx = \lambda(\Omega) \int_\omega u \phi \, dx$$

$$\Rightarrow \quad \int_\omega \nabla U \cdot \nabla \phi \, dx = \lambda(\Omega) \int_\omega U \phi \, dx + \Lambda \int_\omega u \phi \, dx,$$

where $\Lambda = \lambda'(\Omega)(\theta)$ is the derivative of the eigenvalue (assumed to be simple).

$$\Rightarrow \quad -\Delta U - \lambda(\Omega) U = \Lambda u \quad \text{in } \Omega.$$

To deduce the boundary condition for U we derive

$$\int_{\partial\Omega} u \psi \, ds = 0 \quad \forall \psi \in C^\infty(\mathbb{R}^2).$$

$$\Rightarrow \quad \int_{\partial\Omega} \left(U \psi + \theta \cdot n \left(\frac{\partial (u\psi)}{\partial n} + H u \psi \right) \right) ds = 0,$$

which yields $U = -\frac{\partial u}{\partial n} \theta \cdot n$ since $u = 0$ on $\partial\Omega$.

Multiplying the equation for U by u and integrating by parts leads to

$$\int_\Omega \nabla U \cdot \nabla u \, dx = \lambda \int_\Omega U u \, dx + \Lambda \int_\Omega u^2 \, dx.$$

Multiplying the equation for u by U and integrating by parts leads to

$$\int_\Omega \nabla U \cdot \nabla u \, dx = \lambda \int_\Omega U u \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} U \, ds.$$

Thus, we deduce

$$\Lambda \int_\Omega u^2 dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} U \, ds = - \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 \theta \cdot n \, ds.$$

The derivative of the objective function is (self-adjoint problem)

$$J'(\Omega)(\theta) = \Lambda + \ell \int_{\partial\Omega} \theta \cdot n \, ds = \int_{\partial\Omega} \left(\ell - \frac{\left(\frac{\partial u}{\partial n}\right)^2}{\int_\Omega u^2 dx} \right) \theta \cdot n \, ds.$$

If $\ell = 0$ we have $J'(\Omega)(\theta) \leq 0$ as soon as $\theta \cdot n \geq 0$, i.e., we minimze $J(\Omega)$ if the domain Ω is enlarged.

Lagrangian method

For $\mu \in \mathbb{R}$, $v, q, z \in H^1(\mathbb{R}^N)$, we introduce the Lagrangian

$$\mathcal{L}(\Omega, \mu, v, q, z) = \mu - \int_\Omega (\Delta v + \mu v) q \, dx + \int_{\partial\Omega} z v \, ds$$

where z is the Lagrange multiplier for the boundary condition. Since the 5 variables are independent it is possible to differentiate.

The partial derivative $\frac{\partial \mathcal{L}}{\partial q} = 0$ gives the state equation.

The partial derivative $\frac{\partial \mathcal{L}}{\partial z} = 0$ gives the Dirichlet boundary condition for the state.

The partial derivative $\frac{\partial \mathcal{L}}{\partial v} = 0$ gives three relationships including the adjoint:

$$-\Delta p = \lambda p \quad \text{in} \quad \Omega, \quad p = 0 \quad \text{on} \quad \partial\Omega, \quad \frac{\partial p}{\partial n} + z = 0 \quad \text{on} \quad \partial\Omega.$$

The partial derivative $\frac{\partial \mathcal{L}}{\partial \mu} = 0$ yields

$$\int_\Omega up\, dx = 1$$

Since the eigenvalue λ is simple, p is a multiple of u. Thus

$$p = \frac{u}{\int_\Omega u^2 dx}.$$

Eventually, the shape partial derivative is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, \lambda, u, p, z)(\theta) = \int_{\partial\Omega} \theta \cdot n \Big(p\Delta u + \lambda p u + \frac{\partial (uz)}{\partial n} + Huz \Big) ds$$

Knowing that $u = p = 0$ on $\partial\Omega$ and $z = -\frac{\partial p}{\partial n}$ we deduce

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, \lambda, u, p, z)(\theta) = \int_{\partial\Omega} \theta \cdot n \Big(- \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \Big) ds = J'(\Omega)(\theta)$$