ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Written exam, March 16th, 2011 (2 hours)

1 Parametric optimization: 12 points

We consider an elastic membrane with a variable thickness $h(x)$, clamped on its boundary, occupying at rest a plane domain Ω (a smooth bounded open set of $\mathbb{R}^2$). This membrane is loaded by a time-dependent force $f(t,x) \in C\left([0,T]; L^2(\Omega)\right)$ where $T > 0$ is a given final time. Its vertical displacement $u(t,x)$ is the unique solution of a dissipative evolution equation

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}\,(h\nabla u) = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(t=0,x) = u_{init}(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $u_{init} \in L^2(\Omega)$ is a given initial data. The thickness belongs to the following space of admissible designs

$$\mathcal{U}_{ad} = \left\{ h \in L^\infty(\Omega)\ , \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega \right\}.$$

The goal is to minimize the objective function

$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J(h) = \int_0^T \int_\Omega j_1(u(t,x))\,dt\,dx + \int_\Omega j_2(u(T,x))\,dx \right\}, \tag{2}$$

where j_1 and j_2 are two smooth functions satisfying

$$|j_k(v)| \leq C(|v|^2+1), \quad |j_k'(v)| \leq C(|v|+1), \quad |j_k''(v)| \leq C.$$

For an integer $N \geq 1$ we define a time step $\Delta t = T/N$, discrete times $t^n = n\Delta t$ and we discretize the previous problem by the following scheme

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta t} - \mathrm{div}\,(h\nabla u^n) = f^n & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J_{\Delta t}(h) = \sum_{n=1}^{N} \Delta t \int_\Omega j_1(u^n(x))\,dx + \int_\Omega j_2(u^N(x))\,dx \right\}, \tag{4}$$

where, for $1 \leq n \leq N$, $f^n(x) = f(t^n, x)$, $u^0(x) = u_{init}(x)$ and $u^n(x)$ is an approximation of $u(t^n, x)$.

1. Write the variational formulation for the unknown u^n in terms of u^{n-1}.

2. Write the Lagrangian associated to the objective function $J_{\Delta t}(h)$.

3. By differentiating the Lagrangian with respect to u^n, deduce the adjoint variational formulation for an adjoint state p^n, for $1 \leq n \leq N$. Write down explicitly the boundary value problem satisfied by p^n. In which order shall we compute the adjoints p^n ?

4. Compute (formally) the derivative of $J_{\Delta t}(h)$.

5. Show that the partial differential equations for $\{p^n(x)\}_{1\leq n\leq N}$ (possibly multiplied by a suitable coefficient) are a time discretization of an evolution equation for a formal limit $p(t,x)$. Give explicitly the boundary value problem satisfied by $p(t,x)$. Similarly, give the formal limit of the derivative of $J_{\Delta t}(h)$ as Δt goes to 0.

6. Explain what is the main computational cost or "bottle-neck" (in terms of CPU and/or memory requirement) in the numerical evaluation of the derivative of $J_{\Delta t}(h)$ when the number N of time steps is large (compared to the steady-state case).

2 Topology optimization: 8 points

In a smooth bounded domain $\Omega \subset \mathbb{R}^N$ we consider a mixture of three conducting phases with conductivities $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3$ occupying three disjoint complementary subsets of Ω with characteristic functions χ_1, χ_2, χ_3, satisfying for any $x \in \Omega$

$$\chi_i(x) = 0 \text{ or } 1 \quad \text{and} \quad \sum_{i=1}^{3} \chi_i(x) = 1. \tag{5}$$

We denote by $a_\chi(x)$ the mixture conductivity

$$a_\chi(x) = \alpha_1\chi_1(x) + \alpha_2\chi_2(x) + \alpha_3\chi_3(x)$$

and for a given source term $f(x) \in L^2(\Omega)$ we solve

$$\begin{cases} -\mathrm{div}\,(a_\chi \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

We maximize the compliance, i.e. we want to obtain the worst possible conducting mixture,

$$\inf_{\chi \in \mathcal{U}_{ad}} \left\{ J(\chi) = -\int_\Omega f(x)\, u(x)\, dx \right\}, \tag{7}$$

where, for $0 \leq c_i \leq 1$ such that $\sum_{i=1}^{3} c_i = 1$, the set of admissible designs is

$$\mathcal{U}_{ad} = \left\{ \chi = (\chi_1, \chi_2, \chi_3) \text{ satisfying (5) and } \int_\Omega \chi_i(x) dx = c_i |\Omega| \right\}.$$

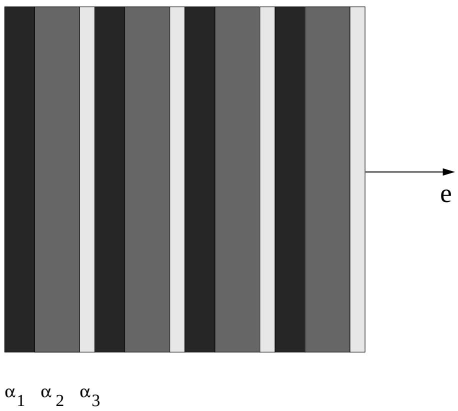
We recall that minus the compliance can be rewritten as

$$-\int_\Omega f(x)\,u(x)\,dx = \min_{v\in H_0^1(\Omega)} \int_\Omega \left(a_\chi|\nabla v|^2 - 2fv\right)dx. \tag{8}$$

To relax the optimization problem (7) we enlarge the space $\mathcal{U}_{ad}$ of designs by allowing composite materials which are simple laminates of the three phases in respective proportions $\theta_1, \theta_2, \theta_3$ satisfying

$$0 \le \theta_i \le 1, \quad \sum_{i=1}^{3} \theta_i = 1, \tag{9}$$

with the lamination direction e (a unit vector, see Figure 2). Of course, the proportions and the direction of lamination may vary with $x \in \Omega$.



1. Compute the homogenized conductivity tensor $A^*(\theta_1, \theta_2, \theta_3)$ for such a simple laminate when the direction of lamination is the first vector e_1 of the canonical basis.

2. Denoting by $A^*(x)$ any possible simple laminate, i.e.,

$$A^*(x) = R(x)\,A^*\left(\theta_1(x), \theta_2(x), \theta_3(x)\right)\,R^T(x),$$

where $A^*(\theta_1, \theta_2, \theta_3)$ is the tensor computed in the first question and $R(x)$ is a rotation matrix ($R^T(x)$ being its transposed), write the relaxed state equation and relaxed objective function.

3. By using the minimization principle (8) for the relaxed state equation, prove that the relaxed formulation is equivalent to

$$\inf_{\theta\in\mathcal{U}^*_{ad}, v\in H_0^1(\Omega)} \left\{ J^*(\theta, v) = \int_\Omega \left(\lambda_\theta^-|\nabla v|^2 - 2fv\right)dx \right\}, \tag{10}$$

where

$$\lambda_\theta^- = \left(\sum_{i=1}^{3} \frac{\theta_i}{\alpha_i} \right)^{-1}$$

and the set of admissible densities is

$$\mathcal{U}_{ad}^* = \left\{ \theta = (\theta_1, \theta_2, \theta_3) \text{ satisfying (9) and } \int_\Omega \theta_i(x) dx = c_i |\Omega| \right\}.$$

4. Show that this relaxed formulation admits at least one optimal solution. Hint: show that $J^*(\theta, v)$ is a convex function.