ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Written exam, March 19th, 2014 (2 hours)

1 Parametric optimization: 10 points

We consider an elastic membrane with a variable thickness $h(x)$, occupying at rest a plane domain Ω (a smooth bounded open set of $\mathbb{R}^2$). The membrane is clamped on its boundary and is loaded by a given force $f(x) \in L^2(\Omega)$. Its vertical displacement $u(x)$ is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}\,(h\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

To emphasize its dependence with respect to h the solution of (1) is also denoted by $u(h)$. The thickness belongs to the following space of admissible designs

$$\mathcal{U}_{ad} = \left\{ h \in L^\infty(\Omega)\,, \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega, \quad \int_\Omega h(x)\,dx = h_0|\Omega| \right\}.$$

The goal is to minimize the compliance

$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J(h) = \int_\Omega f\,u(h)\,dx \right\}. \tag{2}$$

1. Let $k \in L^\infty(\Omega)$ be a given function. We denote by $v = \langle u'(h), k\rangle$ the directional derivative of $u(h)$, solution of (1), in the direction k. Recall the boundary value problem satisfied by v. In the sequel we will write $v = v(h)$ if we want to emphasize the dependence of v on h.

2. Let $\tilde{k} \in L^\infty(\Omega)$ be another given function. We denote by $w = \langle v'(h), \tilde{k}\rangle$ the directional derivative of $v(h)$, solution of the p.d.e. defined in the previous question, in the direction $\tilde{k}$. Similarly to the first question, we denote by $\tilde{v} = \langle u'(h), \tilde{k}\rangle$ the directional derivative of $u(h)$ in the direction $\tilde{k}$.

   Determine the boundary value problem satisfied by w (p.d.e. and boundary conditions).

   By definition, the function w is also the second order derivative of $u(h)$, namely $w = \langle u''(h), (k, \tilde{k})\rangle$. Check that w is symmetric with respect to $(k, \tilde{k})$.

3. Compute the first and second order derivatives, $\langle J'(h), k\rangle$ and $\langle J''(h), (k, \tilde{k})\rangle$, of $J(h)$ in terms of v and w.

4. Give a formula for $\langle J''(h), (k, \tilde{k})\rangle$ which does not depend on w and is symmetric in $(k, \tilde{k})$.

5. Prove that, for any $k \in L^\infty(\Omega)$,

   $$\langle J''(h), (k, k)\rangle \geq 0.$$

   What can be said about possible local minimizers of (2) ?

2 Geometric optimization: 10 points

We consider a thermal conductivity problem in a bounded smooth domain $D \subset \mathbb{R}^N$. Inside the domain D, there is a "default", i.e. a smooth subset $\Omega_0 \subset D$, where some heat "leakage" takes place. We consider the so-called "inverse" problem to find the default Ω_0 by comparing physical measurements with numerical simulations.

The domain boundary is decomposed in two disjoint parts, $\partial D = \Gamma_D \cup \Gamma_N$, such that a known heat flux $g \in L^2(\Gamma_N)$ is imposed on Γ_N while the temperature is set to 0 on Γ_D. By a physical experiment we measure the temparature u_0 on Γ_N corresponding to the true and unknown default Ω_0. The heat leakage is modeled by a constant (normalized to 1) adsorption in a "candidate" default Ω. Therefore, our model for numerical computations is to find the temperature $u \in H^1(D)$, solution of

$$\begin{cases} -\Delta u + \chi_\Omega u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N, \end{cases} \tag{3}$$

where $\chi_\Omega(x)$ is the characteristic function of Ω which takes the value 1 inside Ω and 0 outside. For a given measured temperature $u_0 \in L^2(\Gamma_N)$, the goal is to minimize the objective function

$$\inf_{\Omega \subset D} \left\{ J(\Omega) = \frac{1}{2} \int_{\Gamma_N} |u - u_0|^2 \, ds \right\}, \tag{4}$$

where u depends on Ω through equation (3). The hope is to find a u for which the objective function (4) vanishes and expect that the corresponding Ω is the true location of the unknown default Ω_0. We use Hadamard's method of shape variations to compute derivatives.

1. Write the Lagrangian $\mathcal{L}(\Omega, v, q)$ corresponding to (4).
2. Deduce the variational formulation of the adjoint problem. Write explicitly the boundary value problem for the adjoint p (p.d.e. and boundary conditions).
3. Compute (formally) the shape derivative of (4).
4. Prove that, if $g \geq 0$, then the solution of (3) satisfies $u \geq 0$. Hint: multiply the equation by $u^- = \min(u, 0)$ which (assumably) belongs to $H^1(D)$ and has a gradient given by
$$\nabla u^- = \begin{cases} 0 & \text{if } u \geq 0, \\ \nabla u & \text{if } u \leq 0. \end{cases}$$
5. We assume that $g \geq 0$. Deduce that, if the predicted temperature u satisfies $u \geq u_0$ on Γ_N, then the objective function will decrease if Ω is enlarged (and the converse if instead $u \leq u_0$ on Γ_N).