MASTER M2 E.D.P. ET ANALYSE NUMERIQUE
UNIVERSITE PARIS 6 - ECOLE POLYTECHNIQUE
Cours de G. Allaire et F. Coquel
Written exam, January 7th 2010 (3 hours)

The evaluation procedure will pay attention to the quality of the dissertation and most particularly to the clarity and the conciseness in the proposed argumentation. The exam is made of two parts : an excercise and a problem. Only the result of the first question in the exercise is used in the course of the problem. The notations and definitions below are not needed to solve the exercise.

The forthcoming Problem deals with nonlinear systems of conservation laws :

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

with unknown $\mathbf{u} \in \Omega \subset \mathbb{R}^p$, $p \geq 1$. Here, the phase space Ω is an open convex subset of $\mathbb{R}^p$ which coincides with $\mathbb{R}^p$ in the all forthcoming examples except in the case of gas dynamich. The flux function $\mathbf{f} : \Omega \to \mathbb{R}^p$ is assumed to be $\mathcal{C}^1$, its Jacobian matrix $\nabla \mathbf{f}(\mathbf{u})$ is assumed to be diagonalizable with only real eigenvalues for all $\mathbf{u} \in \Omega$ so that (1) is a hyperbolic system.

Let be given a pair of $\mathcal{C}^1$ flux functions $\mathbf{f}^+, \mathbf{f}^- : \Omega \to \mathbb{R}^p$. Assume that their Jacobian matrices $\nabla \mathbf{f}^+(\mathbf{u})$ and $\nabla \mathbf{f}^-(\mathbf{u})$ are diagonizable with real eigenvalues for all $\mathbf{u} \in \Omega$. The pair $(\mathbf{f}^+, \mathbf{f}^-)$ is then said to be a *flux splitting* associated with (1) if the following two conditions are satisfied :

– (1) a consistency relation

$$\mathbf{f}(\mathbf{u}) = \mathbf{f}^+(\mathbf{u}) + \mathbf{f}^-(\mathbf{u}), \quad \forall \mathbf{u} \in \Omega, \tag{2}$$

– (2) an upwind condition : all the eigenvalues of $\nabla \mathbf{f}^+(\mathbf{u})$ (respectively of $\nabla \mathbf{f}^-(\mathbf{u})$) are supposed to be nonnegative (resp. nonpositive), *i.e.* :

$$\lambda_1^-(\mathbf{u}) \leq \lambda_2^-(\mathbf{u}) \leq \cdots \leq \lambda_p^-(\mathbf{u}) \ \leq 0 \leq \ \lambda_1^+(\mathbf{u}) \leq \lambda_2^+(\mathbf{u}) \leq \cdots \leq \lambda_p^+(\mathbf{u}), \quad \forall \mathbf{u} \in \Omega. \tag{3}$$

Observe that by contrast the eigenvalues of $\nabla \mathbf{f}(\mathbf{u})$ are not assumed to keep a constant sign.

Hereafter, $\{\mathbf{r}_i^+(\mathbf{u})\}_{1 \leq i \leq p}$ (respectively $\{\mathbf{r}_i^-(\mathbf{u})\}_{1 \leq i \leq p}$) denotes a basis of right eigenvectors associated with $\nabla \mathbf{f}^+(\mathbf{u})$ (resp. $\nabla \mathbf{f}^-(\mathbf{u})$).

In practice, a given flux splitting $(\mathbf{f}^+, \mathbf{f}^-)$ of the flux $\mathbf{f}$ allows to introduce the following two hyperbolic systems :

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}^+(\mathbf{v}) = 0, \tag{4}$$

and

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}^-(\mathbf{w}) = 0. \tag{5}$$

Roughly speaking, the system (4) is intended to approximate the waves propagating with nonnegative speed in the solutions of the system (1) while the system (5) is devoted to the approximation of the waves with nonpositive speeds. This heuristic claim is grounded in Section 3 devoted to the numerical approximation of the solutions of (1). Section 2 studies the existence of flux splittings (2)–(3) for various hyperbolic models. Section 4 investigates consistency conditions of a flux splitting $(\mathbf{f}^+, \mathbf{f}^-)$ with existing entropy pairs for (1). In that aim, the examples introduced in Section 2 are further analyzed.

1 Exercise (6/20)

The Exercise deals with the Euler equations for barotropic gases

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p(\rho)) = 0, \end{cases} \tag{6}$$

equipped with the particular pressure law :

$$p(\rho) = \frac{1}{3}\rho^3, \quad \rho > 0. \tag{7}$$

The phase space associated with (6)–(7) is the usual one :

$$\Omega = \{\mathbf{u} = (\rho, \rho u) \in \mathbb{R}^2 / \rho > 0, \rho u \in \mathbb{R}\}. \tag{8}$$

Questions (II.3.i) to (II.3.iv) in the Problem will use the results of question (I.1) below.

- (I.1) Reformulate the system (6)–(7) in the new variable $\mathbf{z} = (\rho, u)$. Find the eigenvalues associated with system (6)–(7) and a basis of right eigenvectors expressed in the variable $\mathbf{z}$. Study the nonlinearity (genuine nonlinearity or linear degeneracy) of each fields.
- (I.2) Write the governing equation satisfied by $\mathcal{R}^+(\rho, u) = u + \rho$ for the smooth solutions of (6)–(7). Solve the same question concerning $\mathcal{R}^-(\rho, u) = u - \rho$. Deduce from the lectures devoted to the scalar conservation laws the validity of the following maximum principles for $\mathcal{R}^+$ and $\mathcal{R}^-$:

$$\min_{y\in\mathbb{R}} \mathcal{R}^\pm(\rho_0, u_0)(y) \le \mathcal{R}^\pm(\rho, u)(x, t) \le \max_{y\in\mathbb{R}} \mathcal{R}^\pm(\rho_0, u_0)(y), \quad \text{for all } x,\ t. \tag{9}$$

- (I.3) From the above question, express a sufficient condition on the initial data $\mathbf{u}_0$ so that the associated Cauchy problem for (6)–(7) admits a smooth solution. Observe that the solution under consideration must have a positive density ρ in order to belong to the phase space Ω introduced in (8).
- (I.4) Being given two constant states $\mathbf{u}_L$ and $\mathbf{u}_R$ in Ω, let us consider the Riemann problem for (6)–(7) with initial data :

$$\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_L, & x < 0, \\ \mathbf{u}_R, & x > 0. \end{cases} \tag{10}$$

  Express the conditions on the states $\mathbf{u}_L$ and $\mathbf{u}_R$ so that the associated Riemann solution is smooth and belongs to the phase space Ω. Depict such a solution in the (x, t) plane.
- (I.5) Consider the case of an initial data $\mathbf{u}_0$ giving rise to a discontinuous solution for the Cauchy problem (6)–(7), write the jump conditions to be considered (it is not required to solve these jump conditions!).

Problem (14/20)

2 Some examples of flux splittings

- (II.1) Consider the scalar conservation law setting $p = 1$ for a strictly convex flux function $f(u)$. The equation $f'(u) = 0$ is assumed to admit an unique bounded solution denoted by $u^\star \in \mathbb{R}$. Since a flux function is defined up to an additive constant, it is also supposed without loss of generality that $f(u^\star) = 0$. Let us then consider the pair :

$$f^+(u) = f(max(u, u^\star)), \quad f^-(u) = f(min(u, u^\star)), \quad \forall u \in \mathbb{R}.$$

  Observe that the proposed two functions achieve the required $\mathcal{C}^1$ smoothness. Explain why the pair (f^+, f^-) yields a flux splitting of the flux f.
- (II.2) Let us address the general case of hyperbolic systems of conservation laws (1). Let be given $\mathcal{K}$ a compact subset of $\mathbb{R}^p$, we introduce :

$$\mathbf{f}^+(\mathbf{u}) = \frac{1}{2}\Big(\mathbf{f}(\mathbf{u}) + b_{\mathcal{K}}\mathbf{u}\Big), \quad \mathbf{f}^-(\mathbf{u}) = \frac{1}{2}\Big(\mathbf{f}(\mathbf{u}) - b_{\mathcal{K}}\mathbf{u}\Big), \quad \forall \mathbf{u} \in \mathcal{K} \tag{11}$$

  where $b_{\mathcal{K}}$ denotes a positive real number which choice may depend on $\mathcal{K}$. Express the condition to be satisfied by $c_{\mathcal{K}}$ so that the pair $(\mathbf{f}^+, \mathbf{f}^-)$ yields a flux splitting of $\mathbf{f}$ for all states $\mathbf{u} \in \mathcal{K}$.
- (II.3) This question is concerned with the Euler equations (6)–(7), studied in the Exercice. Let us introduce the notation $m = u/\rho$ in order to define the change of variables $(\rho, m) \in \mathbb{R}^+_\star \times \mathbb{R} \to (\rho, u(\rho, m)) \in \mathbb{R}^+ \times \mathbb{R}$. The flux function $\mathbf{f}(\mathbf{u})$ associated with (6)–(7) is given the following condensed form :

$$\mathbf{f}(\rho, m) = (\rho^2 a(m),\ \rho^3 b(m)), \quad a(m) = m, \quad b(m) = (m^2 + \frac{1}{3}). \tag{12}$$

  Consider a pair $(\mathbf{f}^+, \mathbf{f}^-)$ whith $\mathbf{f}^+$ defined as follows :

$$\mathbf{f}^+(\rho, m) = (\rho^2 a^+(m),\ \rho^3 b^+(m)), \tag{13}$$

  where the pair $(a^+(m), b^+(m))$ are $\mathcal{C}^1$ functions given by :

$$(a^+(m), b^+(m)) = \left\{ \begin{array}{rc} (0,0), & m < -1, \\ (\frac{1}{4}(m+1)^2, \frac{1}{6}(m+1)^3), & -1 \le m \le 1, \\ (m, (m^2 + \frac{1}{3})) & m > 1. \end{array} \right. \tag{14}$$

  The flux $\mathbf{f}^-$ is deduced from $\mathbf{f}^+$ thanks to the consistency condition (2). We wish to prove that the pair $(\mathbf{f}^+, \mathbf{f}^-)$ under consideration actually yields a flux splitting of $\mathbf{f}$ obeying the sign requirement (3) on the eigenvalues. The spectral properties of $\nabla\mathbf{f}^+$ are first analyzed, those of $\nabla\mathbf{f}^-$ will then easily follow. **Hint :** Use the variable $\mathbf{z} = (\rho, u)$ to carry out the required analysis.
  - (II.3.i) Owing to the results of the Exercise, question (I.1), give without any calculation in the case $m > 1$ the eigenvalues associated with $\mathbf{f}^+$ and a basis of right eigenvectors. Explain why the same basis can be kept unchanged in the case $m < -1$.

- (II.3.ii) Compute the eigenvalues and a basis of right eigenvectors for (4)–(14) when $m \in [-1, 1]$. Deduce the definition of a basis of right eigenvectors when m covers the full range $\mathbb{R}$.
- (II.3.iii) Express the relationship between the proposed basis of right eigenvectors for (4)–(14) and the one obtained for (6)–(7) in question (I.1). Explain why this property does not depend on a given change of variable.
- (II.3.iv) Deduce from the above question a basis of right eigenvectors expressed in the $\mathbf{z}$ variables and associated with $\nabla \mathbf{f}^-(\mathbf{u})$. Express the associated eigenvalues and analyze their sign for $m \in \mathbb{R}$. Conclude.

3 Application to the numerical approximation of (1)

This section is devoted to the numerical approximation of the solutions of system (1) thanks to a given flux splitting $(\mathbf{f}^+, \mathbf{f}^-)$ of the flux $\mathbf{f}$. For that matter, the Riemann solutions associated with both systems (4) and (5) will be used. Admissible shocks in these Riemann solutions are selected thanks to the Lax interlacing conditions of the eigenvalues of $\nabla \mathbf{f}^+$ (resp. $\nabla \mathbf{f}^-$) with the shock speed. Each field of both systems (4) and (5) is assumed to be either genuinely nonlinear or linearly degenerate, this property guarantees (thanks to a theorem proved in the lectures) the existence of solutions to the Riemann problems under consideration for sufficiently close left and right states $\mathbf{u}_L$ and $\mathbf{u}_R$.

Being given two states $\mathbf{u}_L$ and $\mathbf{u}_R$, we denote by $\omega_+(\frac{x}{t}, \mathbf{u}_L, \mathbf{u}_R)$ (respectively by $\omega_-(\frac{x}{t}, \mathbf{u}_L, \mathbf{u}_R)$) the solution of the Riemann problem for system (4) (resp. for system (5)).

- (III.1) Prove that the self-similar function $\mathbf{f}^+(\omega_+(\xi, \mathbf{u}_L, \mathbf{u}_R))$ is continuous at $\xi = 0$. Give its value at $\xi = 0$ in terms of the states $\mathbf{u}_L$ and $\mathbf{u}_R$ involved in the initial data. Solve the same question concerning $\mathbf{f}^-(\omega_-(\xi, \mathbf{u}_L, \mathbf{u}_R))$.

We propose to approximate the solutions $\mathbf{u}$ of (1) by piecewise constant functions $\mathbf{u}_h$, defined according to the lectures from given constant time step $\Delta t > 0$ and constant space step $\Delta x > 0$:

$$\mathbf{u}_h(x, t^n) = \mathbf{u}_j^n, \quad x \in (x_{j-1/2}, x_{j+1/2}), \quad x_{j+1/2} = (j + 1/2)\Delta x,\ t^n = n\Delta t, \tag{15}$$

with $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. The discrete initial data $\mathbf{u}_h^0$ is defined as usual by averaging the exact initial data $\mathbf{u}_0$ over each cell $(x_{j-1/2}, x_{j+1/2})$. Assuming the approximate solution $\mathbf{u}_h(x, t^n)$ to be known at time t^n, this solution is updated at the next time level t^{n+1} under the following CFL condition :

$$\frac{\Delta t}{\Delta x} \max_{\mathbf{u}}\big(\max_{1 \le j \le p} |\lambda_j^+(\mathbf{u})|, \max_{1 \le j \le p} |\lambda_j^-(\mathbf{u})|\big) \le \frac{1}{2}, \tag{16}$$

for all the states $\mathbf{u}$ under consideration. Let be given at each cell interfaces $x_{j+1/2}$, $j \in \mathbb{Z}$, the Riemann solutions $\omega_+(\xi, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$ and $\omega_-(\xi, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$. We define the following partial averages :

$$\begin{aligned} (\mathbf{u}^-)_{j+1/2}^{n+1} &= \frac{2}{\Delta x} \int_{x_j}^{x_{j+1/2}} \omega_-\Big(\frac{x - x_{j+1/2}}{\Delta t}, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n\Big) dx, \\ (\mathbf{u}^+)_{j+1/2}^{n+1} &= \frac{2}{\Delta x} \int_{x_{j+1/2}}^{x_{j+1}} \omega_+\Big(\frac{x - x_{j+1/2}}{\Delta t}, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n\Big) dx. \end{aligned} \tag{17}$$

(Observe the distinct stencils of integration !). The updated value $\mathbf{u}_j^{n+1}$ in the cell $(x_{j-1/2}, x_{j+1/2})$ is then defined setting :

$$\mathbf{u}_j^{n+1} = \frac{1}{2}\Big((\mathbf{u}^+)_{j-1/2}^{n+1} + (\mathbf{u}^-)_{j+1/2}^{n+1}\Big), \quad j \in \mathbb{Z}. \tag{18}$$

- (III.2) Compute, under the CFL condition (16), the two averages $(\mathbf{u}^+)_{j+1/2}^{n+1}$ and $(\mathbf{u}^+)_{j+1/2}^{n+1}$ in terms of $\mathbf{u}_j^n$ and $\mathbf{u}_{j+1}^n$.
- (III.3) Deduce that the proposed update (18) yields a consistent and conservative finite volume method for (1) under the classical form

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\Big(\mathbf{g}_{j+1/2}^n - \mathbf{g}_{j-1/2}^n\Big), \quad j \in \mathbb{Z}, \ n \geq 0. \tag{19}$$

  Give the associated numerical flux function $\mathbf{g}_{j+1/2}^n$.

4 Consistency with the entropy condition

From now on, it is assumed that the hyperbolic system (1) is equipped with an entropy pair $(\mathcal{U}, \mathcal{F})$ for a strictly convex function $\mathbf{u} \in \Omega \to U(\mathbf{u}) \in \mathbb{R}$. The pair under consideration allows to select the admissible weak solutions of (1) according to the condition :

$$\partial_t \mathcal{U}(\mathbf{u}) + \partial_x \mathcal{F}(\mathbf{u}) \leq 0, \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}). \tag{20}$$

The flux splitting $(\mathbf{f}^-, \mathbf{f}^+)$ (2) is said to be consistent with the entropy pair $(\mathcal{U}, \mathcal{F})$ if there exists two entropy fluxes $\mathcal{F}^-, \mathcal{F}^- : \Omega \to \mathbb{R}$ so that the smooth solutions $\mathbf{v}$ of (4) (respectively smooth solutions $\mathbf{w}$ of (5)) satisfy the additional conservation law :

$$\partial_t \mathcal{U}(\mathbf{v}) + \partial_x \mathcal{F}^+(\mathbf{v}) = 0,$$

respectively

$$\partial_t \mathcal{U}(\mathbf{w}) + \partial_x \mathcal{F}^-(\mathbf{w}) = 0.$$

The entropy pair $(\mathcal{U}, \mathcal{F}^+)$ (respectively $(\mathcal{U}, \mathcal{F}^-)$) allows us to select the weak solutions of (4) thanks to the differential inequality :

$$\partial_t \mathcal{U}(\mathbf{v}) + \partial_x \mathcal{F}^+(\mathbf{v}) \leq 0, \quad \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}), \tag{21}$$

and resp. the weak solutions of (5) according to :

$$\partial_t \mathcal{U}(\mathbf{w}) + \partial_x \mathcal{F}^-(\mathbf{w}) \leq 0, \quad \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}). \tag{22}$$

- (IV.1) Prove that the entropy fluxes $\mathcal{F}^+$ and $\mathcal{F}^-$ necessarily obey (up to some additive constant) the relation :

$$\mathcal{F}(\mathbf{u}) = \mathcal{F}^+(\mathbf{u}) + \mathcal{F}^-(\mathbf{u}), \quad \mathbf{u} \in \Omega. \tag{23}$$

– (IV.2) The solutions of the Riemann problems for (4) (respectively for (5)) are selected thanks to the entropy inequality (21) (resp. (22)). Establish under the CFL condition (16) that the finite volume method (17)–(18) is consistent with the entropy inequality (20), namely that the following discrete inequalities hold true :

$$\mathcal{U}(\mathbf{u}_j^{n+1}) - \mathcal{U}(\mathbf{u}_j^n) + \frac{\Delta t}{\Delta x}\Big(\mathcal{G}_{j+1/2}^n - \mathcal{G}_{j-1/2}^n\Big) \leq 0, \quad j \in \mathbb{Z},\ n \geq 0, \tag{24}$$

for some numerical entropy flux fonction flux $\mathcal{G}$ to be determined.

Hereafter, we analyze the consistency of the flux splitting examples introduced in Section 2 with respect to the available entropy pairs for the hyperbolic models under consideration.

– (IV.3) Explain why the flux splitting (II.1) is consistent with any given entropy pair $(\mathcal{U}, \mathcal{F})$ associated with the scalar flux function $f(u)$. In that aim, give the precise definition of the split entropy fluxes $\mathcal{F}^+$ and $\mathcal{F}^-$. Which property of the numerical flux function g of the finite volume method allows to directly prove the required entropy consistency ?

– (IV.4) Prove that the flux splitting (11) introduced in the case of general hyperbolic systems in question (II.2) is consistent with any given avalaible entropy pair $(\mathcal{U}, \mathcal{F})$ for system (1). To that purpose, show that systems (4) and (5) admit entropy flux functions $\mathcal{F}^+$ and $\mathcal{F}^-$ for any of the entropy $\mathcal{U}$ under consideration.

– (IV.5) Let us consider the flux splitting (13)–(14) associated with the Euler equations for barotropic gases (6)–(7). Arguing about the relationship between the right eigenvector basis for $\nabla\mathbf{f}^+$, $\nabla\mathbf{f}^-$ and $\nabla\mathbf{f}$ (*cf* question (II.3)), deduce that for any given entropy pair $(\mathcal{U}, \mathcal{F})$ associated with (1), the matrices $\nabla^2\mathcal{U}\nabla\mathbf{f}^+$ and $\nabla^2\mathcal{U}\nabla\mathbf{f}^-$ are symmetric. Infer that the flux splitting (11) is consistent with all the available entropy pairs for (6)–(7).