Functional analysis and applications
MASTER "Mathematical Modelling"
École Polytechnique and Université Pierre et Marie Curie
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See also the course webpage:
http://www.cmap.polytechnique.fr/ allaire/master/course-funct-analysis.html

Exercise 1 Sequence spaces ℓ^p are Banach spaces

Given a sequence $(x_1, x_2, \dots, x_k, \dots)$, set

$$\|x\|_p = \left[\sum_k |x_k|^p\right]^{1/p} \quad \text{for } 1 \le p < \infty \text{ and } \|x\|_\infty = \sup_k |x_k|.$$

Prove that the spaces $\ell_p = \{x, \|x\|_p < \infty\}$ with $1 \le p \le \infty$. are Banach space.

Answer of exercise 1

We start with the case $p < \infty$. Let u^n be a Cauchy sequence in ℓ^p: for all $\epsilon > 0$ there exists $N \in \mathbb{N}^*$ such that

$$\|x^n - x^m\|_p = \left[\sum_k |x_k^n - x_k^m|^p\right]^{1/p} \le \epsilon, \quad \forall n, m \ge N.$$

This implies that for a fixed $k \in \mathbb{N}^*$, the sequence x_k^n is a Cauchy sequence. Since $\mathbb{R}$ is complete, there exists $x_k^\infty \in \mathbb{R}$ such that $x_k^n \xrightarrow{n\to\infty} x_k^\infty$.

(i) $x^\infty \in \ell_p$**:** Since x^n is a Cauchy sequence it is bounded by some constant $C > 0$ (in l^p). Let $K, N \in \mathbb{N}^*$ then

$$\left[\sum_{k=1}^K |x_k^n|^p\right]^{1/p} \le \|x^n\|_p \le C.$$

Let $n \to \infty$ to obtain $\left[\sum_{k=1}^K |x_k^\infty|^p\right]^{1/p} \le C$.

Let $K \to \infty$ to obtain $\left[\sum_{k=1}^\infty |x_k^\infty|^p\right]^{1/p} = \|x^\infty\|_p \le C$.

(ii) x^n **converges toward** x^∞ **in** ℓ_p: Let $\epsilon > 0$. For $K \in \mathbb{N}^*$ and n, m sufficiently large

$$\left[\sum_{k=1}^K |x_k^n - x_k^m|^p\right]^{1/p} \le \|x^n - x^m\|_p \le \epsilon.$$

Letting $n \to \infty$ and then $K \to \infty$ allows to conclude ($\|x^\infty - x^m\|_p \le \epsilon$).

For $p = \infty$ we assume that x^n is a Cauchy sequence and easily deduce pointwise convergence toward some sequence x^∞ (i.e. for all $k \in \mathbb{N}^*$, $x_k^n \xrightarrow{n\to\infty} x_k^\infty$). Since it is a Cauchy sequence, x^n is bounded in ℓ_∞ and in turn $x^\infty \in \ell_\infty$. Finally $|x_k^\infty - x_k^n| \le \epsilon$ for all $k \in \mathbb{N}^*$ implies $\|x^\infty - x^n\|_\infty = \sup_k\{|x_k^\infty - x_k^n|\} \le \epsilon$.

Exercise 2 The theorems of Egorov and Vitali

Assume $|\Omega| < \infty$ Let (f_n) be a sequence of measurable functions such that such that $f_n \to f$ a.e. (with $|f| < \infty$ a.e.).

1. Let $\alpha > 0$ be fixed. Prove that
$$\text{meas}[|f_n - f| > \alpha] \xrightarrow[n\to\infty]{} 0.$$
2. More precisely, let
$$S_n(\alpha) = \bigcup_{k\ge n} [|f_k - f| > \alpha].$$
Prove that $|S_n(\alpha)| \xrightarrow[n\to\infty]{} 0$.
3. *(Egorov)* Prove that
$$\left\{ \begin{array}{l} \forall\delta > 0, \exists A \subset \Omega \text{ mesurable such that} \\ |A| < \delta \text{ and } f_n \to f \text{uniformely on } \Omega \setminus A. \end{array} \right.$$
4. *(Vitali)* Let (f_n) be a sequence in $L^p(\Omega)$ with $1 \le p < \infty$. Assume that
   (i) $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $\int_A |f_n|^p < \varepsilon$, $\forall n$ and $\forall A \in \Omega$ measurable with $|A| < \delta$.
   (ii) $f_n \to f$ a.e.

   Prove that $f \in L^p(\Omega)$ and that $f_n \to f$ in $L^p(\Omega)$.

Answer of exercise 2

1. Let $\alpha > 0$ and $g_n \in L^\infty(\Omega)$ defined by
$$g_n(x) = \left\{ \begin{array}{ll} 1 & \text{if } |f_n - f| > \alpha \\ 0 & \text{if } |f_n - f| \le \alpha \end{array} \right.$$
As $f_n \to f$ a.e., g_n converges toward 0 a.e. Moreover, it is bounded by the constant map 1, which belongs to $L^1(\Omega)$ as Ω is of finite measure. Thus, from the Lebesgue Theorem, g_n converges toward 0 in $L^1(\Omega)$. Finally,
$$\text{meas}[|f_n - f| > \alpha] = \int_\Omega g_n \to 0.$$

2. We set
$$F_n = \sup_{k \geq n} |f_k - f|.$$
F_n converges toward 0 almost everywhere, thus, from the previous question,
$$|S_n(\alpha)| = \text{meas}[|F_n| > \alpha] \to 0.$$

3. For every integer $m \geq 1$, there exists N_m such that
$$|S_n(1/m)| < \delta/2^m.$$
for every $n \geq N_m$. Setting $\Sigma_m = S_{N_m}(1/m)$, we have
$$|f_k(x) - f| < \frac{1}{m}, \forall k \geq N_m, \forall x \in \Omega \setminus \Sigma_m.$$
Let $\Sigma = \cup_m \Sigma_m$. We have $|\Sigma| < \delta$. Moreover, f_n does uniformly converge toward f on $\Omega \setminus \Sigma$. Indeed, for all m, for all $x \in \Omega \setminus \Sigma$ and for all $k \geq N_m$, we have
$$|f_k(x) - f(x)| < \frac{1}{m}.$$

4. For every $\varepsilon > 0$, let δ as in (i). From the Egorov Theorem, there exists a measurable subset A of Ω such that $|A| < \delta$ and f_n converges toward f uniformly on $\Omega \setminus A$. First, notice that
$$\int_A |f|^p \leq \liminf \int_A |f_n|^p \leq \varepsilon.$$
We have
$$\int_\Omega |f_n - f|^p = \int_{\Omega\setminus A} |f_n - f|^p + \int_A |f_n - f|^p \leq \int_{\Omega\setminus A} |f_n - f|^p + 2\varepsilon.$$
Finally, as f_n converges toward f uniformly on $\Omega \setminus A$, for n large enough,
$$|f_n - f| < |\Omega|^{-1}\varepsilon$$
and
$$\int |f_n - f|^p < 3\varepsilon.$$
We conclude that f_n does converge toward f in $L^p(\Omega)$.

Exercise 3

Let $j : \mathbb{R} \to (-\infty, \infty]$ be a convex function. The domain of j is defined by
$$D(j) = \{x \in \mathbb{R} \ : \ j(x) < \infty\}.$$

1. Prove that for all $x^- < x < x^+$, such that x^- and $x^+ \in D(j)$, we have

$$\frac{j(x^-) - j(x)}{x^- - x} \leq \frac{j(x^+) - j(x)}{x^+ - x}.$$

2. Let x be an element of the interior of j. We set

$$\alpha = \inf_{x^+ > x} \frac{j(x^+) - j(x)}{x^+ - x}.$$

   Prove that $\alpha \in \mathbb{R}$ (that is $|\alpha| \neq \infty$).

3. Prove that for every x of the interior of the domain of j, there exists $\alpha \in \mathbb{R}$ such that

$$j(x + y) \geq \alpha y + j(x) \quad \forall y \in \mathbb{R}. \tag{1}$$

Answer of exercise 3

In a first step, we are going to prove that for all $x^- < x < x^+$ such that $x^-, x^+ \in D(j)$, we have

$$\frac{j(x^-) - j(x)}{x^- - x} \leq \frac{j(x^+) - j(x)}{x^+ - x} \tag{2}$$

There exists $\theta \in [0, 1]$, such that $x = \theta x^+ + (1 - \theta)x^-$. As j is convex,

$$j(x) \leq \theta j(x^+) + (1 - \theta)j(x^-),$$

thus

$$(\theta - 1)(j(x^-) - j(x)) \leq \theta(j(x^+) - j(x)). \tag{3}$$

Moreover, we have

$$x - x^- = \theta(x^+ - x^-)$$

and

$$x - x^+ = (\theta - 1)(x^+ - x^-).$$

Hence, by multiplying (3) by $(x^+ - x^-)$ we get

$$(x - x^+)(j(x^-) - j(x)) \leq (x - x^-)(j(x^+) - j(x))$$

and (2) as claimed. Let

$$\alpha = \inf_{x^+ > x} \frac{j(x^+) - j(x)}{x^+ - x}.$$

As x belongs to the interior of the domain of j, $\alpha < \infty$ and from (2), $\alpha > -\infty$. Finally, from the definition of α, for every $x^+ > x$, we have

$$\alpha \leq \frac{j(x^+) - j(x)}{x^+ - x}$$

that is

$$\alpha(x^+ - x) + j(x) \le j(x^+). \tag{4}$$

and from (2), for every $x^- < x$,

$$\alpha \ge \frac{j(x_-) - j(x)}{x^- - x},$$

that is

$$\alpha(x^- - x) + j(x) \le j(x^-). \tag{5}$$

Finally, (1) follows from (4) and (5).

Exercise 4 Jensen's inequality

Assume that $|\Omega| < \infty$. Let $j : \mathbb{R} \to (-\infty, \infty]$ be a convex l.s.c. function, $j \not\equiv \infty$. Let $f \in L^1(\Omega)$ be such that $j(f(x)) < \infty$ a.e. and $j(f) \in L^1(\Omega)$. Prove that

$$j\left(\frac{1}{|\Omega|}\int_\Omega f\right) \le \frac{1}{|\Omega|}\int_\Omega j(f).$$

Answer of exercise 4

Firstly, let us remark that as j is a convex function, its domain is also convex. Thus, as $f(x) \in D(j)$ a.e., $m = |\Omega|^{-1}\int_\Omega f \in D(j)$. Secondly, without lost of generality, we can assume that m belongs to the interior of the domain $D(j)$ (if m belongs to the boundary of the domain, f is constant and the result is obvious). The function j being convex, there exists $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$j(x + m) \ge \alpha x + j(m).$$

In particular,

$$j(f(x)) \ge \alpha(f(x) - m) + j(m).$$

By integration over Ω, we get

$$\int_\Omega j(f) \ge \alpha\left(\int_\Omega f - |\Omega| m\right) + |\Omega| j(m) = |\Omega| j(m),$$

as desired.