Functional analysis and applications
MASTER "Mathematical Modelling"
École Polytechnique and Université Pierre et Marie Curie
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See also the course webpage:
http://www.cmap.polytechnique.fr/ allaire/master/course-funct-analysis.html

Exercise 1 Duality in ℓ^p

Let $1 < p < \infty$ and p' such that $1/p + 1/p' = 1$.

1. *(Young's inequality)* Prove using the concavity of the ln that for every $a, b > 0$,
$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}.$$
2. Prove that for every $x \in \ell^p$ and $y \in \ell^{p'}$, $xy \in \ell^1$ and that
$$\|xy\|_{\ell^1} \leq \frac{1}{p}\|x\|_{\ell^p}^p + \frac{1}{p'}\|y\|_{\ell^{p'}}^{p'}$$
3. Prove that for every $x \in \ell^p$ and $y \in \ell^{p'}$,
$$\sum_{n=0}^{\infty} x_n y_n \leq \|x\|_{\ell^p}\|y\|_{\ell^{p'}}$$
4. Prove that for every $y \in \ell^{p'}$ the map
$$x \to \sum_{n=0}^{\infty} x_n y_n,$$
is correctly defined, linear and continuous on ℓ^p.
5. Let $L \in (\ell^p)^*$ prove that there exists $y \in \ell^{p'}$ such that for every $x \in \ell^p$,
$$L(x) = \sum_{n=0}^{\infty} y_n x_n.$$
Moreover, show that
$$\|y\|_{\ell^{p'}} = \|L\|_{(\ell^p)^*}$$

Answer of exercise 1

1. As ln is concave, for all $a, b > 0$, we have
$$\ln(ab) = \frac{1}{p}\ln(a^p) + \frac{1}{p'}\ln(b^{p'}) \leq \ln\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right)$$

Taking the exponential of this inequality leads to

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}a^{p'}.$$

2. Let $x \in \ell^p$ and $y \in \ell^{p'}$. We have, from the Young's inequality

$$\sum_n |x_n||y_n| \leq \frac{1}{p}\sum_n |x_n|^p + \frac{1}{p'}\sum_n |y_n|^{p'}.$$

3. We already now that $\sum x_n y_n$ is absolutely convergent. Moreover, applying the previous inequality to $x/\|x\|_{\ell^p}$ and $y/\|y\|_{\ell^{p'}}$ instead of x and y leads to

$$\|x\|_{\ell^p}^{-1}\|y\|_{\ell^{p'}}^{-1}\sum_n |x_n||y_n| \leq \frac{1}{p} + \frac{1}{p'} = 1.$$

4. Firstly, the sum $\sum x_n y_n$ is convergent as already mentioned. Moreover the map $x \mapsto \sum x_n y_n$ is obviously linear and as

$$\sum x_n y_n \leq \|y\|_{\ell^{p'}}\|x\|_{\ell^p},$$

it is continuous.

5. Let (e^n) be a basis of ℓ^p defined by $e^n_k = \delta^n_k$. Let us set $y \in \mathbb{R}^{\mathbb{N}}$, defined by $y_n = L(e_n)$. For all $x_n \in \ell^p(\Omega)$,

$$\sum_n y_n x_n = L(x) \leq \|L\|_{(\ell^p)^*}\|x\|_{\ell^p}$$

Choosing $x_n = |y_n|^{p'-2}y_n$, we get

$$\begin{aligned}\|y\|_{\ell^{p'}}^{p'} &= \sum_n |y_n|^{p'} \leq \|L\|_{(\ell^p)^*}\left(\sum_n |y_n|^{p(p'-1)}\right)^{1/p} \\ &= \|L\|_{(\ell^p)^*}\left(\sum_n |y_n|^{p'}\right)^{1/p} = \|L\|_{(\ell^p)^*}\|y\|_{\ell^{p'}}^{p'/p}\end{aligned}$$

and thus

$$\|y\|_{\ell^{p'}} = \|y\|_{\ell^{p'}}^{p'-p'/p} \leq \|L\|_{(\ell^p)^*}.$$

We have thus obtained that $y \in \ell^{p'}$ and

$$\|L\|_{(\ell^p)^*} \geq \|y\|_{\ell^{p'}}.$$

As we already have proven the converse inequality, we get

$$\|L\|_{(\ell^p)^*} = \|y\|_{\ell^{p'}}.$$

Exercise 2   Decomposition in Banach spaces

Let E be a Banach space. Assume that F and G are closed subspaces of E such that $F+G$ is closed. Then there exists $C>0$ such that for every $z \in F+G$, there exists $x \in F$ and $y \in G$ such that

$$z = x + y$$

and

$$\|x\| \leq C\|z\| \text{ and } \|y\| \leq C\|z\|.$$

Answer of exercise 2

Let $T : F \times G \to F+G$ defined by $T(x,y) = x+y$. The map T is a linear continuous map between Banach space. Moreover, it is onto. Thus, from the open mapping Theorem, there exists $r>0$ such that

$$\{z \in F+G \text{ such that } \|z\|_{F+G} < r\} \subset T\left(\{(x,y) \in F \times G \text{ such that } \|x\|_F < 1 \text{ and } \|y\|_G < 1\}\right)$$

Note, that all the spaces F, G and $F+G$ are all endowed with the norm of E. It follow that, for every $z \in F+G$, let $\widetilde{z} = \alpha z$, with $\alpha = r/(2\|z\|)$. We have $\|\widetilde{z}\| < r$ and from the inclusion given by the open mapping Theorem, there exists $\widetilde{x} \in F$ and $\widetilde{y} \in G$ such that

$$\widetilde{z} = \widetilde{x} + \widetilde{y}$$

and $\|\widetilde{x}\| < 1$, $\|\widetilde{y}\| < 1$. Setting $x = \widetilde{x}/\alpha$ and $y = \widetilde{y}/\alpha$, we get

$$z = x + y$$

with

$$\|x\| \leq \alpha^{-1} = 2\|z\|/r$$

and

$$\|y\| \leq \alpha^{-1} = 2\|z\|/r.$$

Exercise 3   Sum of two closed subspaces

We want to prove that the assumption $F+G$ closed in Exercise 2 is not trivial (meaning that it is not a consequence of the other assumptions) and is necessary.

1. Find E Banach space and F and G closed subspaces of E such that the subspace $F+G$ of E is not closed.
   [**Hint:** Let $E = \ell_1$, $F = \{(x_n)_{n\in\mathbb{N}} \in \ell_1; x_{2n} = 0, \forall n \in \mathbb{N}\}$ and $G = \{(x_n)_{n\in\mathbb{N}} \in \ell_1; x_{2n-1} = n x_{2n}, \forall n \in \mathbb{N}\}$. Prove that $F+G$ is dense in E but $F+G \neq E$.]
2. Using the example found, prove that there is no constant C such that for all $z \in F+G$, there exists $x \in F$ and $y \in G$ such that $z = x+y$ whereas $\|x\| \leq C\|z\|$ and $\|y\| \leq C\|z\|$.

Answer of exercise 3

Write the answer for $E = \ell_1$.

Exercise 4

Let $X \subset L^1(\Omega)$ be a closed vector space in $L^1(\Omega)$. Assume that

$$X \subset \bigcup_{1<q\leq\infty} L^q(\Omega).$$

1. Prove that there exists some $p > 1$ such that $X \subset L^p(\Omega)$. [**Hint:** For every integer $n \geq 1$ consider the set

$$X_n = \left\{ f \in X \cap L^{1+1/n}(\Omega);\ \|f\|_{1+1/n} \leq n \right\}$$

]

2. Prove that there is a constant C such that

$$\|f\|_p \leq C\|f\|_1, \quad \forall f \in X.$$

Answer of exercise 4

The set X_n are closed subsets of X. Indeed, let $f_k \in X_n$ such that $f_k \to f$ in $L^1(\Omega)$, without lost of generality, we can assume that f_k does converge almost everywhere. Then, from Beppo - Levi's Theorem,

$$\|f\|_{1+1/n} \leq \liminf_k \|f_k\|_{1+1/n} \leq n.$$

Moreover, $X \subset \cup_n X_n$. Indeed, for all $f \in X$, there exists $q > 1$ such that $f \in L^1(\Omega) \cap L^q(\Omega)$ and for every $1 \leq r \leq q$ $f \in L^r(\Omega)$ with

$$\|f\|_r \leq \|f\|_1^\alpha \|f\|_q 1 - \alpha,$$

with

$$\alpha + \frac{1-\alpha}{q} = \frac{1}{r}.$$

It follows that for every every $1 \leq r \leq q$,

$$\|f\|_r \leq \max(1, \|f\|_1) \max(\|f\|_q, 1) = C(f).$$

For n great enough, $1 + 1/n \leq q$ and $C(f) \leq n$, so that

$$\|f\|_r \leq n,$$

with $r = 1 + 1/n$ and $f \in X_n$ as claimed.

We thus have $X = \cup_n X_n$, and as X is a Banach space and X_n is a sequence of closed subset of X, from the Baire's Lemma, there exists n such that the interior of X_n in X is not void. There exists $g \in X_n$ and $\beta > 0$, such that

$$\{h \in X \ :\ \|h - g\|_1 \leq \beta\} \subset X_n$$

Thus, for every $f \in X$, let $h = g + \beta f/\|f\|_1$, we have $\|h - g\|_1 \leq \beta$ and

$$\|g + \beta f/\|f\|_1\|_{1+1/n} = \|h\|_{1+1/n} \leq n.$$

We conclude that

$$\beta \frac{\|f\|_{1+1/n}}{\|f\|_1} \leq n + \|g\|_{1+1/n}$$

and

$$\|f\|_{1+1/n} \leq (n + \|g\|_{1+1/n})/\beta < \infty.$$