Exercise 1

Let $E$ be a Banach space and let $(x_n)$ be a sequence such that $x_n \rightharpoonup x$ in the weak $\sigma(E, E^*)$ topology. Set

$$y_n = \frac{1}{n} \sum_{k \leq n} x_k.$$ 

Prove that $y_n \rightharpoonup x$.

Answer of exercise 1

Let $T \in E^*$. We have

$$T(y_n) = \frac{1}{n} \sum_{k \leq n} T(x_k).$$

and

$$|T(y_n) - T(x)| \leq \frac{1}{n} \sum_{k \leq n} |T(x_n) - T(x)|.$$ 

For all $\varepsilon > 0$, there exists $N$ such that for all $n > N$, $|T(x_n) - T(x)| < \varepsilon$. Thus,

$$|T(y_n) - T(x)| \leq \frac{1}{n} \sum_{k \leq N} |T(x_n) - T(x)| + \varepsilon,$$

and for $n$ great enough,

$$|T(y_n) - T(x)| \leq 2\varepsilon.$$

Exercise 2

1. Consider the sequence $(f_n)$ of functions defined by $f_n(x) = ne^{-nx}$. Prove that

   1. $f_n \to 0$ a.e.
   2. $(f_n)$ is bounded in $L^1(\Omega)$.
   3. $f_n \to 0$ in $L^1(\Omega)$ strongly.
   4. $f_n \rightharpoonup 0$ weakly in $\sigma(L^1, L^\infty)$.

   More precisely, there is no subsequence that converges weakly $\sigma(L^1, L^\infty)$. 


2. Let \( 1 < p < \infty \) and consider the sequence \((g_n)\) of functions defined by \( g_n(x) = n^{1/p} e^{-nx} \). Prove that

1. \( g_n \to 0 \) a.e.
2. \((g_n)\) is bounded in \( L^p(\Omega) \).
3. \( g_n \not\to 0 \) in \( L^p(\Omega) \) strongly but \( g_n \to 0 \) in \( L^q(\Omega) \) strongly for every \( 1 \leq q < p \).
4. \( g_n \rightharpoonup 0 \) weakly in \( \sigma(L^p, L^p') \).

**Answer of exercise 2**

1. We consider \( f_n = ne^{-nx} \). For all \( x \in (0,1) \), \( f_n(x) = e^{-n(x+\ln(n)/n)} \) and converges toward 0.

\[
kf_n k_{L^1(\Omega)} = \int_0^1 |f_n| = \int_0^1 ne^{-nx} = -\int_0^1 (e^{-nx})' = -[e^{-nx}]_0^1 = 1 - e^{-n}.
\]

Thus, \( f_n \) is bounded in \( L^1(\Omega) \) and does not converge toward 0 in \( L^1(\Omega) \). Finally, let \( u \in C^1([0,1]) \),

\[
\int_0^1 f_n u = \int_0^1 (e^{-nx})' u = [e^{-nx} u]_0^1 - \int_0^1 e^{-nx} u' \to u(0).
\]

Thus, \( f_n \) does not converge toward 0 in \( \sigma(L^1, L^\infty) \) (and even no subsequence).

2. We have defined \( g_n \) by

\[
g_n(x) = n^{1/p} e^{-nx},
\]

with \( 1 < p < \infty \). Obviously, \( g_n(x) \) goes to zero for every \( x \in (0,1) \). Moreover,

\[
k_g k_{L^p} = \int_0^1 ne^{-nx} = \frac{1}{p}(1 - e^{-np}).
\]

Thus, \( g_n \) is bounded in \( L^p(0,1) \) and does not converge toward 0 in \( L^p(0,1) \). Now, let \( u \in C^\infty_0(0,1) \). As \( g_n \) does converge uniformly toward 0 on the support of \( u \), we have

\[
\int_0^1 g_n u \to 0.
\]

Let \( v \in L^{p'}(0,1) \). For all \( \varepsilon > 0 \), there exists \( u_\varepsilon \in C^\infty_0(0,1) \) such that \( ku_\varepsilon - v k_{L^{p'}(\Omega)} < \varepsilon \). It follows that

\[
\int_0^1 g_n v = \int_0^1 g_n (u_\varepsilon - v) + \int_0^1 g_n u_\varepsilon \leq kg_n k_{L^p} ku_\varepsilon - v k_{L^{p'}} + \int_0^1 g_n u_\varepsilon.
\]

As \( g_n \) is bounded in \( L^p(0,1) \), we get that

\[
\int_0^1 g_n v \leq C \varepsilon + \int_0^1 g_n u_\varepsilon.
\]
For $n$ great enough, we obtain that

$$\left| \int_0^1 g_n v \right| \leq (C + 1)\varepsilon.$$  

It follows that $\int_0^1 g_n v$ converges toward 0 as $n$ goes to infinity, that is $g_n$ converges weak toward 0 in $L^p(0, 1)$.

**Exercise 3**

Assume that $|\Omega| < \infty$. Let $1 < p < \infty$. Let $(f_n)$ be a sequence in $L^p(\Omega)$ such that

1. $(f_n)$ is bounded in $L^p(\Omega)$.
2. $f_n \to f$ a.e. on $\Omega$.

1. Prove that $f_n * f$ weakly in $\sigma(L^p, L^{p'})$.
2. Same conclusion if assumption (2) is replaced by

$$k f_n - f k_i \to 0.$$  

3. Assume now (1) and (2) and $|\Omega| < \infty$. Prove that $k f_n - f k_i \to 0$ for every $q$ with $1 \leq q < p$.

**Answer of exercise 3**

1. To simplify the proof, we assume that $|\Omega| < \infty$. It can be easily adapt to the case where $\Omega$ is $\sigma$-finite. First, let us notice that that from Fatou’s Lemma, $f \in L^p(\Omega)$. In a first step, we are going to prove that up to a subsequence, $f_n$ weakly converges toward $f$ in $L^p(\Omega)$. As $f_n$ is bounded in $L^p(\Omega)$, it admits a weakly convergent subsequence. That is there exists $\phi$ monotone map from $\mathbb{N}$ into $\mathbb{N}$ and $\tilde{f} \in L^p(\Omega)$ such that $f_{\phi(n)}$ weakly converges toward $\tilde{f}$. Moreover, from the Egorov’s Theorem, for all integer $m > 0$, there exists a measurable subset $A_m$ of $\Omega$ such that $f_{\phi(n)}$ converges toward $f$ uniformly. It follows that for all $g \in L^{p'}(\Omega)$,

$$\int_{\Omega \setminus A_m} f_{\phi(n)} g \to \int_{\Omega \setminus A_m} f g$$

and

$$\int_{\Omega \setminus A_m} f_{\phi(n)} g \to \int_{\Omega \setminus A_m} \tilde{f} g.$$

Thus,

$$\int_{\Omega \setminus A_m} (f - \tilde{f}) g = 0,$$
for every \( g \in L^{p'}(\Omega) \). Choosing \( g = \text{sign}(f - \tilde{f}) \), it follows that \( f = \tilde{f} \) a.e. in \( \Omega \setminus A_m \). In particular, \( f = \tilde{f} \) a.e. in \( \Omega \setminus (n_m A_m) \). As \( |n_m A_m| = 0 \), we deduce that \( f = \tilde{f} \) almost everywhere. It remains to prove that the whole sequence \( f_n \) weakly converges toward \( f \) in \( L^p(\Omega) \). Assume this is not the case. Then, there exists \( h \in L^{p'}(\Omega) \) and \( \psi : \mathbb{N} \to \mathbb{N} \) monotone such that

\[
\left| \int_{\Omega} (f_{\psi(n)} - f)h \right| > \delta > 0.
\]

Replacing \( f_n \) by \( f_{\psi(n)} \) in the first part of the proof, we conclude that there exists \( \Phi : \mathbb{N} \to \mathbb{N} \) monotone such that

\[
f_{\psi(n)} \star f \text{ in } L^p(\Omega)
\]

and

\[
\left| \int_{\Omega} (f_{\psi(n)} - f)h \right| > \delta > 0,
\]

what is contradictory. We conclude as the whole sequence \( f_n \) weakly converges toward \( f \) in \( L^p(\Omega) \).

2. The proof is exactly the same as in the previous case. It departs only in to establish that \( f = \tilde{f} \). In this case, we have immediately that

\[
\int_{\Omega} (f - \tilde{f})g = 0,
\]

for all \( g \in L^\infty(\Omega) \). Choosing once again \( g = \text{sign}(f - \tilde{f}) \), we get that \( f = \tilde{f} \) a.e.

3. For every \( \varepsilon > 0 \), from the Egorov’s Theorem, there exists a measurable subset \( A \) of \( \Omega \), such that \( |A| < \varepsilon \) and \( f_n \) converges uniformly toward \( f \) in \( \Omega \setminus A \). We have

\[
\int_{\Omega} |f_n - f|^q = \int_A |f_n - f|^q + \int_{\Omega \setminus A} |f_n - f|^q.
\]

From Hölder’s inequality, we have

\[
\int_A |f_n - f|^q \leq \left( \int_A |f_n - f|^p \right)^{q/p} |A|^{q/p - q} < C \varepsilon^{q/p - q}.
\]

Moreover, as \( f_n \) uniformly converges toward \( f \) on \( \Omega \setminus A \), for \( n \) great enough, we have

\[
\int_{\Omega \setminus A} |f_n - f|^q < \varepsilon.
\]

We conclude that for \( n \) great enough,

\[
\int_{\Omega} |f_n - f|^q < C \varepsilon^{q/p - q} + \varepsilon,
\]

and that \( f_n \) converges toward \( f \) in \( L^q(\Omega) \) for all \( 1 \leq q < p \).
Exercise 4

Let $E$ be a Banach space and $x_0 \in E$ be fixed. Prove that there exists $T \in E^*$ such that

$$T(x_0) = kx_0k_E^2$$

and $kTk_{E^*} = kx_0k_E$.

**Answer of exercise 4**

Let $G = \mathbb{R}x$ and $T$ be the linear continuous map defined on $G$ by

$$T(tx_0) = tkx_0k^2.$$

From the Hahn-Banach Theorem, there exists an extension of $T$ on $E$ such that $kTk_{E^*} = kTk_{E^*} = kx_0k_E$.

Exercise 5

Let $E$ be a Banach space and let $A \subset E$ be a subset that is sequentially compact for the weak topology of $E$. Prove that $A$ is bounded.

**Answer of exercise 5**

Let $(x_n)$ be a sequence of elements of $A$. As $A$ is sequentially compact, there exists $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi$ is increasing and $x \in A$, with

$$x_{\phi(n)} \to x.$$

In particular, for all $T \in E^*$, $T(x_{\phi(n)})$ is bounded and, from the Banach-Steinhaus Theorem, there exists $C$ such that for all $T \in E^*$,

$$T(x_{\phi(n)}) \leq CkTk_{E^*}.$$

From the Exercise 4 there exists $T$ such $T(x_{\phi(n)}) = kx_{\phi(n)}k^2$ and $kTk = kx_{\phi(n)}k$. It follows that

$$kx_{\phi(n)}k \leq C.$$

If $A$ was not bounded, we could construct a sequence $x_n$ of elements of $A$ such that $kx_nk_E \geq n$, what is impossible from the last inequality.

Exercise 6  Rademacher’s functions

Let $1 \leq p \leq \infty$ and let $f \in L^p_{\text{loc}}(\mathbb{R})$. Assume that $f$ is $T$-periodic, i.e., $f(x + T) = f(x)$, a.e. on $\mathbb{R}$. Set

$$T = |T|^{-1} \int_0^T f(t)dt.$$

Consider the sequence $(u_n)$ in $L^p(0,1)$ defined by

$$u_n(x) = f(nx), \quad x \in (0,1).$$
1. Prove that $u_n \rightharpoonup f$ with respect to the topology $\sigma (L^p, L^{p'})$.
2. Determine $\lim_{n \to \infty} ku_n - f k_p$.
3. Examine the following examples:
   1. $u_n (x) = \sin(nx)$.
   2. $u_n (x) = f_n (x)$ where $f$ is 1-periodic and
      $$f (x) = \begin{cases} 
      \alpha & \text{for } x \in (0, 1/2), \\
      \beta & \text{for } x \in (1/2, 1).
      \end{cases}$$

   The functions of (2) are called Rademacher’s functions.

**Answer of exercise 6**

1. Let $0 < a < b < 1$ and $v$ be the indicator function of $(a, b)$ on $(0, 1)$, that is
   $$v(x) = \begin{cases} 
      1 & \text{if } a < x < b \\
      0 & \text{if } x \in (0, 1) \setminus (a, b).
      \end{cases}$$

   We have
   $$\int_0^1 u_n v = \int_0^1 f(nx)v(x) \, dx = n^{-1} \int_0^n f(x)v(x/n) \, dx = n^{-1} \int_{na}^{nb} f(x) \, dx.$$ 

   We set $k$ and $l$ to be the integers such that
   $$(k - 1)T < na \leq kT, \quad lT < nb \leq (l + 1)T.$$ 

   We have
   $$\int_0^1 u_n v = \int_0^1 f(nx)v(x) \, dx = n^{-1} \int_0^n f(x)v(x/n) \, dx = n^{-1} \int_{na}^{nb} f(x) \, dx.$$ 

   From the definition of $k$ and $l$, we have
   $$\frac{l - k}{n} \leq \frac{b - a}{T} \leq \frac{l - k + 2}{n}.$$ 

   Thus, $(l - k)/n \to (b - a)/T$. Moreover,
   $$\frac{1}{n} \left| \int_{na}^{kT} f(x) \, dx + \int_{T}^{nb} f(x) \, dx \right| \leq \frac{2}{n} k f_{L^1(0,T)} \to 0.$$
It follows that
\[ \int_0^1 u_n v \to f(b-a), \]
as \( n \) goes to infinity. We deduce that for any step function \( v \), we have
\[ \int_0^1 u_n v \to \int_0^1 v. \]

As the set of step functions is dense in \( L^{p'}(0,1) \), with \( 1 \leq p' < \infty \), we deduce that if \( f \in L^{p'}_{loc}(0,T) \), with \( 1 < p \leq \infty \), \( u_n \) does converge toward \( f \) in \( \sigma(L^p, L^{p'}) \). Indeed, for every \( \epsilon > 0 \), there exists a step function \( w \) such that \( \| v - w \|_{L^{p'}(0,1)} \leq \epsilon \). We then have
\[
\left| \int_0^1 u_n v - f \int_0^1 v \right| \leq \left| \int_0^1 u_n w - f \int_0^1 w \right| + \int_0^1 |u_n||v-w| + |f| \int_0^1 |v-w|.
\]
From Hölder inequality,
\[
\int_0^1 |u_n||v-w| \leq k u_n k_{L^p(0,1)} k v - w k_{L^{p'}(0,1)}
\]
and
\[
|f| \int_0^1 |v-w| \leq f k v - w k_{L^{p'}(0,1)}.
\]
Moreover, from the previous analysis, we have
\[
k u_n k_{L^p(0,1)}^2 = \int_0^1 u_n^p dx \to T^{-1} \int_0^T f^p.
\]
In particular, \( u_n \) is bounded in \( L^p(0,1) \). We have obtained that
\[
\left| \int_0^1 u_n v - f \int_0^1 v \right| \leq \left| \int_0^1 u_n w - f \int_0^1 w \right| + C k v - w k_{L^{p'}(0,1)}.
\]
Finally, has \( w \) is a step function, for \( n \) great enough,
\[
\left| \int_0^1 u_n w - f \int_0^1 w \right| \leq \epsilon
\]
and
\[
\left| \int_0^1 u_n v - f \int_0^1 v \right| \leq (1 + C)\epsilon.
\]
It remains to consider the case \( p = 1 \) and \( f \in L^1_{loc}(\mathbb{R}) \). For every \( \epsilon > 0 \), there exists a \( T \) periodic function, \( g \in L^\infty \) such that
\[
T^{-1} kf - g k_{L^1(0,T)} \leq \epsilon
\]
For every \( v \in L^\infty(0, 1) \), we have from the previous analysis,
\[
\int_0^1 g(nx)v(x) \, dx \to g \int_0^1 v.
\]

On the hand, we have
\[
\left| \int_0^1 u_nv - f \int_0^1 v \right| \leq kf(nx) - g(nx)k_{L^1(0,1)}kv_{\infty} + \left| \int_0^1 g(nx)v - f \int_0^1 v \right|
\]

We have
\[
kf(nx) - g(nx)k_{L^1(0,1)} \to \frac{1}{T} \int_0^T |f - g| \leq \epsilon.
\]
Thus, for \( n \) great enough, we have
\[
kf(nx) - g(nx)k_{L^1(0,1)} \leq 2\epsilon
\]
and
\[
\left| \int_0^1 g(nx)v - f \int_0^1 v \right| \leq \epsilon + |f - g|kv_{L^1(0,1)}.
\]
Furthermore
\[
|f - g| \leq T^{-1} \int_0^T |f - g| \leq \epsilon.
\]

We thus have proved that for \( n \) great enough
\[
\left| \int_0^1 u_nv - f \int_0^1 v \right| \leq 2\epsilon kv_{\infty} + \epsilon kv_{L^1},
\]
and that \( \int u_nv \to f \int v \) as claimed.

2. We set \( g(s) = |f(s) - f|^p \). As \( g \) is \( T \)-periodic ans \( g \in L^1_{\text{loc}}(\mathbb{R}) \), we have from the previous question
\[
\int_0^1 g(nx) \, dx \to g,
\]
that is
\[
\lim ku_n - \bar{f}k^p = \frac{1}{T^2} \int_0^T \left| \int_0^T (f(s) - f(t)) \, ds \right|^p dt.
\]

3. 1. \( u_n = \sin(nx) \). We have \( u_n \) weakly-* in \( L^\infty \),
2. \( u_n = f(nx) \) where \( f \) is one periodic and
\[
f(x) = \begin{cases} 
\alpha & \text{if } x \in (0, 1/2) \\
\beta & \text{if } x \in (1/2, 1).
\end{cases}
\]

Then \( u_n \to (\alpha + \beta)/2 \) for the weak-* topology of \( L^\infty(0, 1) \).