Functional analysis and applications
MASTER "Mathematical Modelling"
École Polytechnique and Université Pierre et Marie Curie
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See also the course webpage:
http://www.cmap.polytechnique.fr/~allaire/master/course-funct-analysis.html

Exercise 1

Let $E = L^2(0,1)$. Given $u \in E$, set

$$Tu(x) = \int_0^x u(t)\,dt.$$

1. Prove that $T \in \mathcal{K}(E)$. [**Hint:** Use Ascoli-Arzelà Theorem]
2. Determine the set $EV(T)$ of eigenvalues of T.
3. Determine T^*.

Answer of exercise 1

1. Let (u_n) be a bounded sequence in $L^2(0,1)$. Let $0 < y < x < 1$. We want to prove that Tu_n is compact in E. From Hölder inequality, we have for all $u \in E$,

   $$|Tu(x) - Tu(y)| = \left| \int_y^x u(s)\,ds \right| \leq |x-y|^{1/2} \left(\int_y^x |u|^2 \right)^{1/2} \leq |x-y|^{1/2} \|u\|_E.$$

   It follows that the sequence Tu_n is uniformly equicontinuous and from Ascoli-Arzelà Theorem, there exists a subsequence $Tu_{\varphi(n)}$ (where φ is an increasing map from $\mathbb{N}$ into $\mathbb{N}$) converging in $C([0,1])$. In particular, it converges in $L^2(0,1)$ (for the strong topology).
2. Let $\lambda \in EV(T)$, there exists $u \not\equiv 0$ in E such that

   $$\int_0^x u(s)\,ds = \lambda u(x)$$

   a.e. in Ω. Not that Tu admits a weak derivative and that

   $$(Tu)' = u.$$

   It follows that

   $$u = \lambda u'.$$

   The solution of this equation are $u = Ce^{x/\lambda}$. But, a $u(0) = 0$ we get that $u = 0$ is the only possible solution. Thus, $VP(T) = \emptyset$. Finally, as T is compact, we have $\sigma(T) \setminus \{0\} = VP(T) \setminus \{0\}$ and $0 \in \sigma(T)$. Thus, $\sigma(T) = 0$.

3. Let $u, v \in E$, $(u, T^*v) = \int_t^1 v(x)dx$.

Exercise 2

1. Let $-\infty \leq a < b \leq \infty$ and $T \in \mathcal{D}'((a,b)^n)$ such that $\partial_i T = 0$ for all $i \in \{1, \cdots, n\}$. Prove that there exists a constant $C \in \mathbb{R}$ such that $T = C$.
2. Extend the previous result to the distribution $\mathcal{D}'(\Omega)$, where Ω is an open and connected subset of $\mathbb{R}^n$.

Answer of exercise 2

1. We will only treat the case $a = -\infty$ and $b = \infty$ (in fact, the proof is exactly the same for every interval). Let us first consider the case $n = 1$. Let T be a distribution such that $T' = 0$. We have for all $\psi \in C_0^\infty(\mathbb{R})$,

$$\langle T, \partial_1 \psi \rangle = 0.$$

Let $\theta \in C_0^\infty(\mathbb{R})$ such that $\int \theta = 1$ Let $\varphi \in C_0^\infty(\mathbb{R})$. We set

$$\psi(x) = \int_{-\infty}^{x} \varphi(s) - C\theta(s)\, ds.$$

We have $\psi \in C^\infty(\mathbb{R})$ and $\psi' = \varphi + C\theta$. We are going to choose C for ψ to be of compact support. To this end, it suffices to have

$$\int \varphi - C\theta = 0,$$

that is

$$C = \int \varphi.$$

Then,

$$\langle T, \psi' \rangle = 0,$$

that is

$$\langle T, \varphi - C\theta \rangle = 0,$$

and

$$\langle T, \varphi \rangle = C \langle T, \theta \rangle,$$

and finally

$$\langle T, \varphi \rangle = \langle T, \theta \rangle \int \varphi.$$

Thus, T is the constant distribution $\langle T, \theta \rangle$.
Let us now tackle the general case. Assume that the result as been proven in $\mathbb{R}^{n-1}$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. We set for every $x = (\widetilde{x}, x_n) \in \mathbb{R}^n$,

$$\psi(x) = \int_{-\infty}^{s} \varphi(\widetilde{x}, s) - \widetilde{\varphi}(\widetilde{x})\theta(s)\, ds,$$

where

$$\widetilde{\varphi}(\widetilde{x}) = \int \varphi(\widetilde{x}, x_n)\, dx_n$$

It is easy to check that $\psi \in C_0^\infty(\mathbb{R}^n)$. Moreover, $\partial_n \psi(x) = \varphi(x) - \widetilde{\varphi}(\widetilde{x})\theta(x_n)$. It follows that

$$\langle T, \varphi\rangle = \langle T, \widetilde{\varphi}(\widetilde{x})\theta(x_n)\rangle.$$

Let $S \in \mathcal{D}'(\mathbb{R}^{n-1})$ be defined by

$$\langle S, \widetilde{\psi}\rangle = \langle T, \widetilde{\psi}(\widetilde{x})\theta(x_n)\rangle.$$

We have for all $i \in \{1, \cdots, n-1\}$,

$$\langle \partial_i S, \widetilde{\psi}\rangle = -\langle \partial_i S, \partial_i \widetilde{\psi}\rangle = -\langle T, \partial_i(\widetilde{\psi}(\widetilde{x}))\theta(x_n)\rangle = -\langle T, \partial_i(\widetilde{\psi}(\widetilde{x})\theta(x_n))\rangle = 0.$$

From the recursive assumption, we S there exists C such that

$$\langle S, \widetilde{\psi}\rangle = C \int_{\mathbb{R}^{n-1}} \widetilde{\psi}.$$

Thus,

$$\begin{aligned} \langle T, \varphi\rangle &= \langle T, \widetilde{\varphi}(\widetilde{x})\theta(x_n)\rangle = \langle S, \widetilde{\varphi}\rangle = C \int_{\mathbb{R}^{n-1}} \widetilde{\varphi} \\ &= C \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \varphi(\widetilde{x}, x_n)\, dx_n\, d\widetilde{x} = C \int \varphi. \end{aligned}$$

2. Let $T \in \mathcal{D}'(\Omega)$ such that $\partial_i T = 0$ for all $i \in \{1, \cdots, n\}$. From the previous question, we know that the restriction of T to any cube (with edged parallel to the axes) included in Ω can be identified to a constant. Next let x be an element of Ω belonging to two cubes Q_1 and Q_2 included in Ω. There exists a small cube Q' centered at x included in the intersection of Q_1 and Q_2. Let T_i we the restriction of T to Q_i $(i = 1, 2)$. We know that T_1 and T_2 are equal to constants (denoted C_1 and C_2 respectively). Obviously we have

$$C_1 = T_1|_{Q'} = T|_{Q'} = T_2|_{Q'} = C_2.$$

Thus, we can define for all $x \in \Omega$ a real $C(x) = C$, where $C = T_|Q$, Q being any cube included in Ω centered at X. The map $C : \Omega \to \mathbb{R}$ is continuous (it is constant on open cubes) and as Ω is connected, it is a constant map.

Exercise 3

Let $I = (0, 1)$.

1. Prove that for every $1 \leq p \leq \infty$, $W^{1,p}(I)$ is included in $L^\infty(I)$ with continuous injection.
2. Assume that (u_n) is a bounded sequence in $W^{1,p}(I)$ with $1 < p \leq \infty$. Show that there exists a subsequence $(u_{\varphi(n)})$ and $u \in W^{1,p}(I)$ such that

$$\|u_{\varphi(n)} - u\|_\infty \to 0.$$

Moreover, $u'_{\varphi(n)} \to u'$ weakly in $L^p(I)$ if $1 < p < \infty$.
3. Construct a bounded sequence (u_n) in $W^{1,1}(I)$ that does not admit any subsequence converging in $L^\infty(I)$.

Answer of exercise 3

1. If $p = \infty$, the inclusion is obvious. Let $1 \leq p < \infty$. Let $v \in C^\infty([0,1])$, we have for very $x, y \in I$,

$$v(x) - v(y) = \int_x^y v'(s)\, ds.$$

thus,

$$|v(x) - v(y)| \leq \int_0^1 |v'| \leq (\int_0^1 |v'|^p)^{1/p} = \|v\|_{1,p}.$$

Then

$$|v(x)| \leq |v(x) - v(y)| + |v(y)|$$

and

$$|v(x)| \leq \int |v(x) - v(y)| + |v(y)|\, dy \leq 2\|v\|_{1,p}.$$

As the set of $C^\infty([0,1])$ is dense in $W^{1,p}(I)$, it follows that the injection of $W^{1,p}(I)$ into $L^\infty(I)$ is continuous.
2. Let $1 < p < \infty$, and $v \in W^{1,p}(I)$, we have

$$v(x) - v(y) = \int_x^y v'(s)\, ds \leq (\int_x^y 1)^{1/p'} \left(\int_x^y |v'|^p\right)^{1/p} \leq |x-y|^{1/p'} \|v\|_{1,p}.$$

In the case $p = \infty$, we have

$$v(x) - v(y) \leq |x - y| \|v\|_\infty.$$

It follows that any bounded sequence in $W^{1,p}(I)$ $(1 < p <\leq \infty)$ is bounded and equicontinuous in $C([0,1])$ and thus admits a converging subsequence in $C([0,1])$ from the Ascoly-Arzelà Theorem.
3. Let $u_n \in W^{1,1}(I)$ be a sequence defined by

$$u'_n(x) = \begin{cases} n \text{ if } x < 1/n \\ 0 \text{ if } x > 1/n \end{cases}$$

and $u_n(0) = 0$. Assume that it admits a converging subsequence in $L^\infty(I)$ toward an element $u \in L^\infty(I)$. The only possible limit is $u = 1$ but

$$\|u_n - 1\|_\infty = 1.$$

Exercise 4

Let $I = (0,1)$. For every $u \in L^p(I)$, we denote $\overline{u}$ the extension of $u \in L^p(\mathbb{R})$ outside I by 0.

1. Prove that if $1 \le p < \infty$, then $u \in W_0^{1,p}(I) \Rightarrow \overline{u} \in W^{1,p}(\mathbb{R})$.
2. Conversely, let $u \in L^p(I)$ (with $1 \le p < \infty$). Prove that $\overline{u} \in W^{1,p}(I) \Rightarrow u \in W_0^{1,p}(I)$.
3. Let $u \in L^p(I)$ (with $1 \le p < \infty$). Show that $u \in W_0^{1,p}(I)$ iff there exists a constant C such that for every $\varphi \in C_0^1(\mathbb{R})$,

$$\left| \int_0^1 u\varphi' \right| \le C\|\varphi\|_{L^{p'}(\mathbb{R})}$$

Answer of exercise 4

1. Let $u \in W^{1,0}(I)$, then there exists a sequence (u_n) in $C_0^\infty(I)$ that converges toward u in $W^{1,p}(I)$. Obviously, $\overline{u}_n$ is a Cauchy sequence in $W^{1,p}(\mathbb{R})$. Thus, it is converging in $W^{1,p}(\mathbb{R})$ and $\in W^{1,p}(\mathbb{R})$.
2. Let $u \in W^{1,p}(I)$ such that $\overline{u} \in W^{1,p}(I)$. For every integer n, there exits $\chi_n \in C_0^\infty(I)$ such that

$$\chi_n(x) = 1 \text{ for every } x \in (1/n, 1 - 1/n),$$

and

$$|\chi_n'| \le nC,$$

where C is a constant that does not depend on n. We have

$$\|(\chi_n\overline{u})' - \overline{u}'\|_p \le \|\chi_n'\overline{u}\|_p + \|(\chi_n - 1)\overline{u}'\|_p.$$

Moreover,

$$\begin{aligned} \|\chi_n'\overline{u}\|_p &= \left(\int_0^{1/n} |\chi_n'\overline{u}|^p\right)^{1,p} + \left(\int_{1-1/n}^1 |\chi_n'\overline{u}|^p\right)^{1,p} \\ &\le \frac{1}{n}Cn\left(\sup_{x\in(0,1/n)} |\overline{u}(x)| + \sup_{x\in(1-1/n,1)} |\overline{u}(x)|\right) \\ &= C\left(\sup_{x\in(0,1/n)} |\overline{u}(x)| + \sup_{x\in(1-1/n,1)} |\overline{u}(x)|\right). \end{aligned}$$

As $\overline{u}$ is continuous, the right-hand side of this inequality goes to zero when n goes to infinity and

$$\|\chi_n'\overline{u}\|_p \to_{n\to\infty} 0.$$

It follows that $\chi_n\overline{u}$ converges toward $\overline{u}$ as n goes to infinity. Similar, we can prove similarly that the map from $W^{1,p}(\mathbb{R})$ into itself $v \mapsto \chi_n v$ is uniformly continuous in $W^{1,p}(I)$. We deduce than for every $\varepsilon > 0$, there exists n such that

$$\|\chi_n \overline{u} - \overline{u}\|_{1,p} \leq \varepsilon.$$

As $C_0^\infty(\mathbb{R})$ is dens in $W^{1,p}(\mathbb{R})$, there exists $v \in C_0^\infty(\mathbb{R})$ such that

$$\|v - \overline{u}\|_{1,p} \leq \varepsilon.$$

It follows that

$$\begin{aligned} \|\chi_n v - u\|_{W^{1,p}(I)} &= \|\chi_n v - \overline{u}\|_{W^{1,p}(\mathbb{R})} \leq \|\chi_n v - \chi_n \overline{u}\|_{1,p} + \|\chi_n \overline{u} - \overline{u}\|_{1,p} \\ &\leq C\|v - \overline{u}\|_{1,p} + \|\chi_n \overline{u} - \overline{u}\|_{1,p} \leq 2\varepsilon. \end{aligned}$$

3. From the inequality, we have that $\overline{u}$ does belong to the dual of $W^{1,p'}(\mathbb{R})$ which is equal to $W^{1,p}(\mathbb{R})$.