Functional analysis and applications
MASTER "Mathematical Modelling"
École Polytechnique and Université Pierre et Marie Curie
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See also the course webpage:
http://www.cmap.polytechnique.fr/~allaire/master/course-funct-analysis.html

Exercise 1

Let Ω be a bounded regular open subset of $\mathbb{R}^N$.

1. Prove that for every $u \in H_0^2(\Omega)$,
$$\int_\Omega |\Delta u|^2 = \int_\Omega \left(\sum_{|\alpha|=2} |D^\alpha u|^2 \right).$$
2. Prove that there exists a constant C such that for every $u \in H_0^2(\Omega)$,
$$\int_\Omega \left(|\Delta u|^2 + |u|^2 \right) \geq C \|u\|_{H^2}^2.$$
3. Prove that for every $f \in L^2(\Omega)$, there exists a unique $u = T(f) \in H_0^2(\Omega)$ such that for all $v \in H_0^2(\Omega)$,
$$\int_\Omega (\Delta u \Delta v + uv) = \int_\Omega fv.$$
4. Prove that T is a compact and self adjoint operator from $L^2(\Omega)$ into $L^2(\Omega)$.
5. Prove that the eigenvectors u solution of
$$\int_\Omega (\Delta u \Delta v + uv) = \lambda \int_\Omega uv$$
defines a Hilbert basis of $L^2(\Omega)$.

Exercise 2

Let $I = (0,1)$. Let $u \in W^{1,p}(I)$ with $1 \leq p < \infty$. Our goal is to prove that $u' = 0$ a.e. on the set $E = \{x \in I \ : \ u(x) = 0\}$. Fix a function $G \in C^1(\mathbb{R}, \mathbb{R})$ such that $|G(t)| \leq 1$ and $|G'(t)| \leq C$ for every $t \in \mathbb{R}$ for a constant C, and
$$G(t) = \begin{cases} 1 & \text{if } t \geq 1 \\ t & \text{if } |t| \leq 1/2 \\ -1 & \text{if } t \leq -1. \end{cases}$$
Set
$$v_n(x) = \frac{G(nu(x))}{n}.$$

1. Check that $\|v_n\|_{L^\infty} \to 0$ as $n \to \infty$.
2. Show that $v_n \in W^{1,p}(I)$ and compute v_n'.
3. Deduce that $|v_n'|$ is bounded by a fixed function in $L^p(I)$.
4. Prove that $v_n'(x) \to f(x)$ a.e. on I, as $n \to \infty$ and identify f.
   [**Hint:** Consider separately the cases $x \notin E$ and $x \in E$.]
5. Deduce that $v_n' \to f$ in $L^p(I)$.
6. Prove that $f = 0$ a.e. on I and conclude that $u' = 0$ a.e. on E.

Answer of exercise 2

1. $\|v_n\| \leq 1/n \to 0$.
2. Assume first that u is a regular map, then

$$v_n'(x) = G'(nu(x))u'(x).$$

Moreover,

$$|v_n'| \leq C|u'|.$$

Thus, we get

$$\|v_n\|_{1,p} \leq \|v_n\|_p + \|v_n'\|_p \leq 1 + C\|u_n'\|_{1,p}.$$

Now, we only have to extend the previous analysis to every $u \in W^{1,p}(I)$. Let $(u_k) \in C^\infty(\overline{I})^{\mathbb{N}}$ be a sequence converging toward u in $W^{1,p}(I)$. we have that

$$\frac{G(nu_k)}{n}$$

is bounded in $L^\infty(I)$ and converging almost everywhere toward $G(nu)/n$. Thus, from the dominated converge Theorem, it converges in $L^p(I)$. Without lost of generality, we can assume that $|u_k'|$ is bounded by a map $\varphi \in L^p(I)$. The sequence $G'(nu_k)u_k'$ converges a.e. toward $G'(u)u'$ (because G in C^1). Moreover, $|G'(nu_k)u_k'|$ is bounded by $C|u_k'|$ and thus by $C\varphi$. From the dominated convergence Theorem, we deduce that $G'(nu_k)u_k'$ converges toward $G'(nu)u'$ in L^p. It follows that $G(nu_k)/n$ is a Cauchy sequence in $W^{1,p}(I)$ and that is it convergent. Moreover, the limit is $G(nu)/n$ and

$$(G(nu)/n)' = \lim_k (G(nu_k)/n)' = G'(nu)u'.$$

3. We have $v_n' = G'(nu)u'$ and $|v_n'|$ bounded by $C|u'| \in L^p(I)$.
4. If $x \notin E$ then $v_n'(x) = G'(nu(x))u'(x) = 0$ for n sufficiently large.
   If $x \in E$ then $v_n'(x) = u'(x)$.
   Finally, $\lim_{n\to\infty} v_n'(x) \to f(x)$ a.e. in I with $f(x) = 0$ if $x \notin E$ and $f(x) = u'(x)$ if $x \in E$.
5. From the dominated convergence Theorem, v_n' does converge toward f in L^p.

6. The sequence v_n is converging in $W^{1,p}(I)$. Let v its limit. We have $v' = f$. We have proved that $v = 0$, thus $f = 0$. As $f = u'$ almost everywhere on E, we conclude that $u = 0$ a.e. on E.

Exercise 3 Helly's selection theorem

Let (u_n) be a bounded sequence in $W^{1,1}(0,1)$. The goal is to prove that there exists a subsequence (u_{n_k}) such that $u_{n_k}(x)$ converges to a limit for every $x \in [0,1]$.

1. Show that we may always assume in addition that
$$\forall n, u_n \text{ is a nondecreasing on } [0,1]. \tag{1}$$
[**Hint:** Consider the sequences $v_n(x) = \int_0^x |u_n'(t)|\,dt$ and $w_n = v_n - u_n$] In what follows we assume that (1) holds.
2. Prove that there exist a subsequence (u_{n_k}) and a measurable set $E \subset [0,1]$ with $|E| = 0$ such that $u_{n_k}(x)$ convergences to a limit, denoted $u(x)$, for every $x \in [0,1] \setminus E$. [**Hint:** Use the fact that $W^{1,1} \subset L^1$ with compact injection.]
3. Show that u is nondecreasing on $[0,1] \setminus E$ and deduce that there are a countable set $D \subset (0,1)$) and a nondecreasing function $\overline{u} : (0,1) \to \mathbb{R}$ such that $\overline{u}(x+0) = \overline{u}(x-0)$, $\forall x \in (0,1) \setminus D$ and $\overline{u}(x) = u(x)$, $\forall x \in (0,1) \setminus (D \cup E)$.
4. Prove that $u_{n_k}(x) \to \overline{u}(x)$, $\forall x \in (0,1) \setminus D$.
5. Construct a subsequence from the sequence (u_{n_k}) that converges for every $x \in [0,1]$. [**Hint:** Use a diagonal process.]

Answer of exercise 3

1. Let T be the map from $C^\infty([0,1])$ into $W^{1,1}(0,1)$ be defined by
$$T(\varphi) = \int_0^x |\varphi'(t)|\,dt.$$
We have $T(\varphi)' = |\varphi'|$. Moreover,
$$|T(\varphi)| \le \|\varphi\|_{1,1}.$$
Thus, T is a linear map such that
$$\|T(\varphi)\|_{1,1} \le \|\varphi\|_{1,1}.$$
As $C^\infty(0,1)$ is dense in $W^{1,1}(0,1)$ It follows that T can be uniquely extend into a linear continuous map (also denoted T) from $W^{1,1}(0,1)$ into itself and as
$$T(\varphi)' = |\varphi'|,$$
for every $\varphi \in C_0^\infty([0,1])$, we have
$$T(u)' = |u'| \text{ for all } u \in W^{1,1}(0,1).$$

Moreover, for all $u \in W^{1,1}(0,1)$, there exists $\varphi_n \in C^\infty([0,1])$ such that φ_n does converge toward u in $W^{1,1}(0,1)$. By definition, we have

$$T(u) = \lim T(\varphi_n),$$

and

$$\begin{aligned}\left| T(\varphi_n)(x) - \int_0^x |u'|(t)\,dt \right| &= \int_0^x |\varphi_n'(t)| - |u'(t)|\,dt \\ &\leq \int_0^x |\varphi_n'(t) - u'(t)|\,dt \leq \|\varphi_n - u\|_{1,1}.\end{aligned}$$

Thus, $T(\varphi_n)$ converges toward $\int_0^x |u'(t)|\,dt$ in $L^\infty(0,1)$. In particular, it converges in $L^1(0,1)$. As $T(\varphi_n)$ does also converges toward $T(u)$ in $W^{1,1}(0,1)$, and thus in $L^1(0,1)$, we have

$$T(u) = \int_0^x |u'(t)|\,dt.$$

It follows that $w_n = v_n - u_n$ with

$$v_n = \int_0^x |u_n'(t)|\,ds$$

belongs to $W^{1,1}(0,1)$ and that

$$w_n' = |u_n'| - u_n' \geq 0.$$

Thus, w_n is a nondecreasing map. Let us assume that the result is proved for nondecreasing maps. As (u_n) is bounded in $W^{1,1}(0,1)$, (v_n) and (w_n) are both bounded in $W^{1,1}(0,1)$ and nondecreasing. Thus, they admit everywhere converging subsequences $(w_{\varphi(n)})$ and $(v_{\varphi(n)})$ and $(u_{\varphi(n)})$ is everywhere converging.

2. As the injection from $W^{1,1}(0,1)$ into $L^1(0,1)$ is compact, there exists a subsequence $u_{\varphi_1(n)}$ converging toward for the strong topology of $L^1(0,1)$ toward an element $u \in L^1(0,1)$. From the inverse Lebesgue's Theorem, there exists a subsequence $u_{\varphi_1\circ\varphi_2(n)}$ that do converge almost everywhere toward u.
3. We set $\varphi = \varphi_1 \circ \varphi_2$ as in Question 2. For all $x < y \in [0,1] \setminus E$, we have $u_{\varphi(n)}(x) \leq u_{\varphi(n)}(y)$. Passing to the limit, we get $u(x) \leq u(y)$. We set

$$\overline{u}(x) = \sup\{u(y) \ : \ y \leq x,\, x \in [0,1] \setminus E\}.$$

It is correctly defined for all $x \in (0,1]$. If $0 \in E$, we set $\overline{u}(0) = \inf u$. Now, as $\overline{u}$ is an increasing function defined on $[0,1]$. Moreover, it is bounded. Thus, it admits only a finite number of jump greater than a given constant C. It follows that the number of jumps is in fact countable. Finally, it is easy to check that $\overline{u} = u$ on $[0,1] \setminus E$.

4. Let $x \in (0,1)\backslash D$. For every $\varepsilon > 0$, their exits $x^-, x^+ \in [0,1]\backslash E$ such that $x^- \leq x \leq x^+$ such that $|\overline{u}(x^+) - \overline{u}(x^-)| < \varepsilon$. As $u_{\varphi(n)}$ is nondecreasing, we have for all $n, m > 0$,

$$u_{\varphi(n)}(x^-) \leq u_{\varphi(n)}(x) \leq u_{\varphi(n)}(x^+)$$

and

$$-u_{\varphi(m)}(x^+) \leq -u_{\varphi(m)}(x) \leq -u_{\varphi(m)}(x^-).$$

Summing both inequalities leads to

$$u_{\varphi(n)}(x^-) - u_{\varphi(m)}(x^+) \leq u_{\varphi(n)}(x) - u_{\varphi(m)}(x) \leq u_{\varphi(n)}(x^+) - u_{\varphi(m)}(x^-).$$

and

$$|u_{\varphi(n)}(x) - u_{\varphi(m)}(x)| \leq \max(|u_{\varphi(n)}(x^-) - u_{\varphi(m)}(x^+)|, |u_{\varphi(n)}(x^+) - u_{\varphi(m)}(x^-)|)$$

For n and m great enough, we get

$$|u_{\varphi(n)}(x) - u_{\varphi(m)}(x)| \leq |\overline{u}(x^+) - \overline{u}(x^-)| + \varepsilon \leq 2\varepsilon.$$

Hence, $u_{\varphi(n)}(x)$ is a Cauchy sequence and is convergent. Finally, we have for every $y, z \in E$ that $y < x < z$,

$$\overline{u}(y) \leq \lim u_{\varphi(n)}(x) \leq \overline{u}(z),$$

and thus

$$\overline{u}(x - 0) \leq \lim u_{\varphi(n)}(x) \leq \overline{u}(x + 0).$$

As $x \notin D$, $\overline{u}(x) = \overline{u}(x^-) = \overline{u}(x^+)$ and

$$\lim u_{\varphi(n)}(x) = \overline{u}(x).$$

5. If D is finite, the proof is almost trivial. Otherwise, let (x_n) be a sequence in $(0,1)$ such that

$$D = \{x_n \ : \ n \in \mathbb{N}\}.$$

Assume that we have construct a subsequence $(u_{\Psi_k(n)})$ of $u_{\varphi(n)}$ such that $(u_{\Psi_k(n)}(x_l))_n$ is converging for every $l < k$. The sequence $(u_{\Psi_k(n)}(x_k))_n$ is bounded in $\mathbb{R}$, so there exists an increasing map $\psi_{k+1} : \mathbb{N} \to \mathbb{N}$ such that $(u_{\Psi_k(n)} \circ \psi_{k+1}(x_k))_n$ is converging. Setting $\Psi_{k+1} = \Psi_k \circ \psi_{k+1}$, we have construct a sequence of subsequences $(u_{\Psi_k(n)}$ such that $(u_{\Psi_k(n)}(x_l))_n$ is converging for every $k < l$. Finally, setting $\Psi(n) = \Psi_n(n)$, the sequence $(u_{\Psi(n)})_n$ is a subsequence of $(u_{\varphi(n)})_n$ that converges for every $x \in D$ and thus for every $x \in [0,1]$ from Question 4.