TWO-SCALE CONVERGENCE ON PERIODIC SURFACES AND APPLICATIONS

Grégoire ALLAIRE
Commissariat à l'Energie Atomique
DRN/DMT/SERMA, C.E. Saclay
91191 Gif sur Yvette, France
Laboratoire d'Analyse Numérique, Université Paris 6

Alain DAMLAMIAN
Centre de Mathématiques
Ecole Polytechnique
91128 Palaiseau, France

Ulrich HORNUNG
Department of Computer Science
University of the Federal Armed Forces Munich
D-85577 Neubiberg, Germany

Abstract

This paper is concerned with the homogenization of model problems in periodic porous media when important phenomena occur on the boundaries of the pores. To this end, we generalize the notion of two-scale convergence for sequences of functions which are defined on periodic surfaces. We apply our results to two model problems : the first one is a diffusion equation in a porous medium with a Fourier boundary condition, the second one is a coupled system of diffusion equations inside and on the boundaries of the pores of a porous medium.

Key words : homogenization, two-scale convergence, periodic structures, porous medium.

1 Introduction

In porous media, there are (at least) two length scales : a microscopic scale (for example, the size of a single pore), and a macroscopic scale (the size of a typical sample of porous media). Quite often, the partial differential equations describing a physical phenomenon are posed at the microscopic level whereas only macroscopic quantities are of interest for the engineer or the physicist. Therefore, effective or homogenized equations have to be derived from the microscopic ones by an asymptotic process. To this end, it is convenient to assume that porous media have a periodic microstructure. Although it is far from being the case, it is perfectly legitimate as far as deriving homogenized models is concerned. There is a vast body of literature on periodic homogenization (see e.g. [3], [4], [11]). In this context, the homogenization process is divided in two steps. In a first step, two-scale asymptotic expansions are used to formally obtain the homogenized problem. In a second step, another method (usually the so-called energy method of Tartar [9], [12]) is applied to prove convergence to the homogenized equation guessed from the first step. Recently, a new method, called two-scale convergence, has appeared which replaces these two steps by a single process (see [1], [10]). It relies on a new type of convergence as recalled in the next theorem.

Theorem 1.1 *Let Ω be a bounded open set in $I\!R^N$, and $Y = [0,1]^N$ the unit cube. Let u_ϵ be a bounded sequence in $L^2(\Omega)$. Then, there exist a subsequence (still denoted by ϵ) and a function $u_0(x,y) \in L^2(\Omega \times Y)$ such that u_ϵ* two-scale converges *to $u_0(x,y)$ in the sense that*

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x)\phi(x, \frac{x}{\epsilon})dx = \int_\Omega \int_Y u_0(x,y)\phi(x,y)dxdy,$$

for any continuous function $\phi(x,y) \in C[\bar{\Omega}; C_\#(Y)]$.

The goal of this paper is to generalize this previous result for sequences of functions which are defined on a periodic surface instead of in a fixed domain. Section 2 is devoted to a generalization of the two-scale convergence in this setting. In Sections 3 and 4 these results are applied to some simple problems in porous media in order to demonstrate the relevance of the method. These model problems are derived from two more complex, and physically sound, systems studied in great details in [5] and [8]. Here our purpose is just to illustrate our method : original examples will appear elsewhere.

2 Presentation of the main results

Let Ω be a bounded open set in $I\!R^N$. As usual in periodic homogenization, Y is the unit periodicity cell $[0,1]^N$ which is identified to the unit torus $I\!R^N/\mathbb{Z}^N$. Let T be an open subset of Y with a smooth boundary Γ, and $Y^* = Y \setminus \bar{T}$. We also identify T, Y^*, and Γ with their images by the universal covering map, i.e. their extension by Y-periodicity to the whole space $I\!R^N$. Note that the periodic extension of T may or may not be connected (in other words, the inclusion T is not necessarily strictly included in the unit cell Y). Then, for a sequence ϵ of positive numbers going to zero, we define a perforated domain Ω_ϵ by

$$\Omega_\epsilon = \left\{ x \in \Omega | \frac{x}{\epsilon} \in Y^* \right\}. \tag{1}$$

We further define a $N-1$ dimensional periodic surface Γ_ϵ by

$$\Gamma_\epsilon = \left\{ x \in \Omega | \frac{x}{\epsilon} \in \Gamma \right\}, \tag{2}$$

which is nothing else than the part $\partial\Omega_\epsilon$ lying inside Ω. It is easily seen that

$$\lim_{\epsilon\to 0} \epsilon|\Gamma_\epsilon|_{N-1} = |\Gamma|_{N-1}\frac{|\Omega|_N}{|Y|_N}, \tag{3}$$

where $|\cdot|_p$ is the p-dimensional Hausdorff measure. We denote by $d\sigma(y), y \in Y$, and $d\sigma_\epsilon(x), x \in \Omega$, the surface measure on Γ, and Γ_ϵ respectively. The spaces of squared integrable functions, with respect to these measures on Γ and Γ_ϵ, are denoted by $L^2(\Gamma)$, and $L^2(\Gamma_\epsilon)$ respectively.

The main result of two-scale convergence (see [1], [10]) can be generalized to the case of sequences defined in $L^2(\Gamma_\epsilon)$.

Theorem 2.1 *Let u_ϵ be a sequence in $L^2(\Gamma_\epsilon)$ such that*

$$\epsilon\int_{\Gamma_\epsilon} |u_\epsilon(x)|^2 d\sigma_\epsilon(x) \leq C, \tag{4}$$

where C is a positive constant, independent of ϵ. There exist a subsequence (still denoted by ϵ) and a two-scale limit $u_0(x,y) \in L^2(\Omega; L^2(\Gamma))$ such that $u_\epsilon(x)$ two-scale converges *to $u_0(x,y)$ in the sense that*

$$\lim_{\epsilon\to 0} \epsilon\int_{\Gamma_\epsilon} u_\epsilon(x)\phi(x,\frac{x}{\epsilon})d\sigma_\epsilon = \int_\Omega\int_\Gamma u_0(x,y)\phi(x,y)dxd\sigma(y),$$

for any continuous function $\phi(x,y) \in C[\bar{\Omega}; C_\#(Y)]$.

Remark 2.2 *Note that the* surface two-scale limit *$u_0(x,y)$ is defined in the whole domain Ω for the macroscopic variable x, and on the surface Γ for the microscopic variable y.*

Remark 2.3 *In Theorem 2.1 the set Γ_ϵ is a periodic $(N-1)$-dimensional surface. Of course, it could be generalized to lower dimensional periodic manyfolds, like curves in 3-D. The same methodology could then be applied to homogenization problems such as fluid flow through small pipes or electric currents through wires.*

The proof of Theorem 2.1 is very similar to the usual two-scale convergence theorem [1]. It relies on the following lemma, the proof of which is left to the reader.

Lemma 2.4 *Let $B = C[\bar{\Omega}; C_\#(Y)]$ be the space of continuous functions $\phi(x,y)$ on $\bar{\Omega}\times Y$ which are Y-periodic in y. Then, B is a separable Banach space (i.e. it contains a dense countable family), which is dense in $L^2(\Omega; L^2(\Gamma))$, and such that any function $\phi(x,y) \in B$ satisfies*

$$\epsilon\int_{\Gamma_\epsilon} |\phi(x,\frac{x}{\epsilon})|^2 d\sigma_\epsilon(x) \leq C\|\phi\|_B^2,$$

and

$$\lim_{\epsilon \to 0} \epsilon \int_{\Gamma_\epsilon} |\phi(x, \frac{x}{\epsilon})|^2 d\sigma_\epsilon(x) = \int_\Omega \int_\Gamma |\phi(x,y)|^2 dx d\sigma(y).$$

Proof of Theorem 2.1. By Schwarz inequality, we have

$$\left| \epsilon \int_{\Gamma_\epsilon} u_\epsilon(x) \phi(x, \frac{x}{\epsilon}) d\sigma_\epsilon \right| \leq C \left| \epsilon \int_{\Gamma_\epsilon} \phi(x, \frac{x}{\epsilon}) d\sigma_\epsilon \right|^{\frac{1}{2}} \leq C \|\phi\|_B. \quad (5)$$

This implies that the left hand side of (5) is a continuous linear form on B which can be identified to a duality product $\langle \mu_\epsilon, \phi \rangle_{B',B}$ for some bounded sequence of measures μ_ϵ. Since B is separable, one can extract a subsequence and there exists a limit μ_0 such μ_ϵ converges to μ_0 in the weak * topology of B' (the dual of B). On the other hand, Lemma 2.4 allows us to pass to the limit in the middle term of (5). Combining these two results yields

$$|\langle \mu_0, \phi \rangle_{B',B}| \leq C \left| \int_\Omega \int_\Gamma |\phi(x,y)|^2 dx d\sigma(y) \right|^{\frac{1}{2}}. \quad (6)$$

Equation (6) shows that μ_0 is actually a continuous form on $L^2(\Omega; L^2(\Gamma))$, by density of B in this space. Thus, there exists $u_0(x,y) \in L^2(\Omega; L^2(\Gamma))$ such that

$$\langle \mu_0, \phi \rangle_{B',B} = \int_\Omega \int_\Gamma u_0(x,y) \phi(x,y) dx d\sigma(y),$$

which concludes the proof of Theorem 2.1.

The following result is an easy generalization of the corrector result of the usual two-scale convergence (Theorem 1.8 in [1]).

Proposition 2.5 *Let u_ϵ be a sequence of functions in $L^2(\Gamma_\epsilon)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega; L^2(\Gamma))$. Then, the measure $u_\epsilon d\sigma_\epsilon$ converges, in the sense of distributions in Ω, to the function $u(x) = \int_\Gamma u_0(x,y) d\sigma(y)$ belonging to $L^2(\Omega)$, and we have*

$$\lim_{\epsilon \to 0} \epsilon \int_{\Gamma_\epsilon} |u_\epsilon|^2 d\sigma_\epsilon \geq \int_\Omega \int_\Gamma |u_0(x,y)|^2 dx d\sigma(y) \geq \int_\Omega |u(x)|^2 dx.$$

Assume further that $u_0(x,y)$ is smooth and that

$$\lim_{\epsilon \to 0} \epsilon \int_{\Gamma_\epsilon} |u_\epsilon|^2 d\sigma_\epsilon = \int_\Omega \int_\Gamma |u_0(x,y)|^2 dx d\sigma(y),$$

then

$$\lim_{\epsilon \to 0} \epsilon \int_{\Gamma_\epsilon} |u_\epsilon(x) - u_0(x, \frac{x}{\epsilon})|^2 d\sigma_\epsilon(x) = 0.$$

In the case where u_ϵ is the trace on Γ_ϵ of some function in $H^1(\Omega)$, a link can be established between its usual and surface two-scale limits.

Proposition 2.6 *Let u_ϵ be a sequence of functions in $H^1(\Omega)$ such that*

$$\|u_\epsilon\|_{L^2(\Omega)} + \epsilon\|\nabla u_\epsilon\|_{L^2(\Omega)} \leq C,$$

where C is a positive constant independent of ϵ. Then, the trace of u_ϵ on Γ_ϵ satisfies the estimate

$$\epsilon \int_{\Gamma_\epsilon} |u_\epsilon(x)|^2 d\sigma_\epsilon(x) \leq C,$$

and, up to a subsequence, it two-scale converges in the sense of Theorem 2.1 to a limit $u_0(x,y)$ which is the trace on Γ of the usual two-scale limit, a function in $L^2(\Omega; H^1_\#(Y))$. More precisely,

$$\lim_{\epsilon\to 0} \epsilon \int_{\Gamma_\epsilon} u_\epsilon(x)\phi(x, \frac{x}{\epsilon}) d\sigma_\epsilon = \int_\Omega \int_\Gamma u_0(x,y)\phi(x,y) dx d\sigma(y),$$

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(x)\phi(x, \frac{x}{\epsilon}) dx = \int_\Omega \int_Y u_0(x,y)\phi(x,y) dx dy,$$

$$\lim_{\epsilon\to 0} \epsilon \int_\Omega \nabla u_\epsilon(x)\phi(x, \frac{x}{\epsilon}) dx = \int_\Omega \int_Y \nabla_y u_0(x,y)\phi(x,y) dx dy,$$

for any continuous function $\phi(x,y) \in C[\bar{\Omega}; C_\#(Y)]$.

Proof. By rescaling and summation over the ϵ-cells of Ω, the trace inequality in the unit cell yields

$$\epsilon \int_{\Gamma_\epsilon} |u_\epsilon(x)|^2 d\sigma_\epsilon(x) \leq C\|u_\epsilon\|^2_{L^2(\Omega)} + \epsilon^2 \|\nabla u_\epsilon\|^2_{L^2(\Omega)}.$$

Thus, up to a subsequence, u_ϵ two-scale converges in the sense of Theorem 2.1 to a limit $v_0(x,y) \in L^2(\Omega; L^2(\Gamma))$. On the other hand, by virtue of Proposition 1.14 in [1], and up to another subsequence, u_ϵ two-scale converges in the sense of Theorem 1.1 to a limit $u_0(x,y) \in L^2(\Omega; H^1_\#(Y))$. To prove that v_0 is just the trace of u_0 on Γ, the sequence u_ϵ is first restricted to the perforated domain Ω_ϵ defined by (1). For any vector-valued smooth test function $\psi(x,y)$, integrating by parts gives

$$\epsilon \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \psi(x, \frac{x}{\epsilon}) dx = -\epsilon \int_{\Omega_\epsilon} u_\epsilon \operatorname{div}_x \psi(x, \frac{x}{\epsilon}) dx - \int_{\Omega_\epsilon} u_\epsilon \operatorname{div}_y \psi(x, \frac{x}{\epsilon}) dx$$
$$+\epsilon \int_{\Gamma_\epsilon} u_\epsilon \psi(x, \frac{x}{\epsilon}) \cdot \vec{n} d\sigma_\epsilon(x). \tag{7}$$

Passing to the two-scale limit in each term, (7) becomes

$$\int_{\Omega}\int_{Y^*}\nabla_y u_0\cdot\psi dxdy=-\int_{\Omega}\int_{Y^*}u_0\,\mathrm{div}_y\psi dxdy+\int_{\Omega}\int_{\Gamma}v_0\psi\cdot\vec{n}dxd\sigma(y). \quad (8)$$

Integrating by parts in (8) gives

$$\int_{\Omega}\int_{\Gamma}(v_0-u_0)\psi\cdot\vec{n}dxd\sigma(y)=0.$$

It is not difficult to check that smooth functions are dense in $L^2(\Omega;L^2_{\#}(Y,\mathrm{div}))$ and that any function of $L^2(\Omega;L^2(\Gamma))$ is attained as the normal trace of some function of $L^2(\Omega;L^2(Y,\mathrm{div}))$. This implies that v_0 coincides with the trace of u_0 on Γ.

We establish below a last corollary of surface two-scale convergence concerning a sequence u_ϵ which belongs to $H^1(\Gamma_\epsilon)$. To define the Sobolev spaces $H^1(\Gamma_\epsilon)$, we first define the tangential derivative operator ∇^t_ϵ on Γ_ϵ in the usual way (see e.g. Chapter 16 in [6]) : for a smooth function $u\in C^1(\bar{\Omega})$ $\nabla^t_\epsilon u(x)$ is the projection of $\nabla u(x)$ on the tangent hyperplane to Γ_ϵ at the point x. Then, $H^1(\Gamma_\epsilon)$ is defined by

$$H^1(\Gamma_\epsilon)=\left\{u\in L^2(\Gamma_\epsilon)|\nabla^t_\epsilon u\in L^2(\Gamma_\epsilon)^N\right\}.$$

A similar definition holds for $H^1(\Gamma)$, based on the tangential derivative operator ∇^t on Γ. We further denote by $H^1_{\#}(\Gamma)$ the subspace of Y-periodic functions in $H^1(\Gamma)$.

Proposition 2.7 *Let u_ϵ be a sequence of functions in $H^1(\Gamma_\epsilon)$ such that*

$$\epsilon\int_{\Gamma_\epsilon}|u_\epsilon(x)|^2d\sigma_\epsilon(x)+\epsilon^3\int_{\Gamma_\epsilon}|\nabla^t_\epsilon u_\epsilon(x)|^2d\sigma_\epsilon(x)\le C, \quad (9)$$

where C is a positive constant independent of ϵ. Then, there exists a subsequence and a function $u_0(x,y)\in L^2(\Omega;H^1_{\#}(\Gamma))$ such that the subsequences u_ϵ and $\epsilon\nabla^t_\epsilon u_\epsilon$ two-scale converge, in the sense of Theorem 2.1, to $u_0(x,y)$ and $\nabla^t_y u_0(x,y)$ respectively.

The proof of Proposition 2.7 requires the following elementary lemma on the tangential divergence.

Lemma 2.8 *Let div^t denote the tangential divergence operator on Γ defined as the adjoint operator of ∇^t through the following Green's formula*

$$\int_{\Gamma}\nabla^t u\cdot v d\sigma=-\int_{\Gamma}u\,\mathrm{div}^t v d\sigma,$$

for any $u \in H^1_\#(\Gamma)$ and $v \in L^2_\#(\Gamma)^N$ with $\mathrm{div}^t v \in L^2_\#(\Gamma)$. Assume that Γ is a C^2 smooth compact boundary in the torus Y. Then, the exterior normal vector $\vec{n}$ of Γ can be extended to a neighbourhood of Γ as a C^1 field, and for smooth functions $\psi(y) \in C^1_\#(Y)^N$ the tangential divergence operator is defined by

$$\mathrm{div}^t \psi(y) = \mathrm{div}\left(\psi(y) - (\psi(y)\cdot\vec{n})\vec{n}\right) \quad \textit{for any } y \in \Gamma.$$

Proof of Proposition 2.7. Thanks to the a priori estimate (9), by application of Theorem 2.1, u_ϵ and $\epsilon\nabla^t_\epsilon u_\epsilon$ two-scale converge, up to a subsequence, to some limits $u_0(x,y) \in L^2(\Omega; L^2(\Gamma))$ and $\xi_0(x,y) \in L^2(\Omega; L^2(\Gamma))^N$. Let $\psi(x,y) \in C[\bar{\Omega}; C_\#(Y)]^N$ have a compact support in Ω. By integration by part,

$$\epsilon^2 \int_{\Gamma_\epsilon} \nabla^t_\epsilon u_\epsilon(x)\cdot\psi(x,\frac{x}{\epsilon})d\sigma_\epsilon(x) = -\epsilon^2 \int_{\Gamma_\epsilon} u_\epsilon(x)\,\mathrm{div}^t_\epsilon\left(\psi(x,\frac{x}{\epsilon})\right)d\sigma_\epsilon(x). \tag{10}$$

By Lemma 2.8 the tangential divergence in the right hand side of (10) can be computed as

$$\epsilon\,\mathrm{div}^t_\epsilon\left(\psi(x,\frac{x}{\epsilon})\right) = \left(\mathrm{div}^t\psi\right)(x,\frac{x}{\epsilon}) + \mathcal{O}(\epsilon),$$

where the operator div^t acts only on the y variable of $\psi(x,y)$. Therefore, passing to the two-scale limit in (10) yields

$$\int_\Omega\int_\Gamma \xi_0\cdot\psi dx d\sigma(y) = -\int_\Omega\int_\Gamma u_0\,\mathrm{div}^t\psi dx d\sigma(y). \tag{11}$$

A last integration by parts in (11) implies that ξ_0 coincides with $\mathrm{div}^t u_0$.

Remark 2.9 *In the present context, many other results can also be obtained by generalizing the previous properties of the usual two-scale convergence. We simply mention the possibility of studying non-linear monotone homogenization problems, or multiple-scale problems [2].*

3 A model of diffusion with Fourier boundary conditions.

In this Section the results of Section 2 are applied to the homogenization of a model problem derived from a more complex and pertinent problem, studied in [5], and modeling the condensation of steam in a periodic cooling structure.

Let $f(x)$ belong to $L^2(\Omega)$ and $\alpha(y) \geq 0$ to $L^\infty_\#(Y)$. Our model problem is a diffusion equation in the porous medium Ω_ϵ with a Fourier boundary condition on Γ_ϵ

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} + \epsilon\alpha(\frac{x}{\epsilon})u_\epsilon = 0 & \text{on } \Gamma_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

which admits a unique solution $u_\epsilon \in H^1(\Omega_\epsilon)$ satisfying the a priori estimate

$$\|u_\epsilon\|_{L^2(\Omega_\epsilon)} + \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq \|f\|_{L^2(\Omega)}.$$

The homogenized system for system (12) is

$$\begin{cases} -\operatorname{div}(A\nabla u) + (1+a)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where a is a non-negative constant given by

$$a = \frac{1}{|Y^*|}\int_\Gamma \alpha(y)d\sigma(y),$$

and A is a symetric positive definite matrix defined by

$$A_{ij} = \frac{1}{|Y^*|}\int_{Y^*}(\nabla_y w_i + \vec{e}_i)\cdot(\nabla_y w_j + \vec{e}_j)dy.$$

and $(w_i)_{1\leq i\leq N}$ is the family of solutions of the cell problem

$$\begin{cases} -\operatorname{div}_y(\nabla_y w_i + \vec{e}_i) = 0 & \text{in } Y^* \\ \quad (\nabla_y w_i + \vec{e}_i)\cdot\vec{n} = 0 & \text{on } \partial T \\ y \to w_i(y)\ Y\text{-periodic.} \end{cases} \tag{14}$$

Proposition 3.1 *The sequence u_ϵ of solutions of (12), extended by zero in $\Omega\backslash\Omega_\epsilon$) two-scale converges to $\chi(y)u(x)$, where $\chi(y)$ is the characteristic function of Y^*, and u the unique solution in $H_0^1(\Omega)$ of the homogenized problem.*

Proof. The application of two-scale convergence to the homogenization of problem (12) with a Neumann (instead of Fourier) boundary condition has already been done in [1] (see Theorem 2.9). Therefore, we only give the new arguments required to treat the Fourier boundary condition. In view of the a priori estimate, there exist $u(x) \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$ such that, up to a subsequence, the extensions by zero of u_ϵ and ∇u_ϵ two-scale converge to $\chi(y)u(x)$ and $\chi(y)(\nabla_x u(x) + \nabla_y u_1(x,y))$. In the variational formulation of (12), we choose a test function $\phi_\epsilon(x) = \phi(x) + \epsilon\phi_1(x, \frac{x}{\epsilon})$

$$\int_{\Omega_\epsilon}\nabla u_\epsilon\cdot\nabla\phi_\epsilon dx + \int_{\Omega_\epsilon}u_\epsilon\phi_\epsilon dx + \epsilon\int_{\Gamma_\epsilon}\alpha(\frac{x}{\epsilon})u_\epsilon\phi_\epsilon d\sigma_\epsilon = \int_{\Omega_\epsilon}f\phi_\epsilon dx. \tag{15}$$

The usual two-scale convergence allows us to pass to the limit in all terms of (15) but the third one. For this latter term, we use Proposition 2.6 which implies that the trace of u_ϵ on Γ_ϵ two-scale converges to $u(x)$ in the sense of Theorem 2.1. Finally, passing to the limit in (15) yields

$$\begin{aligned} &\textstyle\int_\Omega\int_{Y^*}(\nabla u + \nabla_y u_1)\cdot(\nabla\phi + \nabla_y\phi_1)dxdy \\ &\quad + \textstyle\int_\Omega\int_{Y^*}u\phi dxdy + \int_\Omega\int_\Gamma\alpha(y)u\phi dxd\sigma(y) = \int_\Omega\int_{Y^*}f\phi dxdy. \end{aligned} \tag{16}$$

It is not difficult to check that (16) is a variational formulation which admits a unique solution $(u, u_1) \in H_0^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*)/\mathbb{R})$. Thus the entire sequence u_ϵ converges. Eventually, the homogenized system (13) is easily recovered from (16) by remarking that

$$u_1(x,y) = \sum_{i=1}^{N} w_i(y) \frac{\partial u}{\partial x_i}(x),$$

where w_i are the solutions of the cell problem (14).

4 A model of diffusion and adsorption in porous media.

We now apply the results of Section 2 to a simplified model derived from a more complete and physical one studied in [7], [8] concerning the diffusion, adsorption, and reaction of chemicals in porous media. Roughly speaking, it is a system of two competing diffusion equations, one inside the pores, and one on their boundaries. For simplicity we assume hereafter that Γ is compactly embeded in Y, considered as an open set in $\mathbb{R}^N$, in order that Γ_ϵ does not meet the boundary $\partial\Omega$. We shall denote by Δ^t_ϵ and Δ^t the Laplace-Beltrami operators on Γ_ϵ and Γ satisfying the usual rule of integration by parts

$$-\int_\Gamma \Delta^t u(y) v(y) d\sigma(y) = \int_\Gamma \nabla^t u(y) \cdot \nabla^t v(y) d\sigma(y)$$

for functions $u, v \in H^1_\#(\Gamma)$ (a similar formula holds for Δ^t_ϵ).

Let $f(x)$ belong to $L^2(\Omega)$ and $\alpha(y) \geq 0$ to $L^\infty_\#(Y)$. The model problem reads as

$$\left\{ \begin{array}{ll} -\Delta u_\epsilon + u_\epsilon = f & \text{in } \Omega_\epsilon \\ -\epsilon^2 \Delta^t_\epsilon v_\epsilon + v_\epsilon = \alpha(\frac{x}{\epsilon})(u_\epsilon - v_\epsilon) & \text{on } \Gamma_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} + \epsilon\alpha(\frac{x}{\epsilon})(u_\epsilon - v_\epsilon) = 0 & \text{on } \Gamma_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{17}$$

which admits a unique solution $(u_\epsilon, v_\epsilon) \in H^1(\Omega_\epsilon)$ satisfying the a priori estimate

$$\left\{ \begin{array}{l} \|u_\epsilon\|_{L^2(\Omega_\epsilon)} + \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C \\ \epsilon \|v_\epsilon\|^2_{L^2(\Gamma_\epsilon)} + \epsilon^3 \|\nabla^t_\epsilon v_\epsilon\|^2_{L^2(\Gamma_\epsilon)} \leq C. \end{array} \right.$$

Proposition 4.1 *The sequences u_ϵ (extended by zero in $\Omega \setminus \Omega_\epsilon$), and v_ϵ two-scale converge to $\chi(y)u(x)$, and $v(x,y)$ respectively, where (u,v) is the unique solution in $H_0^1(\Omega) \times L^2(\Omega; H^1_\#(\Gamma))$ of the homogenized system*

$$\left\{ \begin{array}{ll} -\operatorname{div}(A\nabla u(x)) + (1+a)u(x) = f(x) + \frac{1}{|Y^*|}\int_\Gamma \alpha(y) v(x,y) d\sigma(y) & \textit{in } \Omega \\ -\Delta^t_y v(x,y) + (1+\alpha(y)) v(x,y) = \alpha(y) u(x) & \textit{in } \Omega \times \Gamma \\ u = 0 & \textit{on } \partial\Omega, \end{array} \right. \tag{18}$$

where the matrix A and the non-negative constant are defined as in Section 3.

Remark 4.2 *The homogenized system (18) can be further simplified since $v(x,y)$ is the product of $u(x)$ by a function depending only on y. Corrector results (i.e. strong convergences) can easily be obtained by using Proposition 2.5 in this paper, and Theorem 1.8 in [1].*

Proof. As in Section 3, the a priori estimate implies the existence of $u(x) \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y^*)/I\!R)$ such that, up to a subsequence, the extensions by zero of u_ϵ and ∇u_ϵ two-scale converge to $\chi(y)u(x)$ and $\chi(y)(\nabla_x u(x) + \nabla_y u_1(x,y))$. Furthermore by Proposition 2.6, there exists $v(x,y) \in L^2(\Omega; H^1_\#(\Gamma))$ such that, up to another subsequence, v_ϵ and $\epsilon \nabla^t_\epsilon v_\epsilon$ two-scale converge, in the sense of Theorem 2.1, to $v(x,y)$ and $\nabla^t_y v(x,y)$. In the variational formulation of (17), we choose a test function $(\phi_\epsilon(x), \theta_\epsilon(x)) = \left(\phi(x) + \epsilon\phi_1(x, \frac{x}{\epsilon}), \theta(x, \frac{x}{\epsilon})\right)$

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla \phi_\epsilon dx + \int_{\Omega_\epsilon} u_\epsilon \phi_\epsilon dx + \epsilon \int_{\Gamma_\epsilon} \alpha(\frac{x}{\epsilon})(u_\epsilon - v_\epsilon)\phi_\epsilon d\sigma_\epsilon = \int_{\Omega_\epsilon} f\phi_\epsilon dx,$$

$$\epsilon^3 \int_{\Gamma_\epsilon} \nabla^t_\epsilon v_\epsilon \cdot \nabla^t_\epsilon \theta_\epsilon d\sigma_\epsilon + \epsilon \int_{\Gamma_\epsilon} v_\epsilon \theta_\epsilon d\sigma_\epsilon = \epsilon \int_{\Gamma_\epsilon} \alpha(\frac{x}{\epsilon})(u_\epsilon - v_\epsilon)\theta_\epsilon d\sigma_\epsilon.$$

We can pass to the two-scale limit in all terms which yields

$$\begin{aligned} \int_\Omega \int_{Y^*} (\nabla u + \nabla_y u_1) \cdot (\nabla \phi + \nabla_y \phi_1) dxdy + \int_\Omega \int_{Y^*} u\phi dxdy \\ + \int_\Omega \int_\Gamma \alpha(y)(u - v)\phi dx d\sigma(y) = \int_\Omega \int_{Y^*} f\phi dxdy, \qquad (19) \\ \int_\Omega \int_\Gamma \nabla^t_y v \cdot \nabla^t_y \theta dx d\sigma(y) + \int_\Omega \int_\Gamma v\theta dx d\sigma(y) = \int_\Omega \int_\Gamma \alpha(y)(u - v)\theta dx d\sigma(y) \end{aligned}$$

It is not difficult to check that (19) is a variational formulation which admits a unique solution $(u, u_1, v) \in H_0^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*)/I\!R) \times L^2(\Omega; H^1_\#(\Gamma))$. Thus the entire sequence (u_ϵ, v_ϵ) converges. Eventually, the homogenized system (18) is easily recovered by arguing as in Section 3.

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