TWO-SCALE CONVERGENCE ON PERIODIC SURFACES
AND APPLICATIONS

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Abstract

This paper is concerned with the homogenization of model problems in periodic porous media when important phenomena occur on the boundaries of the pores. To this end, we generalize the notion of two-scale convergence for sequences of functions which are defined on periodic surfaces. We apply our results to two model problems: the first one is a diffusion equation in a porous medium with a Fourier boundary condition, the second one is a coupled system of diffusion equations inside and on the boundaries of the pores of a porous medium.

Key words: homogenization, two-scale convergence, periodic structures, porous medium.

1 Introduction

In porous media, there are (at least) two length scales: a microscopic scale (for example, the size of a single pore), and a macroscopic scale (the size of a typical sample of porous media). Quite often, the partial differential equations describing a physical phenomenon are posed at the microscopic level whereas only macroscopic quantities are of interest for the engineer or the physicist. Therefore, effective or homogenized equations have to be derived from the microscopic ones by an asymptotic process. To this end, it is convenient to assume that porous media have a periodic microstructure. Although it is far from being the case, it is perfectly legitimate as far as deriving homogenized models is concerned.
concerned. There is a vast body of literature on periodic homogenization (see e.g. [3], [4], [11]). In this context, the homogenization process is divided in two steps. In a first step, two-scale asymptotic expansions are used to formally obtain the homogenized problem. In a second step, another method (usually the so-called energy method of Tartar [9], [12]) is applied to prove convergence to the homogenized equation guessed from the first step. Recently, a new method, called two-scale convergence, has appeared which replaces these two steps by a single process (see [1], [10]). It relies on a new type of convergence as recalled in the next theorem.

**Theorem 1.1**

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^N \), and \( Y = [0, 1]^N \) the unit cube. Let \( u_\epsilon \) be a bounded sequence in \( L^2(\Omega) \). Then, there exists a subsequence (still denoted by \( u_\epsilon \)) and a function \( u_0(x;y) \) such that

\[
\lim_{\epsilon \to 0} \int_{\Omega} u_\epsilon(x) \phi(x, \frac{x}{\epsilon}) \, dx = \int_{\Omega} u_0(x;y) \phi(x, y) \, dx dy,
\]

for any continuous function \( \phi(x, y) \in C^\infty(C \# Y) \).

**2 Presentation of the main results**

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^N \), and \( Y = [0, 1]^N \) the unit cube. Let \( u_\epsilon \) be a bounded sequence in \( L^2(\Omega) \). Then, there exists a subsequence (still denoted by \( u_\epsilon \)) and a function \( u_0(x;y) \) such that

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for any continuous function \( \phi(x, y) \in C^\infty(C \# Y) \).
which is nothing else than the part lying inside . It is easily seen that
\[ \lim_{\epsilon \to 0} \epsilon \left[ \frac{1}{N} \right]^N = \frac{1}{|Y|^N}. \]

Theorem 2.1 Let \( u_\epsilon \) be a sequence in \( L^2 \), such that
\[ \epsilon \int_{\Gamma_\epsilon} |u_\epsilon(x)|^2 d\sigma_\epsilon, x \leq C, \]
where \( C \) is a positive constant, independent of \( \epsilon \). There exist a subsequence (still denoted by \( \epsilon \)) and a two-scale limit \( u_0(x,y) \in L^2 \) such that \( u_\epsilon \to u_0(x,y) \) in the sense that
\[ \epsilon \int_{\Gamma_\epsilon} u_\epsilon(x, \phi(x, \frac{x}{\epsilon})) d\sigma_\epsilon - \int_{\Omega} \int u_0(x,y, \phi(x, y)) dx d\sigma y, \]
for any continuous function \( \phi(x, y) \in C \cap \text{C}_y \).

Remark 2.2 Note that the surface two-scale limit \( u_0(x,y) \) is defined in the whole domain for the macroscopic variable \( x \), and on the surface for the microscopic variable \( y \).

Remark 2.3 In Theorem 2.1 the set \( \mathcal{A} \) is a periodic \( N \)-dimensional surface. Of course, it could be generalized to lower dimensional periodic manyfolds, like curves in 3-D. The same methodology could then be applied to homogenization problems such as fluid flow through small pipes or electric currents through wires.

Lemma 2.4 Let \( B = C \cap \text{C}_y \) be the space of continuous functions \( \phi(x, y) \) on \( \times Y \) which are \( Y \)-periodic in \( y \). Then, \( B \) is a separable Banach space (i.e. it contains a dense countable family), which is dense in \( L^2 \), and such that any function \( \phi(x, y) \in B \) satisfies
\[ \epsilon \int_{\Gamma_\epsilon} |\phi(x, \frac{x}{\epsilon})|^2 d\sigma_\epsilon, x \leq C\|\phi\|^2_B, \]
and
\[ \epsilon^{-\alpha} \epsilon \int_{\Gamma_\epsilon} |\phi(x, \frac{x}{\epsilon})|^2 d\sigma \leq \int_{\Omega} \int_{\Gamma} |\phi(x, y)|^2 dx dy. \]

**Proof of Theorem 2.1.**
\[
|\epsilon \int_{\Gamma_\epsilon} u_\epsilon(x, \phi(x, \frac{x}{\epsilon})\int_{\Gamma} d\sigma)\leq C \left| \epsilon \int_{\Gamma_\epsilon} \phi(x, y) \int_{\Gamma} \frac{x}{\epsilon} d\sigma \right| \leq C \| \phi \|_B.
\]

By Schwarz inequality, we have
\[
\int_{\Gamma_\epsilon} u_\epsilon(x, \phi(x, \frac{x}{\epsilon})\int_{\Gamma} d\sigma) \leq C \int_{\Omega} \int_{\Gamma} |\phi(x, y)|^2 dx dy.
\]

This implies that the left hand side of (5) is a continuous linear form on \(B\) which can be identified to a duality product \(h_0, \cdot\) \(B_0; B\) for some bounded sequence of measures. Since \(B\) is separable, one can extract a subsequence and there exists a limit \(0\) such converges to \(0\) in the weak * topology of \(B_0\) (the dual of \(B\)). On the other hand, Lemma 2.4 allows us to pass to the limit in the middle term of (5). Combining these two results yields
\[
\int_{\Gamma_\epsilon} u_\epsilon(x, \phi(x, \frac{x}{\epsilon})\int_{\Gamma} d\sigma) \leq C \int_{\Omega} \int_{\Gamma} |\phi(x, y)|^2 dx dy.
\]

Equation (6) shows that \(0\) is actually a continuous form on \(L^2(\mathbb{R}^2)\), by density of \(B\) in this space. Thus, there exists \(u_0(x, y) \in L^2(\mathbb{R}^2)\) such that
\[
h_0, \cdot (u_0(x, y) = \int_{\Omega} \int_{\Gamma} u_0(x, y) \phi(x, y) dx dy,
\]
which concludes the proof of Theorem 2.1.

The following result is an easy generalization of the corrector result of the usual two-scale convergence (Theorem 1.8 in [1]).

**Proposition 2.5** Let \(u_\epsilon\) be a sequence of functions in \(L^2(\mathbb{R}^2)\) which two-scale converges to a limit \(u_0(x, y) \in L^2(\mathbb{R}^2)\). Then, the measure \(u_\epsilon d\sigma\) converges, in the sense of distributions in \(L^2(\mathbb{R}^2)\), to the function \(u(x) = \int_{\Gamma} u_0(x, y) d\sigma\), and we have
\[
\epsilon^{-\alpha} \epsilon \int_{\Gamma_\epsilon} |u_\epsilon|^2 d\sigma \geq \int_{\Omega} \int_{\Gamma} |u_0(x, y)|^2 dx dy \geq \int_{\Omega} |u(x)|^2 dx.
\]

Assume further that \(u_0(x, y)\) is smooth and that
\[
\epsilon^{-\alpha} \epsilon \int_{\Gamma_\epsilon} |u_\epsilon|^2 d\sigma \geq \int_{\Omega} \int_{\Gamma} |u_0(x, y)|^2 dx dy \geq \int_{\Omega} |u(x)|^2 dx,
\]
then
\[
\epsilon^{-\alpha} \epsilon \int_{\Gamma_\epsilon} |u_\epsilon(x) - u_0(x|\frac{x}{\epsilon})|^2 d\sigma \xrightarrow{\epsilon \to 0} 0.
\]
In the case where $u$ is the trace on $\Gamma$ of some function in $H^1(\mathbb{N})$, a link can be established between its usual and surface two-scale limits.

**Proposition 2.6** Let $u_\varepsilon$ be a sequence of functions in $H^1(\mathbb{N})$ such that

$$\|u_\varepsilon\|_{L^2(\mathbb{N})} + \varepsilon \|
abla u_\varepsilon\|_{L^2(\mathbb{N})} \leq C,$$

where $C$ is a positive constant independent of $\varepsilon$. Then, the trace of $u_\varepsilon$ on $\mathbb{N}$ satisfies the estimate

$$\varepsilon \int_{\Gamma} |u_\varepsilon(\mathbf{x})|^2 d\sigma_\varepsilon(\mathbf{x}) \leq C,$$

and, up to a subsequence, it two-scale converges in the sense of Theorem 2.1 to a limit $u_0(\mathbf{x}, y)$, which is the trace on $\mathbb{N}$ of the usual two-scale limit, a function in $L^2(\mathbb{N})$. More precisely,

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma} u_\varepsilon x \phi x, \frac{x}{\varepsilon} d\sigma_\varepsilon = \int_{\mathbb{N}} u_0 x, y \phi x, y dx d\sigma y,$$

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\mathbb{N}} u_\varepsilon x \phi x, \frac{x}{\varepsilon} dx = \int_{\mathbb{N}} u_0 x, y \phi x, y dy,$$

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\mathbb{N}} \nabla u_\varepsilon x \phi x, \frac{x}{\varepsilon} dx = \int_{\mathbb{N}} \nabla u_0 x, y \phi x, y dy,$$

for any continuous function $\phi x, y \in C(\mathbb{N}; \mathbb{C})$.

**Proof.**

$$\varepsilon \int_{\Gamma} |u_\varepsilon\|_{L^2(\mathbb{N})}^2 + \varepsilon^2 \|\nabla u_\varepsilon\|_{L^2(\mathbb{N})}^2.$$

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma} \nabla u_\varepsilon x \psi x, \frac{x}{\varepsilon} dx = -\varepsilon \int_{\Omega_0} u_\varepsilon x \psi x, \frac{x}{\varepsilon} dx - \varepsilon \int_{\Omega_0} u_\varepsilon y \psi x, \frac{x}{\varepsilon} dx$$

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma} \nabla u_\varepsilon \psi x, \frac{x}{\varepsilon} d\sigma_\varepsilon = \varepsilon \int_{\Gamma} \nabla u_\varepsilon \psi x, \frac{x}{\varepsilon} d\sigma_\varepsilon.$$
Passing to the two-scale limit in each term, (7) becomes
\[
\int_{\Omega} \int_N \nabla_y u_0 \cdot \psi dxdy - \int_{\Omega} \int_Y u_0 \cdot y \psi dxdy \int_{\Omega} \int \nabla \psi \cdot \bar{u} dxdy = 0.
\]

Integrating by parts in (8) gives
\[
\int_{\Omega} \int_{\Gamma} u_0 \cdot \nabla_y \psi \cdot dxdy = 0.
\]

It is not difficult to check that smooth functions are dense in \( L^2(\Omega; L^2(\mathbb{R}^2)) \) and that any function of \( L^2(\Omega; L^2(\mathbb{R}^2)) \) is attained as the normal trace of some function of \( L^2(\Omega; L^2(\mathbb{R}^2)) \). This implies that \( u_0 \) coincides with the trace of \( u_0 \) on \( \Omega \).

We establish below a last corollary of surface two-scale convergence concerning a sequence \( u \) which belongs to \( H^1(\Omega) \). To define the Sobolev spaces \( H^1(\Omega) \), we first define the tangential derivative operator \( \nabla^t \) on \( \Omega \) in the usual way (see e.g. Chapter 16 in [6]): for a smooth function \( u \in C^1(\Omega) \), \( \nabla^t u(x) \) is the projection of \( \nabla u(x) \) on the tangent hyperplane to \( \Omega \) at the point \( x \).

Then, \( H^1(\Omega) \) is defined by
\[
H^1(\Omega) = \left\{ u \in L^2(\Omega) : \nabla^t u \in L^2(\Omega) \right\}.
\]

A similar definition holds for \( H^1(\mathbb{R}) \), based on the tangential derivative operator \( \nabla^t \) on \( \mathbb{R} \). We further denote by \( H^1(\mathbb{R}) \) the subspace of \( L^2(\mathbb{R}) \) periodic functions in \( H^1(\mathbb{R}) \).

**Proposition 2.7** Let \( u_\epsilon \) be a sequence of functions in \( H^1(\Omega) \) such that
\[
\epsilon \int_{\Gamma} \left| u_\epsilon x \right|^2 d\sigma_x \leq C,
\]
where \( C \) is a positive constant independent of \( \epsilon \). Then, there exists a subsequence and a function \( u_0, y \in L^2(\mathbb{R}) \) such that the subsequences \( u_\epsilon \) and \( \nabla^t u_\epsilon \) two-scale converge, in the sense of Theorem 2.1, to \( u_0, y \) and \( \nabla^t u_\epsilon \), respectively.

**Lemma 2.8** Let \( \nabla^t \) denote the tangential divergence operator on \( \mathbb{R} \) defined as the adjoint operator of \( \nabla^t \) through the following Green's formula
\[
\int_{\Gamma} \nabla^t u \cdot v d\sigma - \int u : \nabla v d\sigma,
\]
for any \( u \in H^1 \) and \( v \in L^2 \). Assume that \( \phi \) is a \( C^2 \) smooth compact boundary in the torus \( Y \). Then, the exterior normal vector \( \tilde{n} \) of \( \phi \) can be extended to a neighbourhood of \( \phi \) as a \( C^1 \) field, and for smooth functions \( \psi \in C^1(\overline{Y}^N) \) the tangential divergence operator is defined by
\[
\text{div}_t \psi = \text{div} \left( \psi \cdot \tilde{n} \right)
\]
for any \( \psi \in C^1(\overline{Y}^N) \).

**Proof of Proposition 2.7.** Thanks to the a priori estimate (9), by application of Theorem 2.1, \( u_\varepsilon \) and \( v_\varepsilon \) two-scale converge, up to a subsequence, to some limits \( u_0(\xi, y) \) and \( v_0(\xi, y) \). Let \( \xi, y \in C_\# Y^N \) have a compact support in \( \Omega \). By integration by part,
\[
\epsilon^2 \int_{\Gamma_\varepsilon} \nabla u_\varepsilon \cdot \psi_x \frac{x}{\varepsilon} \, d\sigma_x \nabla_x \cdot \frac{x}{\varepsilon} = \epsilon^2 \int_{\Gamma_\varepsilon} u_\varepsilon \cdot \psi_x \frac{x}{\varepsilon} \, d\sigma_x \nabla_x \cdot \frac{x}{\varepsilon}.
\]

Remark 2.9 In the present context, many other results can also be obtained by generalizing the previous properties of the usual two-scale convergence. We simply mention the possibility of studying non-linear monotone homogenization problems, or multiple-scale problems [2].

### 3 A model of diffusion with Fourier boundary conditions.

\[
\left\{ \begin{array}{ll}
- \frac{\partial u_{\varepsilon}}{\partial x} & = f \quad \text{in } Y^N \\
\epsilon \frac{\partial u_{\varepsilon}}{\partial x} & = \frac{x}{\varepsilon} \quad \text{in } \partial Y^N.
\end{array} \right.
\]
\[ u \in H^1 \quad \text{and} \quad u \in L^2(\Omega) \]

\[ \| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)} \leq \| f \|_{L^2(\Omega)}. \]

The homogenized system for system (12) is

\[ \nabla A \nabla u + (1 + a) u = f \quad \text{in} \quad \Omega, \]

\[ u = 0 \quad \text{on} \quad \partial \Omega. \]

where \( a \) is a non-negative constant given by

\[ a = \frac{1}{j_y j_z(y) \, dz(y)} \]

and \( A \) is a symmetric positive definite matrix defined by

\[ A_{ij} = \frac{1}{j_y j_z(y)} \int_{\Omega} \nabla w_i \cdot \nabla w_j \, dz(y). \]

Proposition 3.1 The sequence \( u_n \) of solutions of (12), extended by zero in \( \Omega \), two-scale converges to \( \chi x u(x) \), where \( \chi \) is the characteristic function of \( \Omega \), and \( u \) the unique solution in \( H^1_0(\Omega) \) of the homogenized problem.

Proof. The usual two-scale convergence allows us to pass to the limit in all terms of (15) but the third one. For this latter term, we use Proposition 2.6 which implies that the trace of \( u_n \) on \( \partial \Omega \) two-scale converges to \( u(x) \) in the sense of Theorem 2.1. Finally, passing to the limit in (15) yields

\[ \int_{\Omega} \nabla u_n \cdot \nabla \phi_1 \, dx \int_{\Omega} u_n \phi_2 \, dx + \int_{\Omega} \alpha \nabla \phi_1 \cdot \nabla u_n \, dx \int_{\Omega} \phi_2 \, dx. \]
It is not difficult to check that (16) is a variational formulation which admits a unique solution \((u; u_1) \in H^1_0 \times L^2(Y)\). Thus the entire sequence \(u\) converges. Eventually, the homogenized system (13) is easily recovered from (16) by remarking that
\[
u_1(x,y) = \sum_{i=1}^N w_i(y) \frac{\partial u_i(x)}{\partial x_i};
\]
where \(w_i\) are the solutions of the cell problem (14).

4 A model of diffusion and adsorption in porous media.

Let \(f(x)\) belong to \(L^2(Y)\) and \((\rho(y))\) to \(L^1(Y)\). The model problem reads
\[
\begin{cases}
-\Delta u + u = f & \text{in } Y, \\
\frac{\partial u}{\partial n} + (1 + \alpha) (-\Delta v) + v = \rho(y) & \text{on } \partial Y;
\end{cases}
\]
which admits a unique solution \((u, v) \in H^1_0(Y) \times L^2(Y)\) satisfying the a priori estimate
\[
\|u\|_{L^2(Y)} + \|\nabla u\|_{L^2(Y)} \leq C.
\]

Proposition 4.1 The sequences \(u_e\) (extended by zero in \(\mathbb{R}^N\)), and \(v_e\) two-scale converge to \(\chi_y u\) and \(v\) respectively, where \(u, v\) is the unique solution in \(H^1_0 \times L^2 \times H^1_0\) of the homogenized system
\[
\begin{cases}
-\Delta u + u = f & \text{in } Y, \\
\frac{\partial u}{\partial n} + (1 + \alpha) \frac{\partial v}{\partial n} + v = \rho(y) & \text{on } \partial Y;
\end{cases}
\]
Remark 4.2 The homogenized system (18) can be further simplified since \( v, x, y \) is the product of \( u, x \) by a function depending only on \( y \). Corrector results (i.e. strong convergences) can easily be obtained by using Proposition 2.5 in this paper, and Theorem 1.8 in [1].

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