Introduction to Continuous optimization
Assessment
(6th January 2021)

Exercise I

We denote $\mathbb{R}^{n\times n}_{\text{sym}}$ the space of dimension $n(n+1)/2$ of symmetric $n\times n$ matrices. We consider the scalar product $X:Y=\sum_{i,j}X_{i,j}Y_{i,j}=\mathrm{Tr}\,(XY)$ (or $\mathrm{Tr}\,(X^TY)$ but it is the same here since X,Y are symmetric).

Let $\mathcal{S}_+\subset\mathbb{R}^{n\times n}_{\text{sym}}$ be the set of $n\times n$ symmetric, positive semidefinite matrices: $X=X^T$, $(X\xi)\cdot\xi\ge 0$ for any $\xi\in\mathbb{R}^n$. Let $\mathcal{S}_{++}$ be the interior of $\mathcal{S}_+$, that is, the set of positive definite matrices: $(X\xi)\cdot\xi>0$ for all $\xi\neq 0$.

We let, for $X\in\mathbb{R}^{n\times n}_{\text{sym}}$:

$$h(X):=\begin{cases}-\ln\det X & \text{if } X\in\mathcal{S}_{++},\\ +\infty & \text{else.}\end{cases}$$

1. Let $X\in\mathcal{S}_{++}$, H a symmetric matrix. Show that for $t\in\mathbb{R}$ with $|t|$ small enough, $X+tH\in\mathcal{S}_{++}$.

For ξ a vector, one has $((X+tH)\xi)\cdot\xi=(X\xi)\cdot\xi+t(H\xi)\cdot\xi\ge(\lambda_1(X)-t\|H\|)|\xi|^2>0$ if $t<\lambda_1(X)/\|H\|$, where $\lambda_1(X)>0$ is the smallest eigenvalue of X.

2. Using $X+tH=X(I+tX^{-1}H)$, show that

$$\nabla h(X)=-X^{-1}.$$

We recall that $\det(I+A)=1+\mathrm{Tr}\,A+o(\|A\|)$.

We have:

$$\begin{aligned}h(X+tH)&=-\ln\det(X(I+tX^{-1}H))\\&=-\ln(\det(X)\det(I+tX^{-1}H))=h(X)-\ln\det(I+tX^{-1}H)\end{aligned}$$
$$=h(X)-\ln(1+t\mathrm{Tr}\,(X^{-1}H)+o(t))=h(X)-t\mathrm{Tr}\,(X^{-1}H)+o(t)=h(X)-tX^{-1}:H+o(t)$$

which shows the claim.

3. One now wants to compute the conjugate $h^*(Y)=\sup_X X:Y-h(X)$.

Let $Y\in\mathbb{R}^{n\times n}_{\text{sym}}$ and assume e is an eigenvector of Y with eigenvalue $\lambda\in\mathbb{R}$ (and $|e|=1$).

Considering first X of the form $te\otimes e+\varepsilon I$ (where for $e\in\mathbb{R}^n\setminus\{0\}$ with $|e|=1$, $e\otimes e$ is the matrix e_ie_j which has eigenvector e with eigenvalue 1), $\varepsilon>0$, $t\to+\infty$, show that $h^*(Y)=+\infty$ if $\lambda\ge 0$.

Deduce that $\mathrm{dom}\,h^*\subset\{Y:-Y\in\mathcal{S}_{++}\}$.

One has, for $X=te\otimes e+\varepsilon I$, $\det X=\varepsilon^{n-1}(t+\varepsilon)$ and $X:Y-h(X)=\varepsilon\mathrm{Tr}\,X+t\lambda+(n-1)\ln\varepsilon+\ln(\varepsilon+t)$ which goes to $+\infty$ as $t\to\infty$ if $\lambda\ge 0$. Hence, $h^*(Y)<+\infty$ only if all eigenvalues of Y are strictly negative, that is $-Y\in\mathcal{S}_{++}$.

4. Now, assuming $-Y > 0$ we admit (even if it is quite easy to show) that $\sup_X X : Y - h(X)$ is reached at some positive matrix X.

Show that $X = -Y^{-1}$. Deduce the expression of h^*. Deduce also that h is convex.

At the maximum X one has $\nabla_X(X : Y - h(X)) = Y - (-X^{-1}) = 0$ so that $Y = (-X^{-1}) \Leftrightarrow X = -Y^{-1}$. Then,

$$X : Y - h(X) = \mathrm{Tr}\,(-Y^{-1}Y) + \ln\det(-Y^{-1}) = -n - \ln\det(-Y).$$

In particular, the function

$$Y \mapsto \begin{cases} -n - \ln\det(-Y) & \text{if } -Y \in \mathcal{S}_{++} \\ +\infty & \text{else.} \end{cases}$$

is convex, and so is $h(X) = n + h^*(-X)$.

5. We consider the problem $\min_{X \in \mathcal{S}_+} C : X$ and the Bregman distance

$$D_h(X, Y) = h(X) - h(Y) - \nabla h(Y) : (X - Y)$$

induced by h, defined for $X, Y \in \mathcal{S}_{++}$. Write the expression of an iteration of non-linear gradient descent for the problem, with step $\tau > 0$, relative to the Bregman distance D_h. Why can we always assume that C is symmetric? What assumption is needed on C in order for the problem to have a solution (and the algorithm to be well defined for all k)?

X^{k+1} is obtained as (if it exists)

$$X^{k+1} = \arg\min_X \frac{1}{\tau} D_h(X, X^k) + C : X$$

and satisfies: $-(X^{k+1})^{-1} = -(X^k)^{-1} - \tau C$, that is

$$X^{k+1} = ((X^k)^{-1} + \tau C)^{-1}$$

If C is not symmetric then $C : X = C^T : X^T = C^T : X = (C + C^T) : X/2$ so one can replace C with its symmetric part. If C has a negative eigenvalue, as in the analysis of the previous question, the minimum problem has no solution (the value is $-\infty$) as soon as $(X^0)^{-1} + \tau k C$ has a negative eigenvalue. If $C \geq 0$, the iterates are $X^k = ((X^0)^{-1} + \tau k C)^{-1}$

Exercice II - prox

Compute the proximity operator (for some parameter $\tau > 0$):

$$\mathrm{prox}_{\tau g}(x) = \arg\min_z g(z) + \frac{1}{2\tau}|z - x|^2$$

for the convex functions:

1. $g_1(x) = -\ln x$ for $x > 0$, $+\infty$ else;

The equation is $z - x - \tau/z = 0$, that is $z^2 - zx - \tau = 0$, that is $z = (x + \sqrt{x^2 + 4\tau})/2$.

2. $g_2(x) = \sum_{i=1}^n \frac{1}{3}|x_i|^3$, $(x \in \mathbb{R}^n)$;

The problem is:

$$\min_z \sum_{i=1}^n \tfrac{1}{3}|z_i|^3 + \frac{1}{2\tau}|z_i - x_i|^2$$

and can be minimized independently for each i: the minimizer satisfies

$$\tau \text{sign}\,(z_i) z_i^2 + z_i - x_i = 0, \quad i = 1, \dots, n.$$

Clearly the sign of z_i is the same as the sign of x_i (as $\tau \text{sign}\,(z_i) z_i^2 + z_i = z_i(\tau|z_i| + 1)$ has the same sign as z_i). Solving the equation one obtains:

$$z_i = \text{sign}\,(x_i) \frac{\sqrt{1 + 4\tau|x_i|} - 1}{2\tau}.$$

3. $g_3(x) = \sum_{i=1}^n \frac{2}{3}|x_i|^{3/2}$, $(x \in \mathbb{R}^n)$;

$g_3 = g_2^*$ and one has the Moreau identity:

$$\text{prox}_{\tau g_3}(x) = x - \tau \text{prox}_{\frac{1}{\tau} g_3^*}(\tfrac{x}{\tau}).$$

Hence,

$$z_i = x_i - \tau \text{sign}\,(x_i) \frac{\sqrt{1 + (4/\tau)|x_i/\tau|} - 1}{2/\tau} = \text{sign}\,(x_i) \left(|x_i| + \frac{\tau^2}{2} - \tau \frac{\sqrt{\tau^2 + 4|x_i|}}{2} \right)$$

$$= \text{sign}\,(x_i) \frac{4|x_i| + \tau^2 + \tau^2 - 2\tau\sqrt{\tau^2 + 4|x_i|}}{4} = \text{sign}\,(x_i) \left(\frac{\sqrt{\tau^2 + 4|x_i|} - \tau}{2} \right)^2$$

The last expression is the one which is obtained directly if one solves the minimization problem (without using Moreau's identity).

4. $g_4(x) = \frac{1}{2}\sum_i x_i^2$ if $x_i \geq 0$, $i = 1, \dots, n$, and $+\infty$ else, defined for $x \in \mathbb{R}^n$ (and with domain $\text{dom}\, g_4 = [0, +\infty)^n$).

One solves:

$$\min_{z_i \geq 0} \sum_i \tfrac{1}{2} z_i^2 + \frac{1}{2\tau}|z_i - x_i|^2$$

which is solved independently for each i. The solution is $z_i = 0$ if $x_i < 0$, otherwise, $(1 + \tau)z_i - x_i = 0$, that is, $z_i = x_i/(1 + \tau)$. Hence, $z_i = x_i^+/(1 + \tau)$.

Exercise III - rate for the proximal point algorithm

We consider M a maximal-monotone operator, defined in a Hilbert space $\mathcal{X}$. Given $x^0 \in \mathcal{X}$, we let for $k \geq 0$:

$$x^{k+1} = (I + M)^{-1} x^k$$

that is, the iterations of the proximal-point algorithm.

1. Let x^* be a zero, that is, a point such that $Mx^* \ni 0$ (we assume the set $M^{-1}(0)$ is not empty). Show that $x^* = (I+M)^{-1}(x^*)$ and that

$$|x^{k+1} - x^*|^2 + |x^k - x^{k+1}|^2 \leq |x^k - x^*|^2.$$

One has $x^* + 0 = x^*$, hence $x^* + Mx^* \ni x^*$, that is $x^* = (I+M)^{-1}(x^*)$. Denoting $J_M = (I+M)^{-1}$ we know that J_M is “firmly non expansive”:

$$|J_M x - J_M x'|^2 + |(I - J_M)x - (I - J_M)x'|^2 \leq |x - x'|^2.$$

With $x = x^k$ and $x' = x^*$ this gives the desired inequality.

2. Show that $|x^{k+1} - x^k|$ is a decreasing function of $k \geq 0$.

This is even easier: if $k \geq 1$, $|x^{k+1} - x^k| = |J_M x^k - J_M x^{k-1}| \leq |x^k - x^{k-1}|$ since J_M is one-Lipschitz.

3. Deduce that

$$|x^{k+1} - x^k| \leq \frac{|x^0 - x^*|}{\sqrt{k+1}}.$$

We sum the inequality of the first question:

$$|x^{k+1} - x^*|^2 + \sum_{l=0}^{k} |x^l - x^{l+1}|^2 \leq |x^0 - x^*|^2$$

then we use the second question to observe that $\sum_{l=0}^{k} |x^l - x^{l+1}|^2 \geq (k+1)|x^k - x^{k+1}|^2$.

4. Let x^{k_l} be a (weakly) converging subsequence, to some point $\bar{x}$. Show that for any $x' \in \mathcal{X}$ and $y' \in Mx'$,

$$\langle x' - \bar{x}, y' \rangle \geq 0.$$

Deduce that $0 \in M\bar{x}$.

Since M is monotone and $x^k - x^{k+1} \in Mx^{k+1}$, $\langle x' - x^{k_l}, y' - (x^{k_l} - x^{k_l+1}) \rangle \geq 0$ and in the limit, thanks to the previous estimate, we obtain the inequality (we have a product (weak convergence)$\times$(strong convergence)).

Since M is maximal-monotone, it means that $0 \in M\bar{x}$ (otherwise one could extend the graph). (Of course, using Opial's lemma, one can then show that $x^k \to \bar{x}$, weakly.)

5. Let $T : \mathcal{X} \to \mathcal{X}$ be a 1-Lipschitz operator and, for $\theta \in (0,1)$, let $T_\theta = (1-\theta)I + \theta T$. Let x^* be a fixed point of T (and therefore also of T_θ for any θ). We now consider the algorithm given by

$$x^{k+1} = T_\theta x^k.$$

Use the parallelogram identity to show that:

$$|x^{k+1} - x^*|^2 \leq |x^k - x^*|^2 - \theta(1-\theta)|Tx^k - x^k|^2$$

One has

$$|x^{k+1}-x^*|^2 = |(1-\theta)(x^k-x^*)+\theta(Tx^k-x^*)|^2$$
$$= (1-\theta)|x^k-x^*|^2+\theta|Tx^k-x^*|^2-\theta(1-\theta)|Tx^k-x^k|^2$$

and one uses $|Tx^k-x^*| = |Tx^k-T^*| \leq |x^k-x^*|$ to conclude.

6. As before, deduce that:

$$|Tx^k-x^k| \leq \frac{|x^0-x^*|}{\sqrt{\theta(1-\theta)}\sqrt{k+1}}.$$

(Remark: in this framework, one can show [Baillon-Bruck 1996] that a similar estimate holds in any metric space, but it is much harder).

As in question **2.**, one has $|x^{k+1}-x^k| \leq |x^k-x^{k-1}|$ for $k \geq 1$, using that T_θ is 1-Lipschitz. We deduce $|Tx^k-x^k| \leq |Tx^{k-1}-x^{k-1}|$. Thus,

$$\theta(1-\theta)(k+1)|Tx^k-x^k|\theta(1-\theta)\sum_{l=0}^{k}|Tx^l-x^l|+|x^{k+1}-x^*|^2 \leq |x^0-x^*|^2$$

7. Application: show that the over-relaxed proximal point algorithm:

$$x^{k+\frac{1}{2}} = (I+M)^{-1}x^k$$
$$x^{k+1} = x^k+\lambda(x^{k+\frac{1}{2}}-x^k)$$

for $1<\lambda<2$ is a converging method.

We know that $(I+M)^{-1} = I/2+R/2$ for a 1-Lipschitz map R. Then,

$$x^{k+1} = x^k+\tfrac{\lambda}{2}(Rx^k-x^k) = (1-\tfrac{\lambda}{2})x^k+\tfrac{\lambda}{2}Rx^k = R_{\frac{\lambda}{2}}x^k$$

is the iteration of an averaged operator, and we can use the previous results to show that $x^{k+1}-x^k \to 0$. Then, we can conclude as in question 4.

Exercise IV - Yosida approximation

Let A be a maximal monotone operator in a Hilbert space, and defined the Yosida approximation, for $\lambda > 0$, as

$$A_\lambda x = \frac{x-J_{\lambda A}x}{\lambda}$$

where $J_{\lambda A} = (I+\lambda A)^{-1}$ is the resolvent.

1. Show that A_λ is a monotone operator.

This is because $J_{\lambda A}$ is 1-Lipschitz. Then, for any x, y,

$$\langle A_\lambda x - A_\lambda y, x-y\rangle = \tfrac{1}{\lambda}(|x-y|^2-\langle J_{\lambda A}x-J_{\lambda A}y, x-y\rangle) \geq 0$$

using that $\langle J_{\lambda A}x-J_{\lambda A}y, x-y\rangle \leq |J_{\lambda A}x-J_{\lambda A}y||x-y| \leq |x-y|^2$.

2. Show that $A_\lambda x = J_{A^{-1}/\lambda}(x/\lambda)$. Deduce that it is $(1/\lambda)$-Lipschitz. Bonus: show that it is λ-co-coercive.

We use Moreau's identity:

$$x = (I+\lambda A)^{-1}x + \lambda(I + \tfrac{1}{\lambda}A^{-1})^{-1}(\tfrac{x}{\lambda}) = J_{\lambda A}x + \lambda J_{A^{-1}/\lambda}(\tfrac{x}{\lambda}).$$

We conclude using that $J_\bullet$ is 1-Lipschitz.

3. Let $x \in \operatorname{dom} A$ (that is, $Ax \neq \emptyset$). Show that

$$\lim_{\lambda\to 0} A_\lambda x = A_0 x := \arg\min_{p\in Ax} |p|.$$

Hint: first, show that if $p_\lambda = A_\lambda x$ then $p_\lambda \in A(x-\lambda p_\lambda)$. Using the monotonicity of A, deduce that for any $p \in Ax$, $|p_\lambda|^2 \leq \langle p_\lambda, p\rangle$, hence that $|p_\lambda| \leq |p|$. Conclude by using that A is maximal.

One has $p_\lambda = (x - J_{\lambda A}x)/\lambda$, hence $(I+\lambda A)(x-\lambda p_\lambda) \ni x$, that is, $p_\lambda \in A(x-\lambda p_\lambda)$. Since A is monotone, for any y and $q \in Ay$,

$$\langle q - p_\lambda, y - x + \lambda p_\lambda\rangle \geq 0. \qquad (*)$$

In particular for $y = x, q = p \in Ax$,

$$\langle p, p_\lambda\rangle \geq |p_\lambda|^2 \ \Rightarrow |p_\lambda| \leq |p|.$$

Hence in the limit, if $p_{\lambda_k} \to \bar p$, we find from $(*)$ that

$$\langle q - \bar p, y - x\rangle \geq 0$$

and $|\bar p| \leq |p|$ for any $p \in Ax$. Since A is maximal, we deduce that $\bar p \in Ax$, so that it is the (unique) element of minimal norm, and the whole sequence p_λ converges to $\bar p$.

4. Contraction semigroup: since A_λ is Lipschitz, by the Cauchy-Lipschitz theorem, one can solve for all $x \in \mathcal{X}$:

$$\begin{cases} \dot X^\lambda(t,x) = -A_\lambda X^\lambda(t,x) & t > 0, \\ X^\lambda(0,x) = x \end{cases}$$

and the solution, which is at least C^1 in time, satisfies:

$$X^\lambda(t,x) = x - \int_0^t A_\lambda X^\lambda(s,x)ds = X^\lambda(t',x) - \int_0^{t-t'} A_\lambda X^\lambda(s, X^\lambda(t',x))ds$$

for any $t' < t$. In particular, $X^\lambda(t,x) = X^\lambda(t-t', X^\lambda(t',x))$.

Show that for any $x, y \in \mathcal{X}$, $t \mapsto |X^\lambda(t,x) - X^\lambda(t,y)|^2$ is non-increasing. Deduce that $|X^\lambda(t,x) - X^\lambda(t,y)| \leq |x-y|$ for all $t \geq 0$.

One simply observes that because A_λ is monotone:

$$\frac{1}{2}\frac{\partial}{\partial t}|X^\lambda(t,x) - X^\lambda(t,y)|^2 = -\left\langle X^\lambda(t,x) - X^\lambda(t,y), A_\lambda X^\lambda(t,x) - A_\lambda X^\lambda(t,y)\right\rangle \leq 0.$$

Then, we deduce

$$|X^\lambda(t,x) - X^\lambda(t,y)| \leq |X^\lambda(0,x) - X^\lambda(0,y)| = |x-y|.$$

5. Show that $t \mapsto |A_\lambda X^\lambda(t,x)|$ is nonincreasing. If $x \in \operatorname{dom} A$, show that $|A_\lambda X(t,x)| \le |A_0 x|$ for all $\lambda > 0$ and $t \ge 0$.
Hint: use that

$$X^\lambda(t+h,x) - X^\lambda(t,x) = X^\lambda(t-t', X^\lambda(t'+h,x)) - X^\lambda(t-t', X^\lambda(t',x))$$

for any $h > 0$, t, $t' < t$, and the previous question.

The contraction semi-group property shows that

$$\begin{aligned}|X^\lambda(t+h,x) - X^\lambda(t,x)| = |X^\lambda(t-t', X^\lambda(t'+h,x)) - X^\lambda(t-t', X^\lambda(t',x))|\\ \le |X^\lambda(t'+h,x)) - X^\lambda(t',x)|\end{aligned}$$

and dividing by h and sending $h \to 0$ it follows

$$|A_\lambda X(t,x)| \le |A_\lambda X(t',x)|.$$

Then from the inequality $|p_\lambda| \le |p|$ of question **2.** we deduce $|A_\lambda X(t,x)| \le |A_0 x|$.

6. Using question **2.**, show that for $\lambda, \mu > 0$ and for any $x \in \mathcal{X}$, $A_\lambda x = A_\mu(x + (\mu - \lambda)A_\lambda x)$. Deduce that:

$$\frac{\partial}{\partial t}|X^\lambda(t,x) - X^\mu(t,x)|^2 \le (\mu - \lambda)(|A_\lambda X^\lambda(t,x)|^2 - |A_\mu X^\mu(t,x)|^2)$$

(or any similar estimate) and in particular that if $x \in \operatorname{dom} A$, $|X^\lambda(t,x) - X^\mu(t,x)| \le C|A_0 x|\sqrt{|\mu - \lambda| t}$ for some constant $C > 0$.

What can you conclude? (Without justifying everything, unless there is still time.)

If $z = A_\lambda x = J_{A_{-1}/\lambda}(x/\lambda)$ (question **2.**), then

$$\begin{aligned} z + \tfrac{1}{\lambda}A^{-1}z \ni \tfrac{x}{\lambda} \Leftrightarrow \tfrac{\lambda}{\mu}z + \tfrac{1}{\mu}A^{-1}z \ni \tfrac{x}{\mu} \\ \Leftrightarrow z + \tfrac{1}{\mu}A^{-1}z \ni \tfrac{x}{\mu} + (1 - \tfrac{\lambda}{\mu})z = \tfrac{x + (\mu-\lambda)z}{\mu} \\ \Leftrightarrow z = J_{A^{-1}/\mu}(\tfrac{x+(\mu-\lambda)z}{\mu}) = A_\mu(x + (\mu - \lambda)A_\lambda x)\end{aligned}$$

As a consequence,

$$\begin{aligned}\frac{\partial}{\partial t}|X^\lambda(t,x) - X^\mu(t,x)|^2 &= -2\left\langle X^\lambda - X^\mu, A_\mu(X^\lambda + (\mu-\lambda)A_\lambda X^\lambda) - A_\mu X^\mu\right\rangle \\ &= -2\left\langle (X^\lambda + (\mu-\lambda)A_\lambda X^\lambda) - X^\mu, A_\mu(X^\lambda + (\mu-\lambda)A_\lambda X^\lambda) - A_\mu X^\mu\right\rangle \\ &\quad + 2(\mu-\lambda)\left\langle A_\lambda X^\lambda, A_\lambda X^\lambda - A_\mu X^\mu\right\rangle \\ &\le 2(\mu-\lambda)\left\langle A_\lambda X^\lambda, A_\lambda X^\lambda - A_\mu X^\mu\right\rangle.\end{aligned}$$

Symmetrically,

$$\frac{\partial}{\partial t}|X^\lambda(t,x) - X^\mu(t,x)|^2 \le 2(\lambda - \mu)\left\langle A_\mu X^\mu, A_\mu X^\mu - A_\lambda X^\lambda\right\rangle$$

and averaging the two estimates we get the answer. Hence the time derivative is bounded by $|\lambda - \mu||A_0 x|^2$ and the estimate follows with $C = 1$ (integrating from 0 to t).

It follows that as $\lambda \to 0$, $X^\lambda(x,t)$ is a Cauchy sequence in $C^0([0,T];\mathcal{X})$ (for any $T > 0$), which converges uniformly to some continuous path $X(t,x)$. This path is a solution of $\partial_t X + AX \ni 0$: precisely one can show that satisfies for all $t \geq 0$:

$$X(t,x) = x - \int_0^t A_0 X(s,x) ds.$$