Continuous optimization, an introduction
Exercises (22th Nov. 2016)

1. We recall that for a convex function \( f : X \to \mathbb{R} \),
   \[
   \text{prox}_f(x) = \arg\min_y f(y) + \frac{1}{2\tau} ||y - x||^2.
   \]
   Evaluate \( \text{prox}_f(x) \) for \( \tau > 0 \), and
   - \( x \in \mathbb{R}^N, f(x) = \frac{1}{2} ||x - x^0||^2 \): what does Moreau’s identity give in this case, for \( x^0 = 0 \)?
   - \( x \in \mathbb{R}^N, f(x) = \delta_{\{|x| \leq 1\}}(x) + \varepsilon ||x||^2 / 2 \).

2. Evaluate the convex conjugate of:
   - \( f(x) = \delta_{\{|x| \leq 1\}}(x) + \varepsilon ||x||^2 / 2 \), \( x \in \mathbb{R}^N \);
   - \( f(x) = \sum_{i=1}^N x_i \log x_i \), \( x \in \mathbb{R}^N \), where \( x \mapsto x \log x \) is \( +\infty \) for \( x < 0 \) and extended by continuity (that is, with the value 0) in \( x = 0 \);
   - \( f(x) = \sqrt{1 + ||x||^2} \), \( x \in \mathbb{R}^N \).

In each case of the three cases above, describe \( \partial f \) and \( \partial f^* \).

3. Show that if \( ||x|| \) is a norm and \( ||y||^o = \sup_{||x|| \leq 1} \langle x, y \rangle \) is the polar or dual norm, then
   \[
   || \cdot ||^o(y) = \delta_{B_{1,\infty}}(0,1)(y) = \begin{cases} 
   0 & \text{if } ||y||^o \leq 1, \\
   +\infty & \text{else}. 
   \end{cases}
   \]
   Hint: write \( \sup_x \langle x, y \rangle - ||x|| \) as \( \sup_{t>0} \left( \sup_{||x|| \leq t} \langle x, y \rangle \right) - t \).

   What is \( || \cdot ||^{o\circ} \)?

4. (Schatten norms) Let \( X \in \mathbb{R}^{n \times p} \) be a matrix.
   a. Show that \( X^TX \) and \( XXX^T \) are a symmetric \( p \times p \) and \( n \times n \) (respectively) matrix and that they have the same nonzero eigenvalues \( (\lambda_1, \ldots, \lambda_k) \) \( (k \leq \min\{p, n\}) \). The values \( \mu_i = \sqrt{\lambda_i} \) are the “singular values” of \( X \).
   b. Show that if \( (e_1, \ldots, e_p) \) is an orthonormal basis of eigenvectors of \( X^T X \) (associated to the eigenvalues \( \lambda_i \), or 0 if \( i > k \)), then \( (Xe_i)_k \) are orthogonal. Show that one can write, for \( \mu_i > 0 \), \( Xe_i = \mu_i f_i \) where \( f_i \) are also orthonormal. Completing \( f_i \) into an orthonormal basis of \( \mathbb{R}^n \), deduce that
   \[
   X = \sum_{i=1}^k \mu_i f_i \otimes e_i = V D^T U
   \]

\[ ^1\delta_C(x) = 0 \text{ if } x \in C, +\infty \text{ if } x \notin C \]
where $U$ is the column vectors $(e_i)_{i=1}^p$, $V$ the column vectors $(f_i)_{i=1}^n$, $D$ is the $n \times p$ matrix with $D_{ii} = \mu_i$, $i = 1, \ldots, k$, $D_{ij} = 0$ for all other entries (just evaluate $Xx = X(\sum_{i=1}^p \langle x, e_i \rangle e_i)$, etc.) What type of matrices are the matrices $U, V$? This is called the “singular value decomposition” (SVD) of $X$ (one usually orders the $\mu_i$ by nonincreasing values).

c. One defines the $p$-Schatten norm of $X$, $p \in [1, \infty]$, as
$$
\|X\|_p = \sum_{i=1}^k \mu_i^p, \quad \|X\|_\infty = \max_i \mu_i.
$$
Show that
$$
\|X\|_2^2 = \sum_{i,j} x_{i,j}^2 = \text{Tr}(XX^T); \quad \|X\|_\infty = \sup_{\|x\| \leq 1} \|Xx\|,
$$
(where in the latter $\|x\|$ is the 2-norm). $\|\cdot\|_\infty$ is called the spectral norm or operator norm.

d. [Exercice 3. is necessary for this question.] Why do we have that
$$
\{X : \|X\|_1 \leq 1\} = \text{conv}\{f \otimes e : f \in \mathbb{R}^n, e \in \mathbb{R}^p, \|f\|_1 \leq 1, \|e\|_1 \leq 1\}?
$$
Deduce that
$$
\|X\|_\infty = \sup_{\{\|Y\|_1 \leq 1\}} \langle Y, X \rangle
$$
where we use the Frobenius (or Hilbert-Schmidt) scalar product $\langle Y, X \rangle = \sum_{i,j} Y_{i,j} X_{i,j} = \text{Tr}(YX)$. Deduce that
$$
\|X\|_1 = \sup_{\{\|Y\|_\infty \leq 1\}} \langle Y, X \rangle.
$$
(Of one also show that $\|\|_o = \|\cdot\|_p, 1/p + 1/p' = 1$.)

e. We want to compute
$$
\hat{Y} = \arg \min_{\|X\|_\infty \leq 1} \|X - \hat{Y}\|_2^2 = \text{prox}_{\delta(\{X : \|X\|_\infty \leq 1\})}(Y). \quad (P_\infty)
$$
Show first that it is equivalent to estimate $\min_{\|X\|_\infty \leq 1} \|X - D\|_2^2$ where $Y = VD^TU$ is the SVD decomposition of $Y$. Show that the matrix $X$ which optimizes this last problem is diagonal, and satisfies $X_{i,i} = \max\{D_{i,i}, 1\}$. Deduce the solution $\hat{Y}$ of ($P_\infty$).

Deduce the proximity operator $\text{prox}_{\|\cdot\|_1}$.

f. A company rents movies and has a file of clients $X_{i,j} \in \{-1, 0, 1\}$ which states for each client $i = 1, \ldots, p$ whether he/she has already rented the film $j = 1, \ldots, n$ (otherwise $X_{i,j} = 0$) and has liked it ($X_{i,j} = 1$), or not ($X_{i,j} = -1$). It wants to determine a matrix of “tastes” for all the clients $Y \in \{-1, 1\}^{p \times n}$. Assuming that the clients can be grouped into few categories, this matrix should have low rank. One could look therefore for an approximation of $Y$ by minimising
$$
\min_Y \|Y\|_1 + \frac{\lambda}{2} \sum_{i,j : X_{i,j} \neq 0} (X_{i,j} - Y_{i,j})^2 + \frac{\varepsilon}{2} \sum_{i,j : X_{i,j} = 0} Y_{i,j}^2
$$
where $\lambda \gg \varepsilon > 0$ are parameters.

Design an iterative algorithm to solve this problem.