PARTIAL DIFFERENTIAL EQUATIONS AND IMAGE PROCESSING

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ABSTRACT

We review in this paper the PDE methods for image processing developed at the CEREMADE, the Universitat de Illes Balears, and the Universidad de Las Palmas de Gran Canaria. We first recall the axiomatic approach of the theory of multiscale analysis of images and try at the same times to see how it can be extended to the problem of color image processing. After stating the theorems that follow from this theory, we discuss some examples of PDEs and show results.

1. AXIOMS FOR THE MULTISCALE ANALYSIS OF IMAGES

An image in what follows will be represented as a function $u(x)$, $x \in \mathbf{R}^2$ with $u(x) \in \mathbf{R}^d$. We will mainly consider the cases where $d = 1$, 2 or 3. $d = 1$ corresponds to grey level images, $d = 3$ to color images (with usually $u(x) = (r(x), g(x), b(x))$, i.e. a red, resp. green, blue, intensity value). $d = 2$ may correspond to a subspace of the color space. We will always assume that u has some regularity (continuous, or twice differentiable).

What we call a multiscale analysis is a family of transforms $(T_t)_{t>0}$ of an image $u^0(x)$, such that $u(t,x) = (T_t \cdot u^0)(x)$ can be seen as the image u^0 at a coarser scale t. (We'll set $T_0 \cdot u^0 = u^0$ for any function u^0.) The simplest example of such a family is given by the gaussian kernel, where

$$T_t \cdot u^0 = G_t * u^0$$

with[1] $G_t(x) = \exp(-|x|^2/4t)/4\pi t$. It is easy to check that the function $u(t,x)$ is the solution of the *heat equation*

$$\frac{\partial u}{\partial t} = \Delta u \qquad (1)$$

with $u(0,\cdot) = u^0$. This linear multiscale analysis was first introduced by Witkin and Koenderink [1, 2]. It is interesting to notice that in this case, if u^0 is a color image, the transformation T_t is performed independently on each channel $u_i, i = 1, \ldots, d$.

Now we shortly review the axioms that should satisfy a multiscale analysis [3, 4, 5, 6]. We will also try to introduce new axioms, specificly oriented towards the analysis of color images.

1.1. Architectural axioms

First, we want the analysis to be causal, which means that we should be able to compute $u(t,\cdot) = T_t \cdot u^0$ from any $u(t',\cdot)$ with $t' \le t$ without necessarily knowing u^0. If this is true, we'll be able to get $u(t+dt,\cdot)$ from $u(t,\cdot)$ and it follows that there may be a PDE such as (1) satisfied by u. The first axiom we can state to guaranty the causality is the following:

A1 [Recursivity] $\forall s,t \ge 0$, $T_s \circ T_t = T_{s+t}$.
This means that T_t is a semigroup, and holds for instance for the analysis defined by equation (1). But for some purpose this assumption is too strong and we need to introduce the following weaker axiom:

A2 [Pyramidal architecture] $\forall s,t \ge 0$, $\exists T_{s+t,t}$, $T_{s+t} = T_{s+t,t} \circ T_t$.

A3 [Regularity] If $u(x) = (u_1(x), \ldots, u_d(x))$ is a quadratic function, i.e. $u_i(x) = c_i + (p_i, x) + (A_i x, x)$, then there exists a function

$F(t,x,u,p,A)$:

$$[0,+\infty) \times \mathbf{R}^2 \times \mathbf{R}^d \times (\mathbf{R}^2)^d \times \mathcal{S}(2)^d \to \mathbf{R}^d$$

(where $\mathcal{S}(2)$ is the set of symmetric 2×2 matrices) such that

$$\lim_{h \to 0} \frac{T_{t+h,t} \cdot u - u}{h} = F(t,x,c,p,A)$$

Moreover, F is supposed to be continuous with respect to each A_i. Observe that if **[Recursivity]** holds, F should not depend on t.

When we deal with grey level images, we consider natural to assume that if one image is brighter than another, then this property remains true at each scale. But if $d > 1$, there is no order in $\mathbf{R}^d$ and therefore no natural way to extend the principle. This will depend on the color representation we're using. Suppose $d = 3$ and $u(x) = (r(x), g(x), b(x))$. We could define an intensity $\langle u(x), \iota \rangle$ where ι is a vector with positive components (for instance $\iota \sim (0.299, 0.587, 0.114)$ if we follow the standard "CIE XYZccir601-1 (D65)", or $\iota \sim (1,1,1)$ if we want to do simpler) and define the principle with respect to this value only.

[1] $|x| \stackrel{\text{def}}{=} \sqrt{(x,x)}$, where $(\cdot,\cdot)$ is the euclidean scalar product in $\mathbf{R}^2$ — and $\langle \cdot,\cdot \rangle$ in $\mathbf{R}^d$.

Following this idea, suppose we assume that

$$\begin{aligned} & \forall x, \quad \langle \iota, u(x)\rangle \le \langle \iota, v(x)\rangle \\ \Rightarrow \quad & \forall x,t, \quad \langle \iota, (T_t\cdot u)(x)\rangle \le \langle \iota, (T_t\cdot v)(x)\rangle. \end{aligned}$$

Then it is easy to check that $\langle \iota, (T_t\cdot u)\rangle$ only depends on $\langle \iota, u\rangle$ and can be considered as a multiscale analysis of the brightness $\tau_t^\iota \cdot \langle \iota, u\rangle$. This is interesting but not general enough for this preliminary study. With a few more axioms, we can even show that if we suppose that the previous property holds for *all* ι with positive components, then T_t is linear, and equivalent — up to a rescaling — to (1). Simpler and quite reasonable is to say that if an image mainly differs from another one because it is "more blue" (or red, or green), this should remain true at coarser scales. We therefore can define an order in $\mathbf{R}^d$ by: $\xi = (\xi_1,\cdots,\xi_d) \le \eta = (\eta_1,\cdots,\eta_d)$ iff. for all $i = 1,\ldots,d$ $\xi_i \le \eta_i$, and state with this partial order the following

A4 [Maximum principle] if $u(x) \le v(x)$ for all $x \in \mathbf{R}^2$ then for all $t > 0$ and all x,

$$(T_t\cdot u)(x) \le (T_t\cdot v)(x).$$

We will often use a local version of this last principle:

A5 [Local maximum principle] (or [local comparison] axiom): for all u,v,t,x, if there is a ball $B(x,r)$ such that $u < v$ on $B(x,r)\setminus\{x\}$, then for h small enough

$$(T_{t+h,t}\cdot u)(x) \le (T_{t+h,t}\cdot v)(x).$$

Now we can state a first theorem (*cf* [6]):

Theorem 1 *If the axioms* **A2**, **A3** *and* **A5** *hold then for all image u and all x where u is C^2*

$$\frac{T_{t+h,t}\cdot u - u}{h}(x) \to F(t,x,u(x),\nabla u(x), d^2u(x)) \qquad (2)$$

as h goes to 0^+, where F is the function introduced in **A3**. *Moreover,* **A5** *implies that, for all t,x,u,p, if the symmetric matrices A_i, B_i $(i = 1,\ldots,d)$ are such that*

$$\textit{for all } i \quad A_i \le B_i,$$

$$\textit{then} \quad F(t,x,u,p,A) \le F(t,x,u,p,B). \qquad (3)$$

The main signification of this theorem is that the regularity we imposed for quadratic functions is automatically extended to any C^2 image, and derivatives of order higher than two play no role in the local evolution of $T_t\cdot u(x)$.

1.2. Geometrical and morphological axioms

1.2.1. geometrical invariances

Let G be a group of affine transformations of $\mathbf{R}^2$. If $g \in G$ and u is an image we will denote $g\cdot u$ the action of g on u, defined by $(g\cdot u)(x) = u(g\cdot x)$. We can define the four following axioms:

A6 [Translation invariance] $\tau\cdot T_t = T_t\cdot\tau$ for any translation τ. This is a very natural and fundamental axiom, which implies that the above defined function F does not depend on x.

A7 [Isometry invariance] $R\cdot T_t = T_t\cdot R$ for any isometry $R \in O(\mathbf{R}^2)$.

A8 [Scale invariance] for any $\lambda > 0$ and t, $\exists t'(t,\lambda)$ such that $D_\lambda\cdot T_{t'} = T_t\cdot D_\lambda$, where $D_\lambda\cdot x = \lambda x$.

A9 [Affine invariance] for any linear transform $A \in Gl(\mathbf{R}^2)$ and any t, $\exists t'(t,A)$ such that $A\cdot T_{t'} = T_t\cdot A$.

It is proved [6] that, up to a rescaling of the analysis T_t (i.e. we substitute T_t with $T_{h(t)}$ where h is an increasing diffeomorphism of $[0,+\infty))$ we may chose $t' = t\lambda = t\sqrt{\det(D_\lambda)}$ in the scale invariance axiom, and $t' = t\sqrt{\det(A)}$ in the affine invariance.

1.2.2. morphological invariances

Here things become complicated. When $d = 1$, it may seem natural to consider the two following axioms, that express the fact that our analysis should be robust to any change of contrast or luminosity in the image.

A10 [Grey level shift invariance] for any real constant C and any image u, we have for all $t > 0$, $T_t\cdot(u+C) = (T_t\cdot u) + C$.

A11 [Morphological invariance] for any image u and any increasing real regular function h, we have for all t $T_t\cdot(h\circ u) = h\circ(T_t\cdot u)$.

However, when $d > 1$, these two axioms do not generalize. We could of course consider any change of any channel and ask T_t to commute with this change, but, for a color image, this would have no perceptual meaning. Here the axioms we need strongly depend on the representation of the color space we're using. In this preliminary study we considered only the most simplified representations. For $d = 3$ we suppose the three channels $u_1(x), u_2(x), u_3(x)$ are the intensity of red, green, and blue. Notice that on most color displays it appears that the axis {red=green=blue} always look grey, and we will try to consider transformations that keep this axis. If we consider the H, S, L[2] values of the color signal, defined by:

$$\begin{aligned} & L(x) = \langle u(x),\sigma\rangle = \frac{red + green + blue}{\sqrt{3}} \\ & \text{with } \sigma = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \\ & S(x) = 1 - \sqrt{3}\frac{\min_{i=1,2,3} u_i(x)}{L(x)} \\ & \text{or (perceptually less correct, but simpler):}^3 \\ & \tilde{S}(x) = \|u(x) - \langle u(x),\sigma\rangle\sigma\| = \|\sigma\times u(x)\| \\ & H(x) = \text{angle}\left(\sigma\times u(x), \sigma\times\begin{pmatrix}1\\0\\0\end{pmatrix}\right), \end{aligned} \qquad (4)$$

[2] the *Hue*, i.e. the "color itself", the *Saturation*, that roughly measures how far the color is from the grey axis, or how "pure" it is, and the *Luminance*, a good approximation for the global intensity of the color (*cf* for instance [7]).

we can state the following axioms, that are roughly the transposition of **[morphological invariance]** in the color domain:

A12 [Luminance scale invariance 1] (correct) for any $u(x)$ and any $h(u) = \frac{\tilde{h}(\langle u,\sigma\rangle)}{\langle u,\sigma\rangle} u$ where $\tilde{h}$ is an increasing real regular function with $\tilde{h}(0) = 0$, we have for all t: $T_t \cdot (h o u) = h \circ (T_t \cdot u)$.
This axiom says that any change of the luminance that preserves the saturation S and the hue H should not modify the analysis.

A13 [Luminance scale invariance 2] (less correct) the same axiom with this time

$$h(u) = u + (\tilde{h}(\langle u, \sigma\rangle) - \langle u, \sigma\rangle)\sigma$$

(Here the change of the luminance h preserves $\tilde{S}$ instead of S that is multiplied by $\langle u,\sigma\rangle/\tilde{h}(\langle u,\sigma\rangle)$, which is perceptually less good.)

Various other axioms may be introduced, for instance:

A14 [Saturation scale invariance] same as the two axioms above but with

$$h(u) = \langle u,\sigma\rangle\sigma + \frac{\tilde{h}(\tilde{S})}{\tilde{S}}(u - \langle u,\sigma\rangle\sigma),$$

which means that the analysis should be robust to any change of $\tilde{S}$. Note that two of the three preceding axioms do not imply the third one.

A15 [Grey image stability] would ensure that if $S(x) = 0$ for all x for an image $u(x)$ (which means the image is grey), then this property remains stable at any scale; i.e.: $u_i^0(x) = u_j^0(x)$ for all x, i, j implies $(T_t \cdot u^0)_i(x) = (T_t \cdot u^0)_j(x)$ for all t and all x, i, j.

A16 [Monochrome image stability] would ensure as well that if an image is all the same color ($H(x)$ is constant over the whole image: note that this is roughly supposed to happen if the image pictures a uniformly coloured object, even with non-constant shading and brightness), then this remains true at any scale: if $u^0(x) = f^0(x)\sigma + g^0(x)h$ where h is a vector orthogonal to σ and g^0 a positive function, then there exist $f(t,x)$ and $g(t,x)$ positive such that $T_t \cdot u = f(t,\cdot)\sigma + g(t,\cdot)h$.

At last an axiom that seems obvious but needs to be stated, is the following:

A17 [Constant image stability] if $u(x) = C \in \mathbf{R}^d$ for all x then for any t, $T_t \cdot u = u$. (A uniformly monochrome image remains the same at any scale!) It implies that $F(t, u, 0, 0) = F(t, 0, 0, 0)$ for any u and t.

For $d = 1$, the above axioms have the following consequence, that $T_t \cdot u$ is the solution of a PDE: (*cf* [6]):

Theorem 2 *If $d = 1$ and* **A2**, **A3**, **A4**, **A5**, **A6** *and* **A10** *hold, and if for any bounded uniformly continuous (BUC) u^0, equation*

$$\frac{\partial u}{\partial t}(t,x) = F(t, \nabla u(x), d^2u(x)) \quad \textit{and} \quad u(0,\cdot) = u^0 \qquad (5)$$

has a unique BUC viscosity solution [8] (this of course depends on F), then $u(t,\cdot) = T_t \cdot u^0$ is this solution.

Remark. For some example given below (namely the mean curvature motion), F is not continuous with respect to d^2u when $\nabla u = 0$. Then theorem 1 only holds at points where $\nabla u \neq 0$, and there is a similar version of theorem 2.

Unfortunately for $d > 1$ (and when there is a coupling between the channels u_i in the equation) there is no such thing as viscosity solutions and it is not clear whether equation (5) has a meaning and a solution for any u^0. In the sequel we will assume that $T_t \cdot u^0$ always exists and is given by PDE (5), as for each proposed PDE the numerical experiments gave stable and good results and it is likely that, at least for a large class of functions, the same kind of result as for $d = 1$ holds.

2. EXAMPLES OF PDEs

2.1. Linear multiscale analyses

It has been proved [3, 6] that if T_t is a multiscale analysis of grey level images that is linear, and satisfy **A2**, **A3**, **A5**, **A6**, and — which in this case of course implies A10 — **A17**, then it is equivalent to the heat equation. The same demonstration may be adapted to show that, if T_t is a linear analysis of color (red, green, blue) images that satisfy the same axioms, plus **A15** and **A16**, then, provided some multi-dimensional equivalent of theorem 2 is true, it is again equivalent to (1). This explains the crucial need to look for *non-linear* PDEs in order to build original filters for image analysis.

2.2. Non-linear multiscale analyses

2.2.1. grey level images

The most interesting multiscale analyses in this case are the ones that satisfy (in addition to **A2** (or A1), **A3**, **A5**, **A6** and **A10**) **A11 [morphological invariance]** and either **A7 [isometry invariance]** or, what is stronger, **A9 [affine invariance]**. In the first case it is proved that $u(t,x) = (T_t \cdot u^0)(x)$ is given by

$$\frac{\partial u}{\partial t} = |\nabla u| G(\mathrm{curv}(u), t) \qquad (6)$$

where $\mathrm{curv}(u)(x) = \mathrm{div}(\nabla u/|\nabla u|)(x)$ is the curvature of the level curve $u^{-1}(u(x))$ at x, and G a nondecreasing function with respect to $\mathrm{curv}(u)$. The main examples are:
- $G(\mathrm{curv}(u), t) = 1$: we get the operators of the classical mathematical morphology [4];
- $G(\mathrm{curv}(u), t) = \mathrm{curv}(u)$: this gives the "mean curvature motion" [9]. Each level curve of function u evolves with a speed that is proportional to its curvature. Equation (6) can also be written as an anisotropic diffusion equation

$$\frac{\partial u}{\partial t} = u_{\xi\xi} \qquad (7)$$

where ξ is the direction normal to the gradient $\nabla u(x)$ and $u_{\xi\xi}$ the second derivative of u in this direction: diffusion

[3] $\|u\| \stackrel{\text{def}}{=} \sqrt{\langle u, u\rangle}$.

hence occurs only along ξ and not in the direction of the gradient. This is related to Malik and Perona's anisotropic diffusion schemes [5, 10]. An interesting problem is to find a morphological algorithm to discretize this equation, i.e. a scheme which really prevents diffusion between the level curves [11].

• $G(\mathrm{curv}(u), t) = (t\mathrm{curv}(u))^{1/3}$: (with $x^{1/3} = \mathrm{sign}(x)|x|^{1/3}$) this is, up to a rescaling, the only morphological and affine invariant multiscale analysis. Now the level curves of u evolve with a speed proportional to the affine curvature along the affine normal to the curve, instead of the euclidean curvature and normal. Note that $|\nabla u|(\mathrm{curv}(u))^{1/3}$ is not singular at $\nabla u = 0$.

2.2.2. color images

The simplest way to handle color images is to treat them as three separate and independent channels, and find the coarser scales for each channel. For instance, the last multiscale analysis presented in the previous section will give an affine invariant analysis of the color image if it is performed independently on the red, green and blue components. For noise removal, this may seem to be the best choice, as the noise on one channel may not depend on the other channels. Moreover, any change of coordinates (not necessarily linear) in the color space will give a new different multiscale analysis and for instance one will get a theoretically correct process by applying the previous affine invariant scheme to the hue, saturation and luminance variables (this would satisfy many of the above mentioned axioms). However, this is hard to implement in practice because a slight diffusion in hue between different level curves will cause much trouble (a yellow line might appear between red and green areas), and it seemed at first more interesting to try to understand how to build totally coupled multiscale analyses of color images. We will discuss two schemes, both inspired from equation (7), and designed for different purposes.

The first equation we propose is still equation (7) but now ξ is the direction orthogonal to η with

$$\eta = \arg \max_{\substack{\eta \in \mathbf{R}^2 \\ |\eta| = 1}} \left\| \begin{array}{c} (\nabla u_1, \eta) \\ (\nabla u_2, \eta) \\ (\nabla u_3, \eta) \end{array} \right\| \tag{8}$$

Notice that an equivalent point of view would be to consider unit vectors $\alpha \in \mathbf{R}^d$ (linear forms in fact) and compute η as the direction of the largest gradient at x of all functions $\langle \alpha, u \rangle$ for all α. The such defined multiscale analysis will satisfy **A1-A5**, **A6-A8**, **A15-A17** but it is not affine invariant, and does not satisfy any of the morphological axioms **A12**, **A13** and **A14**. Its main interest is that it preserves some edges slightly better than when the mean curvature motion (7) is applied independently to each color channel. If an edge is strong in one color only but fuzzy in another, the direction of diffusion that is selected is imposed to the latter channel by the first one. However, as the three color gradients are almost parallel in most images, it is very hard to see the difference, and understand which method is better than the other and why.

The second equation we propose has been designed to help solving color constancy problems (see for instance [12, 13]). Still, we follow a scheme based on (7) but now ξ is perpendicular to

$$(u_1 - u_2)\nabla u_3 + (u_2 - u_3)\nabla u_1 + (u_3 - u_1)\nabla u_2, \tag{9}$$

which is proportional to $\tilde{S}(x)\nabla H(x)$. The idea is to stop the diffusion in the direction that maximizes the color change. Unfortunately, this analysis will satisfy the same axioms as the previous one and no more. What happens in fact is that when any of the transformations $u \mapsto h(u)$ of axioms **A12-A14** is performed, the directions of (9) and thus of ξ do not change (this may be false with the previous scheme) but the local speed of the diffusion may vary according to the local behavior of $u(x)$.

Note that for noisy images, both methods can be improved by passing first a linear filter on the image before estimating the gradient and ξ (of course the diffusion is not done on the filtered image!). Then, the heat equation may be applied in the areas where the gradient is small.

Figure 1 illustrates the first PDE. Figure 3 shows the effect of the second PDE on figure 2. Here the heat equation was applied when the gradient was too low. The result is that the edge between the red nose and the blue cheeks of the baboon is very well preserved whereas the shadows on the cheeks, that are mainly due to variations of luminance but not of hue, are highly diffused.

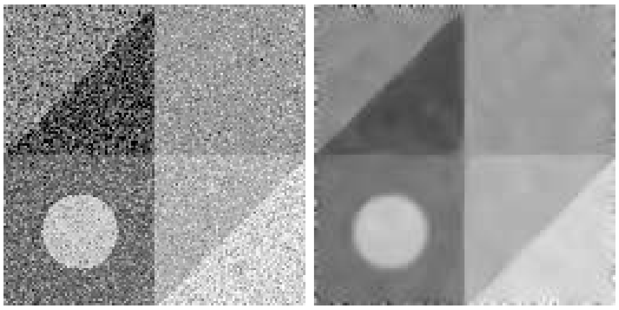

Figure 1: A noisy color test image, and the image after filtering in the direction perpendicular to (8)

3. CONCLUSION

The axiomatic approach to image processing is a powerful tool that leads to the "right" partial differential equations for image analysis. For the time being, the method has not be fully extended to the analysis of color images, but we expect great progress soon. Many other interesting developements of (6)-(7), that cannot be included in the present brief review, have been studied so far. The most interesting is probably the Alvarez-Mazzora PDE [14], based on Osher and Rudin's shocks filters [15], that performs simultaneously denoising and edge enhancement and gives great results. The application of the same method, for instance to the last color filter described in section 2.2.2, is very promising.

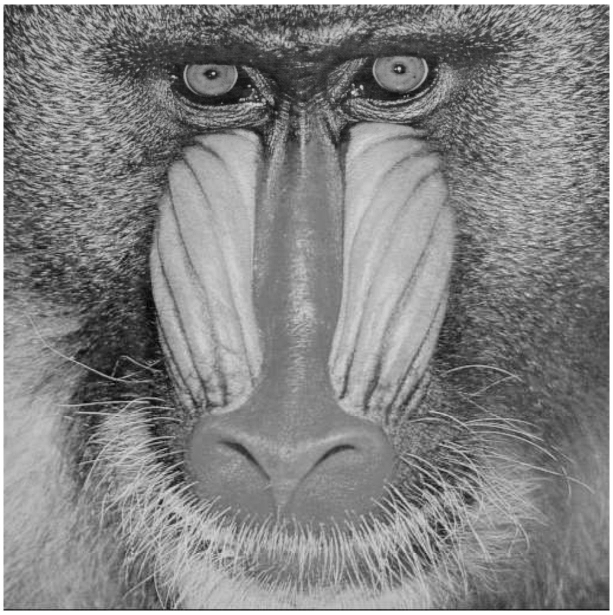

Figure 2: the original baboon

Figure 3: baboon after filtering

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