Finite Element Method and applications

Master 2 Acoustical engineering - Session 6

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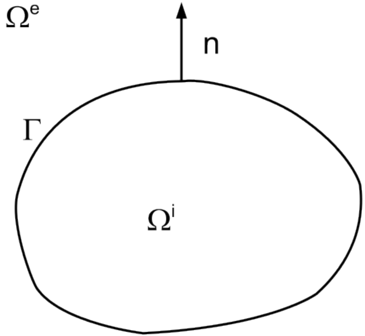
Context

- Ω : full space,
- Ω_i : interior domain,
- Ω_e : exterior domain,
- Γ : boundary smooth and oriented,
- $\mathbf{n}$: normal at the boundary.



Note : Γ is a singularity of $\Omega \Rightarrow$ Distribution theory...

Wave equation with neumann condition

For a wave propagating in Ω_i or Ω_e with **Neumann** condition (sound-hard), relative magnitude of the wave $u(\mathbf{x}, t)$ is governed by a PDE :

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \Delta_{\mathbf{x}} u &= 0, \\
\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x} \in \Gamma, \cdot) &= 0, \\
u(\cdot, t = 0) &= u_0, \\
\frac{\partial u}{\partial t}(\cdot, t = 0) &= v_0,
\end{aligned}$$

with c the sound celerity, t the time, $\mathbf{x}$ the position in Ω and u_0, v_0 the initial position and speed.

Note : Ω_i for room acoustic, Ω_e for underwater acoustic, etc.

Review of numerical methods

To solve this problem, various numerical methods are currently used in industry :

Method	Domain	Math	Advantage	Disadvantage	TP
FDTD	straight grid	Taylor expensions	easy, massively parallel and fast	carthesian grid and no local refinment	X
Lattice-Boltzmann	edge grid	in progress	easy and fast	young, lake of proof	X
FEM	simplex or polygonal	weak formulation (Lax-Milgram)	generic and robust	difficult to implement	(X)
BEM	simplex or polygonal	distribution theory	precise and open domain	very hard to implement and accelerate	-
Ray-tracing	simplex or polygonal	high frequency approximation	unique way for huge domain	not precise (at all)	X

Weak formulation (variational)

Starting with the wave equation :

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0,$$

a weak formulation is obtained multipling by an arbitrary test function $v(\mathbf{x}, t) \in L^2(\mathbb{R})$:

$$v\frac{\partial^2 u}{\partial t^2} - c^2 v \Delta u = 0,$$

and integrating in the whole domain of interest :

$$\int_\Omega v \frac{\partial^2 u}{\partial t^2} dx - c^2 \int_\Omega v \Delta u dx = 0.$$

Note : v not really going to be arbitrary...

Weak formulation (variational)

Using a part integration in space [1] (green theorem) :

$$\int_\Omega v\Delta u dx = -\int_\Omega \nabla u \nabla v dx + \int_\Gamma v\frac{\partial u}{\partial \mathbf{n}},$$

and considering neumann condition :

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x} \in \Gamma, \cdot) = 0,$$

the weak formulation is given by :

$$\int_\Omega v\frac{\partial^2 u}{\partial t^2}dx + c^2\int_\Omega \nabla v \cdot \nabla u dx = 0.$$

Lax-Milgram theorem gives a mathematical environment to ensure existence and unicity of the solution of this equation [2].

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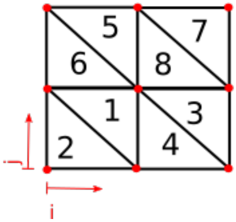
1. $\int_a^b v(x)u'(x)dx = [vu]_a^b - \int_a^b v'(x)u(x)dx$
2. *https ://fr.wikipedia.org/wiki/Theoreme_de_Lax-Milgram*

FEM step 1 : the mesh

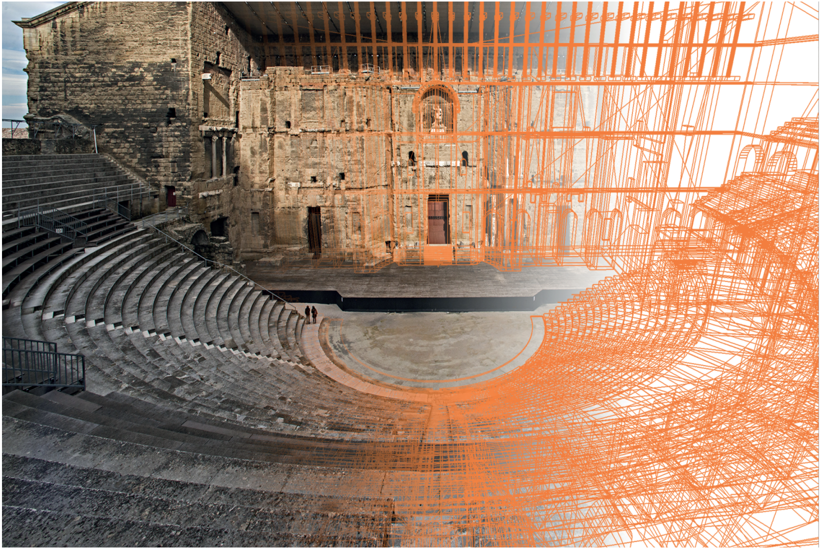
To use weak formulation in scientific computing, we first have to discretize the continuous domain Ω using an arbitrary mesh :

$$vtx = \begin{pmatrix} 0. & 0. \\ 0.5 & 0. \\ 1. & 0. \\ 0. & 0.5 \\ 0.5 & 0.5 \\ 1. & 0.5 \\ 0. & 1. \\ 0.5 & 1. \\ 1. & 1. \end{pmatrix} \qquad elt = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 4 \\ 5 & 6 & 3 \\ 2 & 3 & 5 \\ 7 & 8 & 5 \\ 4 & 5 & 8 \\ 8 & 9 & 5 \\ 5 & 6 & 8 \end{pmatrix}$$



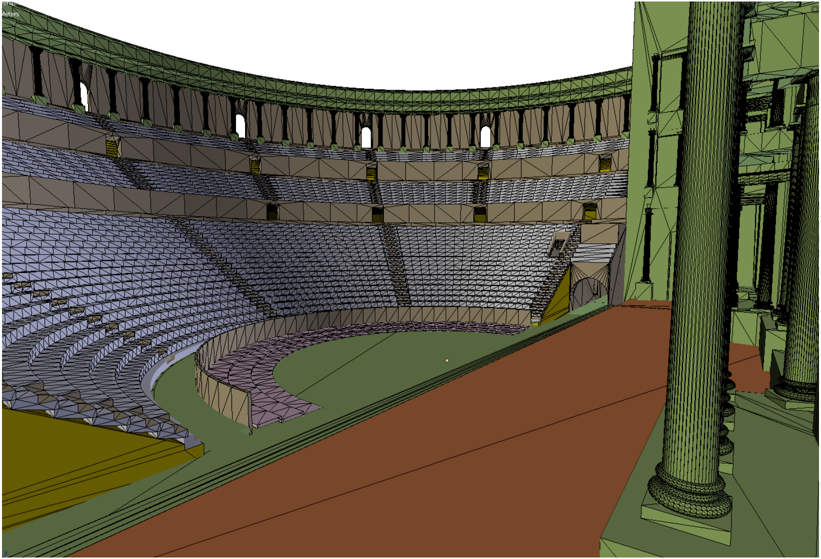
Note : 1 error has crept into the element table. Could you find it ?

FEM step 1 : the mesh



The Orange theater by R. Gueguen and T. Bartet

FEM step 1 : the mesh



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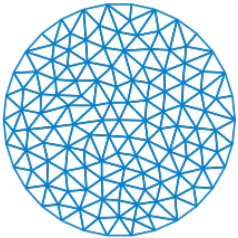
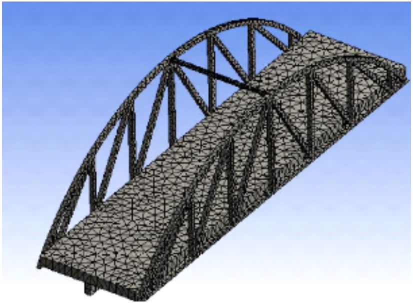
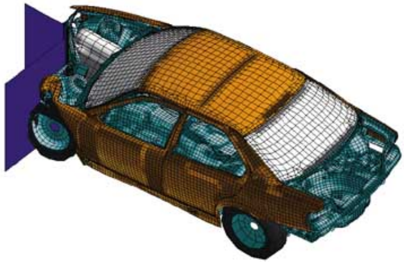
FEM step 1 : the mesh



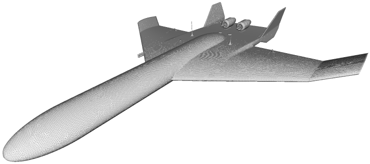
Delaunay mesh



Volumic mesh, apron of a bridge



Quadrangle mesh



Surfacic mesh, combat plane

FEM step 2 : the domain quadrature

As a weak formulation is defined by integration over the domain, we have to define a quadrature rule (rectangle, trapezoidal, simpson, gauss-legendre, etc.) [3] :

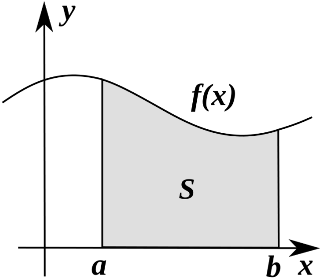
- Regular integration are done using a quadrature $(\mathbf{x}_q, \gamma_q)_{1 \leq q \leq n_q}$ for all the domain Ω :

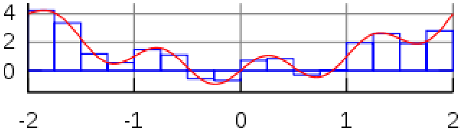
$$\int_\Omega f(\mathbf{x}) dx \approx \sum_{q=1}^{n_q} \gamma_q f(\mathbf{x}_q).$$

- If necessary, singular integrations are done analytically or numerically... and it's could be hard !

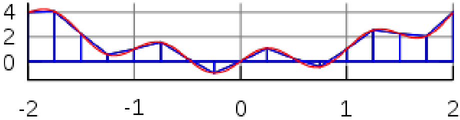
3. https ://en.wikipedia.org/wiki/Numerical_integration#Quadrature_rules_based_c

FEM step 2 : the domain quadrature





Rectangle rule



Trapezoidal rule

FEM step 3 : the Galerkin formulation

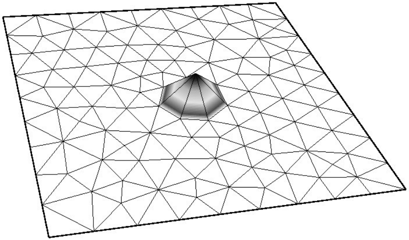
Galerkin formulation is an approximation of the final solution u such that :

$$u(\mathbf{x}, t) \approx \sum_j \alpha_j(t)\phi_j(\mathbf{x}),$$

with $\alpha_j(t) = u(\mathbf{x}_j, t)$ and $\phi_j(\mathbf{x})$:

$$\begin{cases} \phi_j(\mathbf{x}){=}1 & \text{if } \ \mathbf{x} = \mathbf{x}_j, \\ \phi_j(\mathbf{x}){=}0 & \text{else.} \end{cases}$$

It's a base decomposition and ϕ is generally called the basis function.



Basis function ϕ linear per piece.

FEM step 3 : the Galerkin formulation

Defining (not so arbitrarily) the test function v such as :

$$v(\mathbf{x}, t) = \sum_i \phi_i(\mathbf{x}),$$

the weak formulation :

$$\int_\Omega v \frac{\partial^2 u}{\partial t^2} dx + c^2 \int_\Omega \nabla v \cdot \nabla u dx = 0,$$

can be approached for each basis function couple (i, j) :

$$\int_\Omega \phi_i \frac{\partial^2 \alpha_j \phi_j}{\partial t^2} dx + c^2 \int_\Omega \nabla \phi_i \cdot \nabla(\alpha_j \phi_j) dx = 0,$$

and finally :

$$\frac{\partial^2 \alpha_j}{\partial t^2} \int_\Omega \phi_i \phi_j dx + c^2 \alpha_j \int_\Omega \nabla \phi_i \cdot \nabla \phi_j dx = 0.$$

FEM step 4 : the time dicretization

Fixing a final time t_f for the wave propagation, a time discretisation $T = (t_0, ..., t_N)$ can be defined such as :

- $t_0 = 0$,
- $t_N = t_f$,
- $t_n = n\delta_t$ with $\delta_t = \frac{t_f}{N+1}$ and $n \in [0, N]$.

Second order derivative in time leads to centered sheme :

$$\alpha(t)'' = \frac{\alpha(t+\delta_t) - 2\alpha(t) + \alpha(t-\delta_t)}{\delta_t^2} + O(\delta_t^4)$$

$$(\alpha^n)'' \approx \frac{\alpha^{n+1} - 2\alpha^n + \alpha^{n-1}}{\delta_t^2}$$

Note : Stable with the CFL condtion $\iff \frac{\delta_x}{\delta_t} \geq c$.

FEM step 4 : the time dicretization

Including the time discretization to the Galerkin approximation, its conduct to :

$$\frac{\alpha_j^{n+1} - 2\alpha_j^n + \alpha_j^{n-1}}{\delta_t^2} \int_\Omega \phi_i \phi_j dx + c^2 \alpha_j^n \int_\Omega \nabla\phi_i \cdot \nabla\phi_j dx = 0,$$

wich lead to the explicit formulation :

$$\begin{aligned}\alpha_j^{n+1} \int_\Omega \phi_i \phi_j dx &= 2\alpha_j^n \int_\Omega \phi_i \phi_j dx \\ &- \alpha_j^{n-1} \int_\Omega \phi_i \phi_j dx \\ &- (c\delta_t)^2 \alpha_j^n \int_\Omega \nabla\phi_i \cdot \nabla\phi_j dx.\end{aligned}$$

FEM step 4 : the time dicretization

FEM step 5 : the matrix formulation

But, using a matrix formulation for each basis function couple (i, j), we get the linear system :

$$[\mathbf{M}][\alpha^{n+1}] = 2[\mathbf{M}][\alpha^n] - [\mathbf{M}][\alpha^{n-1}] - (c\delta_t)^2[\mathbf{K}][\alpha^n],$$

where :

$$\begin{aligned} M_{ij} &= \int_\Omega \phi_i(\mathbf{x})\phi_j(\mathbf{x})dx, \\ K_{ij} &= \int_\Omega \nabla\phi_i(\mathbf{x}) \cdot \nabla\phi_j(\mathbf{x})dx, \\ \alpha_j^n &= u(\mathbf{x}_j, t_n). \end{aligned}$$

- Matrix terms are numerically evaluated using quadrature rule $(\mathbf{x}_q, \gamma_q)_{1 \leq q \leq n_q}$ on the mesh of the domain Ω,
- The local support of basis functions lead to sparse matrices,
- α^n is computed at each time step by recursion,
- In mechanics, $[\mathbf{M}]$ is traditionally named the mass matrix and $[\mathbf{K}]$ the stiffness matrix (or rigidity matrix).

Practical cases

Download the Gypsilab framework :

https ://github.com/matthieuaussal/gypsilab