The Sparse Cardinal Sine Decomposition applied to Stokes integral equation

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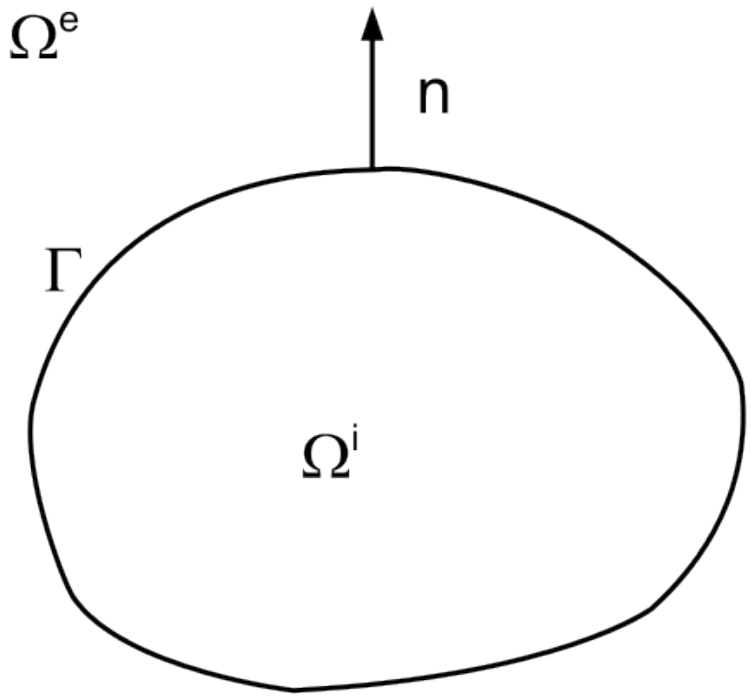
Definitions

Let a fluid flow in full space $\mathbb{R}^3$, moving around solid body Ω :

- Γ boundary of Ω, smooth and oriented,
- $\mathbf{n}$ the unit normal directed to the liquid,
- μ the fluid viscosity,
- $\mathbf{u}$ the fluid velocity,
- σ the stress tensor,
- p the pressure.



Integral representation for Stokes equation

For Newtonian fluids, if $\mathbf{u}$ and p read the stokes equation :

$$\left\{ \begin{array}{rcccl} -\mu\Delta\mathbf{u} + \nabla p & = & 0 & \in & \mathbb{R}^3 \setminus \Omega, \\ \mathrm{div}(\mathbf{u}) & = & 0 & \in & \mathbb{R}^3 \setminus \Omega, \end{array} \right.$$

then u satisfies for each Carthesian component :

$$\begin{aligned} u_j(\mathbf{x}) &= -\frac{1}{4\pi\mu}\int_\Gamma \sum_i (\sigma\cdot\mathbf{n})_i(\mathbf{y})\, G_{ij}(\mathbf{x}-\mathbf{y})\, d\Gamma_y \\ &+ \frac{1}{4\pi\mu}\int_\Gamma \sum_{i,k} u_i(\mathbf{y})\, T_{ijk}(\mathbf{x}-\mathbf{y})\, n_k(\mathbf{y})\, d\Gamma_y, \end{aligned}$$

with the so-called Stokeslet G_{ij} and Stresslet T_{ijk} :

$$G_{ij}(\mathbf{x}) = \frac{\delta_{ij}}{|\mathbf{x}|} + \frac{x_i x_j}{|\mathbf{x}|^3}, \quad T_{ijk}(\mathbf{x}) = -6\frac{x_i x_j x_k}{|\mathbf{x}|^5}.$$

Boundary Element Method with Galerkin approximation

- Single layer potential for each **scalar** Carthesian component :

$$\mathcal{S}\lambda(\mathbf{x}) = \int_\Gamma G_{ij}(\mathbf{x}-\mathbf{y})\lambda(\mathbf{y})d\Gamma_y, \quad [i,j] \in [1,3]^2.$$

- Boundary finite elements $(\phi_n(\mathbf{x}))_{1\leq n\leq N_{dof}}$:

$$\lambda(\mathbf{x}) \sim \lambda_\phi(\mathbf{x}) = \sum_{n=1}^{N_{dof}} \lambda_n \phi_n(\mathbf{x}).$$

- Discretization of Γ with a quadrature $(\mathbf{y}_q, \gamma_q)_{1\leq q\leq N_q}$:

$$\mathcal{S}\lambda(\mathbf{x}) \sim G_{ij} \star \lambda_\phi(\mathbf{x}) = \sum_{q=1}^{N_q} \gamma_q G_{ij}(\mathbf{x}-\mathbf{y}_q)\lambda_\phi(\mathbf{y}_q).$$

- Galerkin formulation (**dense**) :

$$[S]_{a,b} = \int_\Gamma \int_\Gamma \phi_a(\mathbf{x}) G_{ij}(\mathbf{x}-\mathbf{y})\phi_b(\mathbf{y}) d\Gamma_x d\Gamma_y.$$

Final formalism with points to points interactions

Considering $\mathbf{r} = \mathbf{x} - \mathbf{y}$, final Galerkin formalism only needs the fast computation of Stokeslet and Stresslet :

$$G(\mathbf{r}) = \frac{\delta}{|\mathbf{r}|} + \frac{\mathbf{r} \otimes \mathbf{r}}{|\mathbf{r}|^3} \quad , \quad T(\mathbf{r}) = -6\frac{\mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r}}{|\mathbf{r}|^5},$$

for each particles interactions.

- **Direct method** : Computing and storing dense matrix ...
  ... $O(N^2)$ operations, impossible for $N \geq N_0$.
- **Iterative method** : Only computing dense matrix ...
  ... $O(N^2)$ operations, slow for $N \geq N_0$.
- **Fast iterative method** : Split the variables $\mathbf{x}$ and $\mathbf{y}$ in $G(\mathbf{x}, \mathbf{y})$...
  ... $O(N)$ operations, fast for $N \geq N_0$.

Example with a regular kernel

If $G(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2$, then

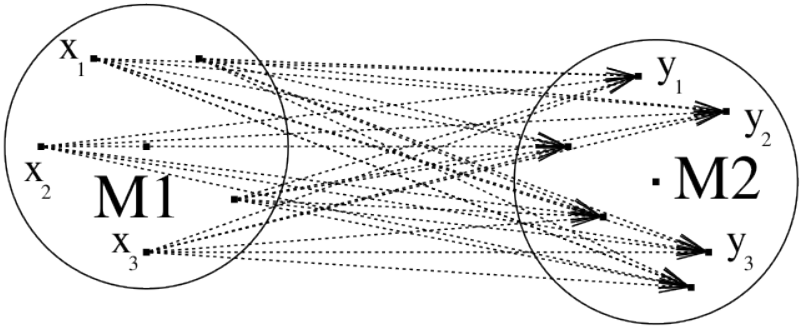
$$\forall i,\ v_i = \sum_j G(\mathbf{x}_i,\mathbf{x}_j)u_j = |\mathbf{x}_i|^2 \sum_j u_j - 2\mathbf{x}_i \cdot \sum_j \mathbf{x}_j u_j + \sum_j |\mathbf{x}_j|^2 u_j$$

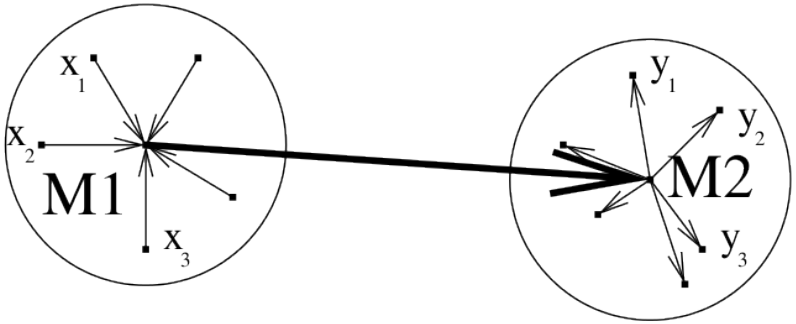
.

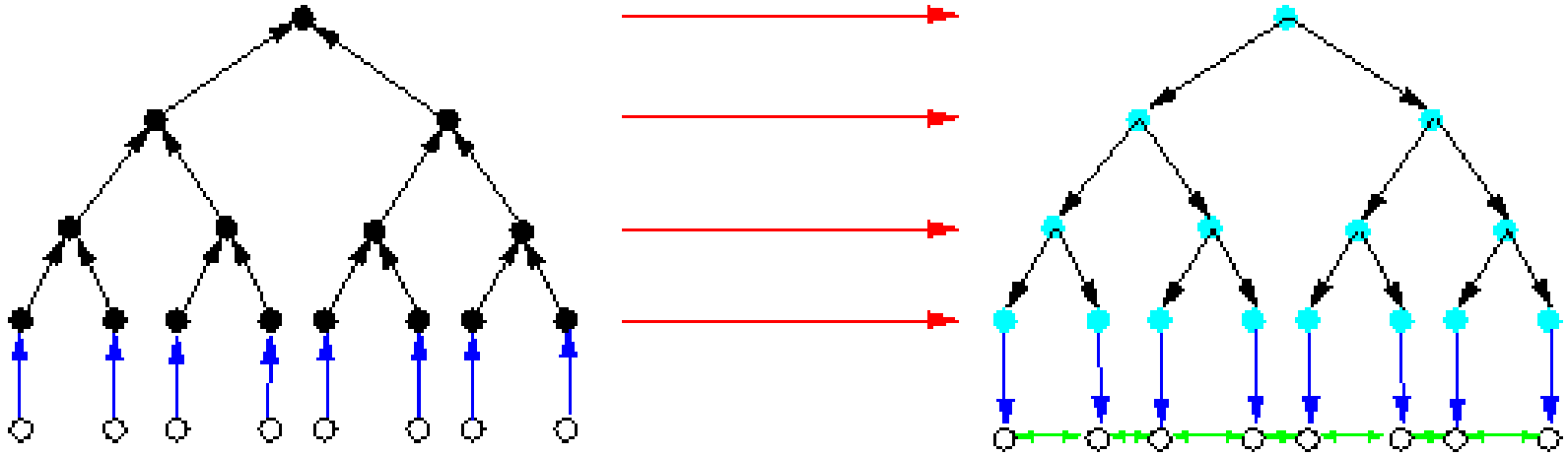
- Separation of the variables $\mathbf{x}$ et $\mathbf{y}$ $\rightarrow$ **Compression !**
- **(A lot of) numerical and technical difficulties** ...
  ... **Complexity** $\rightarrow O(N \log N)$.

The Fast Multipole Method (FMM)

Firstly introduced by L. Greengard in 1987 for Laplace kernel.



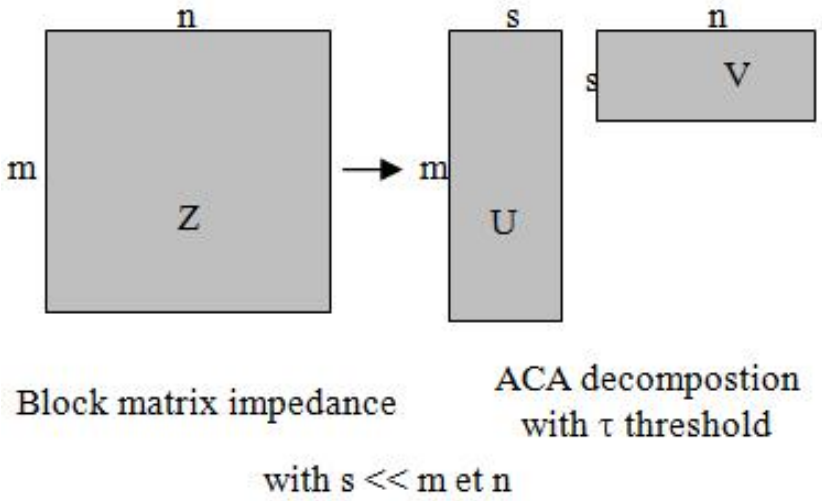


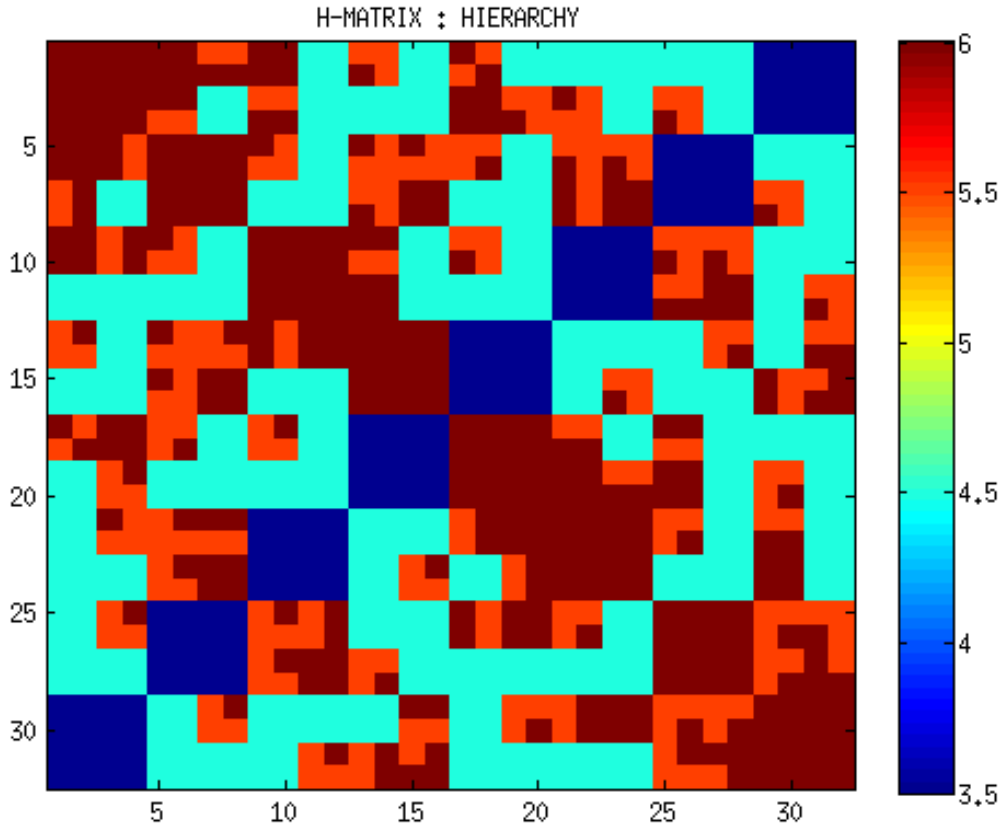


list of particles
multipole expansion
local expansion
direct neighbor interaction
multipole to local conversion
form multipole expansion
evaluate local expansion
translate multipole expansion
translate local expansion

The Hierarchical Matrices ($\mathcal{H}$-Matrix)

Firstly introduced by
W. Hackbusch in 1999





6 level hierarchy for 6 000 particles

The Sparse Cardinal Sine Decomposition (SCSD)

- Firstly introduced by F. Alouges & M. Aussal in 2014 to solve Laplace equation.
- Today extended to Helmholtz, Maxwell and Stokes problems.

Main Trick : For stokes equations, considering :

$$
\begin{aligned}
L(\mathbf{r}) &= \frac{1}{|\mathbf{r}|}, \\
\nabla L(\mathbf{r}) &= -\frac{\mathbf{r}}{|\mathbf{r}|^3}, \\
\nabla^2 L(\mathbf{r}) &= -\frac{\delta}{|\mathbf{r}|^3} + 3\frac{\mathbf{r}\otimes\mathbf{r}}{|\mathbf{r}|^5},
\end{aligned}
$$

Stokeslet $G(\mathbf{r})$ and Stresslet $T(\mathbf{r})$ are seen as linear combination of Laplace kernel $L(\mathbf{r})$ and its derivatives.

$\Rightarrow$ **Let's detail Laplace green kernel convolution with SCSD !**

Fast convolution with Cardinal Sine

- Convolution in space $\Leftrightarrow$ Product in Fourier domain.
- Fourier transform of the cardinal sine :

$$\begin{aligned}
\mathcal{F}\left(\frac{\sin(|\mathbf{r}|)}{|\mathbf{r}|}\right) &= 2\pi^2\delta_{S^2} \qquad \forall \mathbf{r} \in R^3, \\
\frac{\sin(|\mathbf{r}|)}{|\mathbf{r}|} &= \frac{1}{4\pi}\int_{S^2} e^{i\mathbf{s}\cdot\mathbf{r}} ds \qquad \forall \mathbf{r} \in \mathbb{R}^3, \\
\frac{\sin(|\mathbf{r}|)}{|\mathbf{r}|} \star \lambda_\phi(\mathbf{x}) &= \frac{1}{4\pi}\int_{S^2} e^{i\mathbf{s}\cdot\mathbf{x}} \left[\int_\Gamma e^{-i\mathbf{s}\cdot\mathbf{y}} \lambda_\phi(\mathbf{y})\, d\mathbf{y}\right] ds
\end{aligned}$$

- A Quadrature of the unit sphere S^2 $(\mathbf{s}_p; \sigma_p)_{1\leq p\leq N_p}$ leads to **fast computation**, with Non Uniform 3D Fourier Transform in space $\mathbf{z}_n$ and frequencies $\mathbf{s}_p$ (*NUFFT 3D type-III*) :

$$\mathcal{F}_f(\mathbf{s}_p) = \sum_{n=1}^{N} e^{\pm i\mathbf{s}_p\cdot\mathbf{z}_n}\ f_n.$$

- **Complexity** $O(N \log N)$ **!**

Fast convolution with Laplace Green kernel

- $L(\mathbf{r})$ is expressed as cardinal sine, using Fourier transform of the Laplace kernel :

$$\mathcal{F}(L|\mathbf{r}|) = \mathcal{F}\left(\frac{1}{|\mathbf{r}|}\right) = \hat{L}(\xi) = \frac{4\pi}{|\xi|^2} \qquad \forall(\mathbf{r},\xi) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

- **A suitable sparse quadrature** of the full Fourier space $\mathbb{R}^3$ $(\xi_p \in \mathbb{R}^3; \omega_p)_{1\leq p\leq N_p}$ is numerically obtained, strictly restricted to $|\mathbf{r}| \in [r_{min}, r_{max}]$:

$$\begin{aligned} L \star \lambda_\phi(\mathbf{x}) &= \int_{\mathbb{R}^3} e^{i\xi\cdot\mathbf{x}} \hat{L}(\xi) \left[\int_\Gamma e^{-i\xi\cdot\mathbf{y}} \lambda_\phi(\mathbf{y})\, d\mathbf{y}\right] d\xi, \\ &= \mathcal{F}^{-1}\left(\hat{L}(\xi)\mathcal{F}(\lambda_\phi \delta_\Gamma)\right). \end{aligned}$$

- NUFFT are used to evaluate each discrete convolutions, with close corrections for $|\mathbf{r}| \in [0, r_{min}]$.
- **Complexity** $\rightarrow O(N^{6/5} \log N^{6/5})$ **!**

SCSD expression for Stokeslet

$$G(\mathbf{r}) = \frac{\delta}{|\mathbf{r}|} + \frac{\mathbf{r} \otimes \mathbf{r}}{|\mathbf{r}|^3}$$

- Identity part is treated as Laplacian kernel :

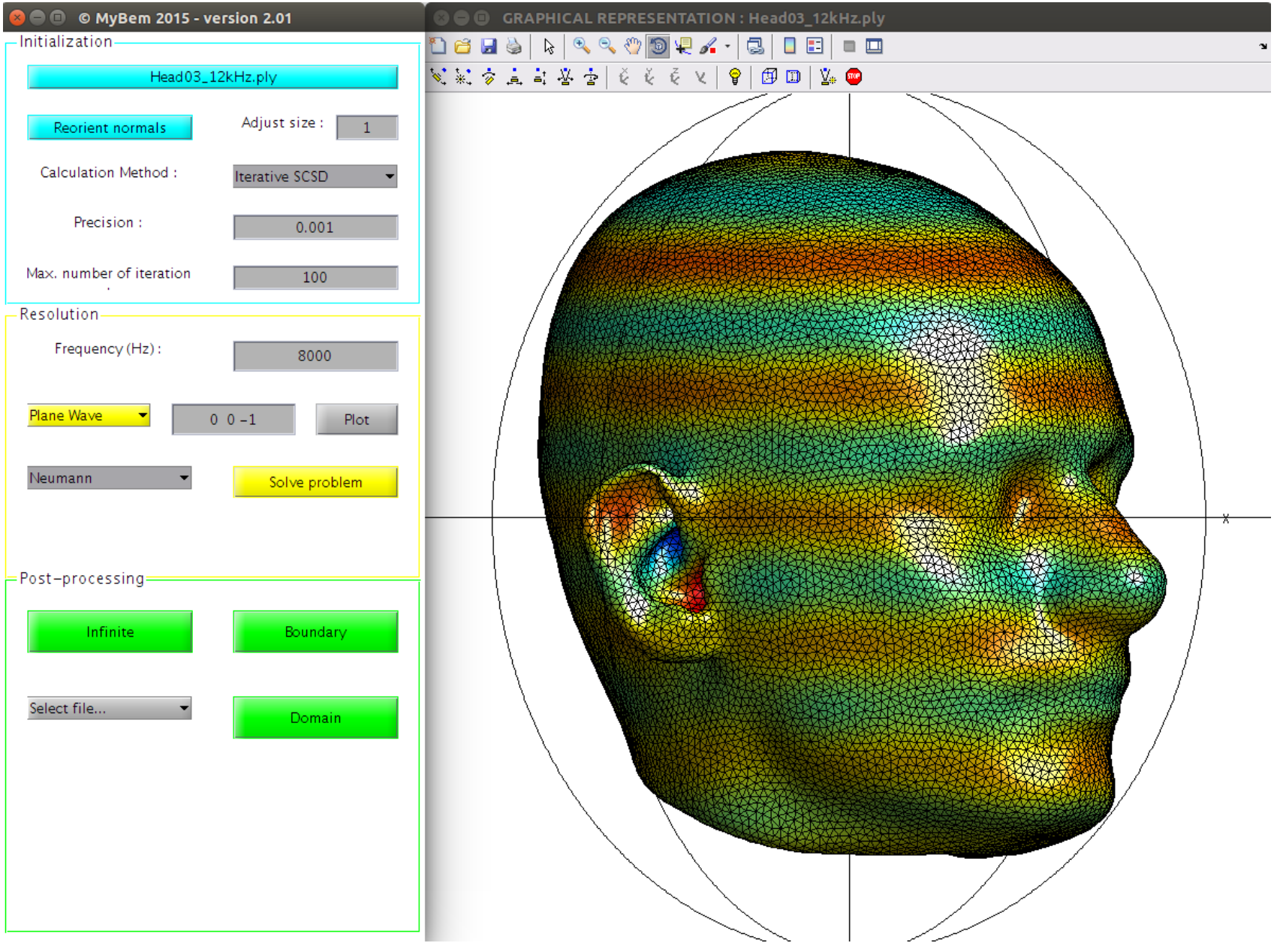
$$\frac{1}{|\mathbf{r}|} \approx \sum_{p=1}^{N_\xi} \omega_p e^{i\xi_p \cdot \mathbf{r}} \approx \sum_{p=1}^{N_\xi} e^{i\xi_p \cdot \mathbf{x}} \omega_p . e^{-i\xi_p \cdot \mathbf{y}}$$

- Tensorial part is obtained by derivation :

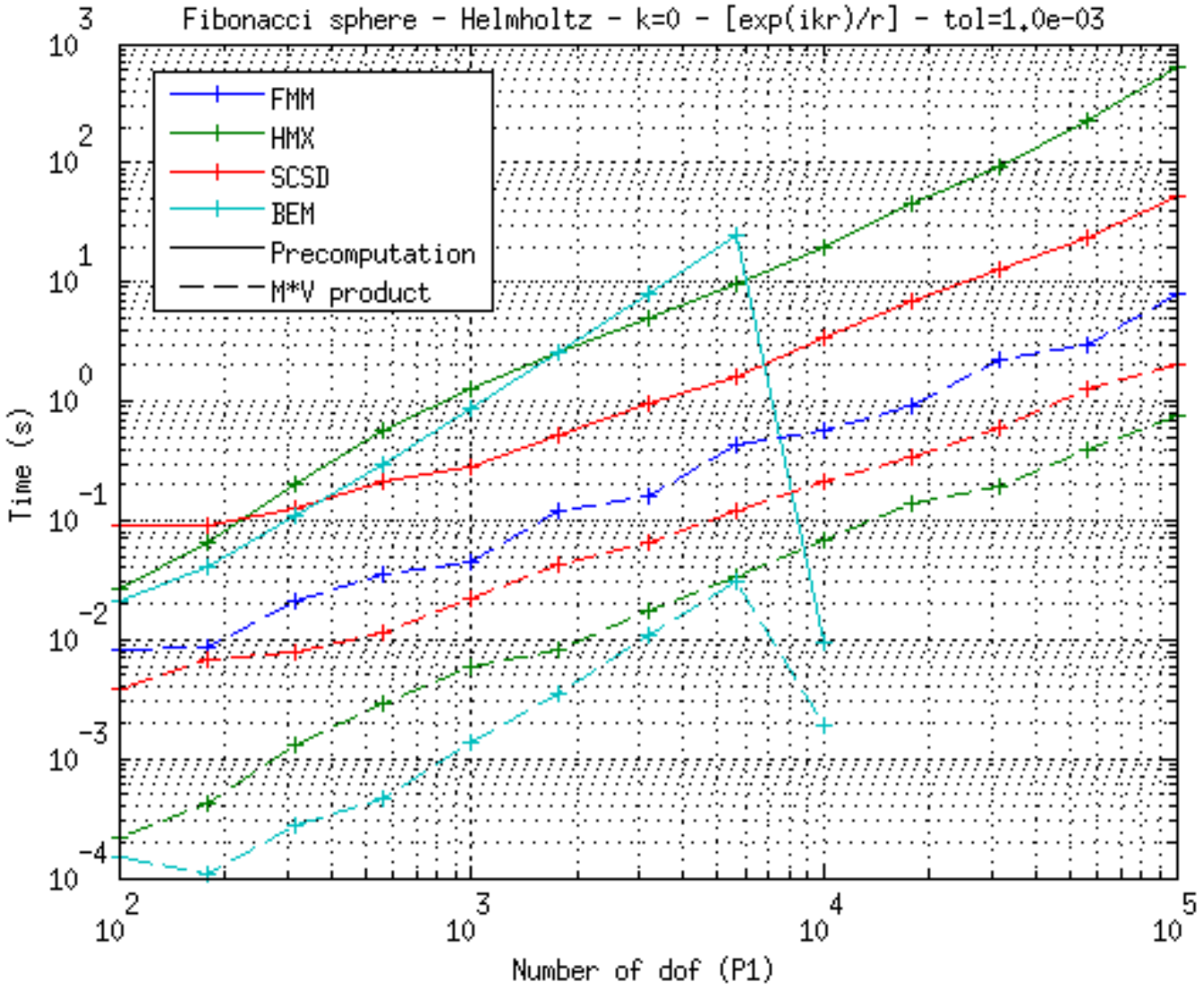
$$\frac{\mathbf{r} \otimes \mathbf{r}}{|\mathbf{r}|^3} \approx \sum_{p=1}^{N_\xi} \left(-i\omega_p \xi_p \otimes \mathbf{x}\right) e^{i\xi_p \cdot \mathbf{r}} + \sum_{p=1}^{N_\xi} \left(i\omega_p \xi_p\right) e^{i\xi_p \cdot \mathbf{r}} \otimes \mathbf{y}.$$

- **x** and **y** are well separated and NUFFT is always available for fast convolution.
- Same approach with 2nd order derivative for Stresslet.

MyBEM - A fast BEM library in native Matlab

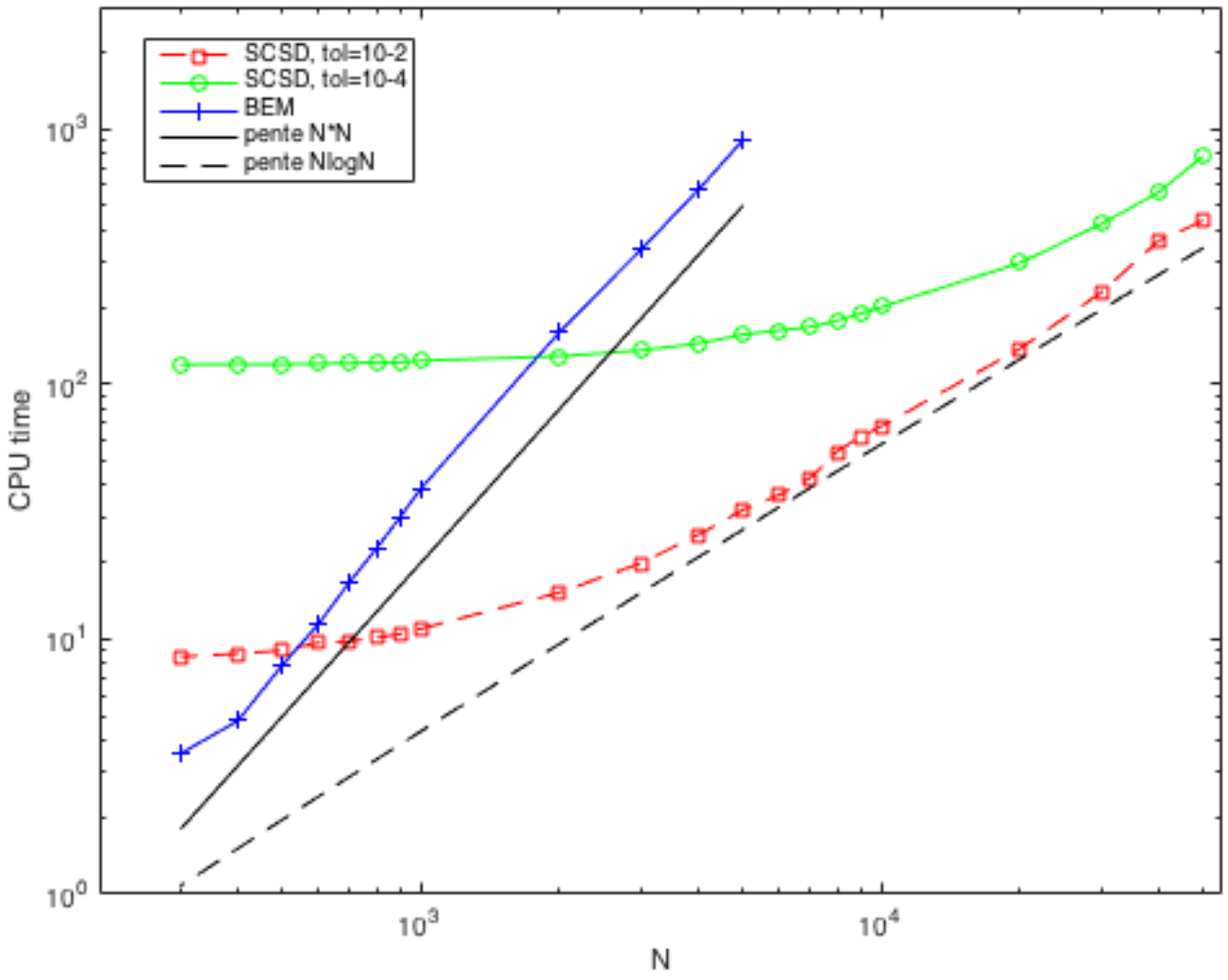


CPU time for Laplace convolution



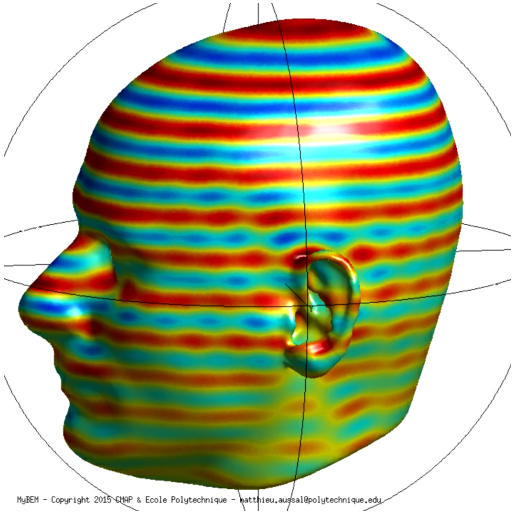
1. MyBEM library, with FMM and NUFFT from www.cims.nyu.edu/cmcl

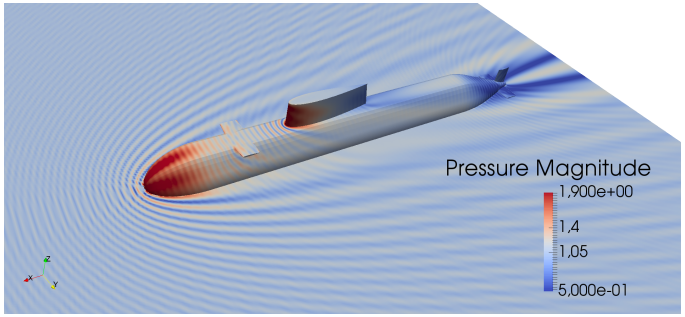
CPU time for Stokes convolution

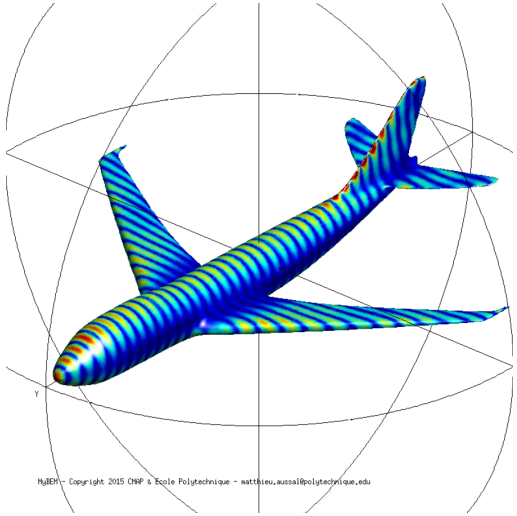


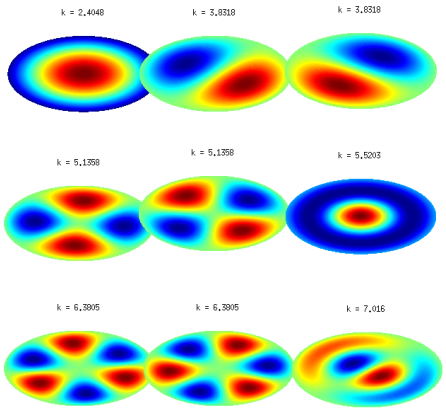
2. Joint work with A. Lefebvre-Lepot

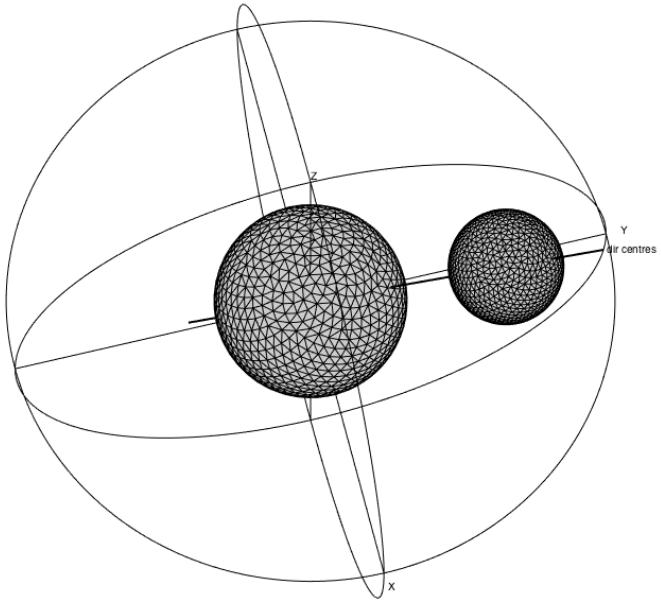
Multi-physical applications

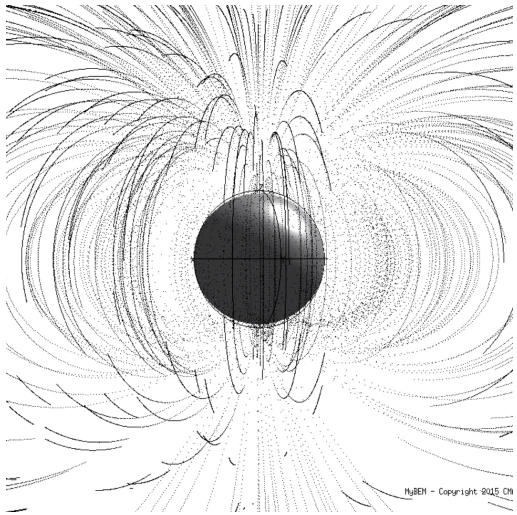












Conclusion and future works

CONCLUSION :

- New SCSD fast convolution for many equation (Laplace, Helmholtz, Maxwell, Stokes),
- Comparison with FMM (in Fortran) and H-Matrix (in Matlab),
- Creation of an object based Matlab library *MyBEM*, for fast and easy BEM prototyping,
- Analytical validation and benchmarks up to 10^6 degrees of freedom for waves equations,
- First step already done for fluids.

FUTURE WORKS :

- Comparison with others BEM codes (LadHyx for example),
- Acceleration for vectorial kernels,
- Rigid spheres and bubbles moving in a fluid (with Saint Gobain research),
- Write paper(s)...

Thanks for your attention

- Alouges, F., Aussal, M. (2015) *The Sparse Cardinal Sine Decomposotion and its application for fast numerical convolution*. Numerical Algorithms, 1-22.
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