MyBEM, a fast boundary element solver by Sparse Cardinal Sine Decomposition

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Abstract

As Fast Multipole Method (FMM), adaptive cross approximation (ACA) or $\mathcal{H}$-matrices, various algorithms for fast convolution on unstructured grids have been developed for many applications (e.g. electrostatics, magnetostatics, acoustics, electromagnetics, etc.). The goal is to reduce the complexity of matrix-vector products, from $O(N^2)$ to $O(N \log N)$.

In [1], we described a new efficient numerical method (SCSD), based on a suitable Fourier decomposition of the Green kernel, sparse quadrature formulae and Type-III Non Uniform Fast Fourier Transform (type-III NUFFT) [5,6]. This talk summarizes the approach and gives results of an application of our new open-source boundary element solver, *MyBEM*.

1 Fast formulation with SCSD

Boundary element formulations lead to the classical single layer potential expression, defined as :

$$\mathcal{S}\lambda(\mathbf{x}) = \int_\Gamma G(\mathbf{x},\mathbf{y})\lambda(\mathbf{y})d\Gamma_y, \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

where $G(\mathbf{x},\mathbf{y})$ is the Green kernel and Γ the boundary. Using a discrete quadrature of Γ, this convolution product needs a fast computation of discrete sums as :

$$G \star f(\mathbf{x}) \sim \sum_{n=1}^{N} G(\mathbf{x},\mathbf{y}_n) f_n, \qquad (1)$$

where the potential $(f_n)_{1\leq n\leq N}$ is known for all $\mathbf{y}_n$.

In the case of the tridimensional Helmholtz Green kernel, defined as :

$$G(\mathbf{x},\mathbf{y}) = \frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},$$

the imaginary part can be evaluated on the unit sphere S^2 by spherical integral representation :

$$\mathfrak{Im}\left(G(\mathbf{x},\mathbf{y})\right) = \frac{k}{(4\pi)^2}\int_{S^2} e^{ik\mathbf{s}\cdot\mathbf{x}} e^{-ik\mathbf{s}\cdot\mathbf{y}} d\mathbf{s}.$$

Since in this formula, the variables $\mathbf{x}$ and $\mathbf{y}$ are well separated, the imaginary part of the discrete Green convolution (1) can obtained by a standard quadrature $(\mathbf{s}_m;\sigma_m)_{1\leq m\leq M}$ on S^2 :

$$\mathfrak{Im}\left(G \star f(\mathbf{x})\right) \sim \frac{k}{(4\pi)^2}\sum_{m=1}^{M} e^{ik\mathbf{x}\cdot\mathbf{s}_m} g_m,$$
$$\text{with} \quad g_m = \sigma_m \sum_{n=1}^{N} e^{-ik\mathbf{s}_m\cdot\mathbf{y}_n} f_n,$$

where each sum is fastly and successively computed using a type-III NUFFT (complexity $N\log N$).

For the real part, we have proposed a quadrature rule to approximate the cosine function as sum of (dilated) sine functions (e.g. [1, 2]), enough sparse on a large interval of $k|\mathbf{x}-\mathbf{y}|$. It leads to a final quadrature $(\xi_l;\omega_l)_{1\leq l\leq L}$ of the full space $\mathbb{R}^3$, constructed as concentric spheres. The final formalism for eq. (1) is :

$$G \star f(\mathbf{x}) \sim \frac{k}{(4\pi)^2}\sum_{l=1}^{L} e^{ik\mathbf{x}\cdot\xi_l} h_l,$$
$$\text{with} \quad h_l = \omega_l \sum_{n=1}^{N} e^{-ik\xi_l\cdot\mathbf{y}_n} f_n,$$

where each sum is evaluated by type-III NUFFT. The complexity has been theoretically studied for the Green Laplace kernel in [1], numerically evaluated for Helmoltz kernel in [2], and the final mono-level algorithm goes as $N^{\frac{6}{5}}\log N$.

2 Test case, a Dirichlet problem

To evaluate the approach, a Matlab solver with Galerkin boundary element approximation has been developed, firstly for Helmholtz equation. This library, called MyBEM, provide direct BEM resolution, iterative FMM (from L. Greengard [3, 4]) and new SCSD computation. This library was parallelized, using the Matlab Parallel Toolbox, and an 8-core computer cadenced at 3GHz was used.

For this validation, analytical results from infinite spherical scattering u^∞ is compared to the numerical solution u provided by MyBEM,

SNR (dB)			*BEM*		*FMM*		*SCSD*	
N_{dof}	f (Hz)	$kr_{\max}$	SNR_2	SNR_∞	SNR_2	SNR_∞	SNR_2	SNR_∞
10^3	300	5	0.017	0.032	0.033	0.067	0.016	0.032
10^4	1000	19	0.002	0.008	0.017	0.060	0.009	0.024
10^5	3200	118	-	-	0.011	0.039	0.019	0.073
10^6	10000	368	-	-	0.021	0.120	0.014	0.090

TOTAL TIMES (s)			*BEM*	*FMM*		*SCSD*	
N_{dof}	f (Hz)	$kr_{\max}$	Time (s)	Time (s)	N_{iter}	Time (s)	N_{iter}
10^3	300	5	2.91	1.76	5	1.67	5
10^4	1000	19	162	15.7	7	8.07	7
10^5	3200	118	-	197	9	95.8	9
10^6	10000	368	-	2700	12	1400	12

FIGURE 1 – Time and accuracy comparison beetween direct BEM, FMM and SCSD computation.

and following signal to noise ratio gives the accuracy :

$$SNR_2 = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[20\log 10\left(\left|\frac{u_i}{u_i^\infty}\right|\right)\right]^2},$$

$$SNR_\infty = \max_{i\in[1,n]}\left|20\log 10\left(\left|\frac{u_i}{u_i^\infty}\right|\right)\right|.$$

In this case, a piecewise linear approximation with Brackage-Werner formulation was used to solve boundary integral equation. As shown in figure 1, MyBEM provides a good accuracy from 10^3 to 10^6 degrees of freedom N_{dof}. Moreover, the SCSD seems to be significantly faster than the FMM.

3 Conclusion

We provide a new promising fast convolution on unstructured grid method, and first results from concrete implementation in numerical solver gives good matching with well known methods as FMM. Same results obtained by MyBEM, not detailed in this abstract, are obtained for the Maxwell equations.

4 References

Références

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