#### Are asset return tail estimations related to volatility long-range correlations?

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We discuss a possible scenario explaining in what respect the observed fat tails of asset returns or volatility fluctuations can be related to volatility long-range correlations. Our approach is based on recently introduced multifractal models for asset returns that account for the volatility correlations through a multiplicative random cascade. Within the framework of these models, it can be shown that the sample size required for a correct estimation of the behavior of extreme return fluctuations is generally huge and outside the range of accessible size of data. Consequently, in many cases, the extreme tail probability appears as a power-law, with a rather small (underestimated) tail exponent. We point out that increasing the amount of data by using smaller and smaller (intraday) scales, does not contribute to reduce the bias and, as observed empirically, the tail exponent turns out to be rather stable across scales.

### I. INTRODUCTION

Heavy tails and volatility clustering are well known empirical features of asset return variations. Since a full understanding of these properties is a major challenge in quantitative finance, they are the heart of a large number of studies [1–4] The debate about the origin and interpretation of these fat tails and correlations in terms of other observables like volume or from market microstructure is still open [5–7]. Our purpose in this short paper is not to enter into such debate but to focus on tail estimations and to show in what respect multifractal fluctuations and correlations can lead to biased tail estimations. More precisely, within the paradigm of cascade models of volatility, we show that the law of extreme value may appear with an underestimated tail exponent, over a large range of scales. While not probably a very accurate model for volatility statistics, we mainly base our discussion on a log-Normal multifractal process. In that case, we show that the law of the maximum value evolves, as one goes from large to small time scales, according to a well known non-linear diffusion equation which solutions are traveling fronts. This approach allows us to obtain the asymptotic (in a sense defined below) shape of the probability law associated with the volatility extreme values.

The paper is organized as follows: In section II we discuss the problem of extreme values of log-Normal continuous cascades. We show that the law of extremes satisfies, as the observation scale and the total sample length are changed, a non-linear partial differential equation that turns out to be a KPP-like equation which solutions are well known to be traveling fronts. The selected exponential rate of these solutions allows us to determine the tail exponent of such volatility cascades. We then briefly discuss how the results for the log-Normal case can be extended to other cascade statistics. In section III we compare our theory to empirical data. We illustrate in what respect multifractal models of volatility fluctuations are able to reproduce the observed power-law distributed extreme events. Conclusion and hints for future works are provided in section IV.

#### II. EXTREME VALUES OF LOG-NORMAL VOLATILITY CASCADES

Since the pioneering work of Mandelbrot and Taylor [8], it is now well admitted that prices variations can be modeled as a subordinated Brownian motion X(t) = B(M(t)) where the stochastic time M(t) variations can be associated with the volatility. In the sequel,  $M(t,\tau) = M(t+\tau) - M(t)$  will denote the volatility at time t and scale  $\tau$ , i.e., the stochastic variance of the return variation between instants t and  $t + \tau$ . Among all the volatility models that have been proposed in the literature, the recently introduced multifractal models [9–13] are appealing models because they

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parsimoniously reproduce most of 'stylized facts' generically observed on empirical data [1, 2, 14]. In this section we will show that the volatility correlations associated with multifractal models have a strong influence on the estimated tails of the return law. More precisely, we will show that the law of extreme volatility values does not correspond to the one expected in the case returns were independent.

The simplest discrete multifractal cascade can be constructed as follows: one starts with an interval of length T (referred to as the *integral scale*) where the volatility is uniform, and split the interval in two equal parts: On each part, the volatility density is multiplied by (positive) i.i.d. random factors W. Each of the two sub-intervals is again cut into two equal parts and the process is repeated infinitely (see e.g. [15, 16] for precise definition and main mathematical properties of discrete cascades). For the sake of simplicity, we will consider in this section essentially log-Normal weights W, i.e. the case when  $\omega = \ln W$  is a Gaussian random variable.

Because the previous construction involves dyadic intervals, and a 'top-bottom' construction, it is far from being stationary. In order to get rid of this drawback and provide a causal stationary model for volatility fluctuations, some continuous cascade constructions have been recently proposed and studied on a mathematical ground [10, 17–19]. Without entering into details, we just want to mention that such continuous cascades involve a family of infinitely divisible stationary random processes  $\omega_l(t)$  (a Gaussian process for the log-Normal case) whose correlation function  $\rho_l(t)$ reproduces the ultrametric structure of discrete cascades down to lag *l*. In the log-Normal case, it is basically of the form [10, 18]

$$\rho_l(t) \simeq \begin{cases} \lambda^2 \ln(T/t) & \text{for } l \le t \le T \\ 0 & \text{for } t \ge T \end{cases}$$
(1)

where l is the smallest construction scale ("infrared" cut-off), T is the decorrelation time (the integral scale) and  $\lambda^2$  is the *intermittency factor* which measures how far from the deterministic function t the processes  $M_l(t)$  are. The final multifractal process M(t) is then obtained as the limit when  $l \to 0^+$  of the process  $M_l(t) = \int_0^t exp(\omega_l(u))du$ . It can be proved [15, 19] that the probability law of the limit process M(t) can have fat tails. However, on the one hand, the tail exponent value associated with the multifractal parameters estimated from financial time series is not compatible with empirical observations (for example, within the log-Normal model, one finds  $\mu = 2/\lambda^2$  leading to a theoretical tail exponent for return probability distribution close to ten times the observed value) and on the other hand, if one does not reach the limit  $l \to 0$ , this theoretical exponent can be even larger (in the log-Normal case, if the construction cut-off l is finite, the volatility probability distribution is indeed decreasing faster than any power law). In the following, we will show that the observed (i.e. estimated) tail exponent turns out to be underestimated because of finite size sample size.

Let us notice that though it is exactly 0 at a lag greater than T, the covariance function  $\rho_l(t)$  (Eq. (1)) has a very slow decay. It is therefore natural to wonder if the asymptotic classical results of extreme value theory [20, 21] established for independent random variables (or under so-called mixing conditions) are reached in the multifractal framework, and, in the case they are, how fast they are reached. In order to answer this question, let us first remark that there are *several ways* to reach the asymptotic limit of infinite number of observations. Indeed, if L is the length of the whole sequence,  $\tau$  the sampling scale and T the integral scale, one can introduce the exponent  $\chi$  as:

$$N_T \sim N_\tau^{\chi} \tag{2}$$

where  $N_T = \frac{L}{T}$  and  $N_{\tau} = \frac{T}{\tau}$ , are respectively the number of integral scales and the number of observations per integral scale. The exponent  $\chi$  controls the relative importance of these two numbers in the asymptotic regime  $N = N_T N_{\tau} \to +\infty$ . If, for instance,  $\chi = 0$ , it means that we are in the case  $\tau \to 0$  (*L* is fixed) while if  $\chi = +\infty$ , it means that we are in the case  $L \to +\infty$  (the observation scale  $\tau$  being fixed). A finite  $\chi$  value interpolates between these two extremes. As shown in ref. [22] and discussed below, the observed law of extreme volatilities continuously depends on the exponent  $\chi$ .

In ref. [22], we have shown that the law of the maximum value associated with a discrete cascade satisfies a non linear iteration equation involving the law of the cascade weights  $\omega_l$ . Instead of reproducing the full analysis of [22], we will consider the analog of this iteration for a continuous log-Normal cascade. This analogy can be obtained along different paths: one can use a replica method on the continuous log-correlated model as in ref. [23] or as in ref. [24], use a proxy of a continuous cascade by a 'Poissonization' trick. Heuristically, one can also directly consider the continuous limit of the discrete iteration of [22]. All these methods lead to the same evolution equation that can be described as follows: Let  $P(x, s, \chi)$  be the cumulative distribution function (cdf) of the maximum value of the volatility logarithm at scale  $\tau = Te^{-s}$  over a sample length  $L = Te^{s\chi}$ :

$$P(x, s, \chi) = Proba\left(\max_{0 \le t \le e^{s\chi}} \ln M(t, e^{-s}) < x\right).$$
(3)

Then by performing an infinitesimal transformation  $s \to s + \epsilon$ ,  $\epsilon \ll 1$ , one can establish that  $P(x, s, \chi)$  satisfies the nonlinear evolution equation in the limit  $s \to +\infty$  ( $\tau \to 0$ ) [23, 25]:

$$\frac{\partial P}{\partial s} = \frac{\lambda^2}{2} \left( \frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial x^2} - P^{-1} (\frac{\partial P}{\partial x})^2 \right) + (1+\chi) P \ln P .$$
(4)

The diffusion term comes from the addition to the log-volatility, as one changes the scale, by an infinitesimal Normal random variable while the non-linear term comes from the fact that one have to consider the maximum over a larger number of variables as one goes to smaller scales <sup>1</sup>. Notice that the extension of this equation to a general infinitely divisible cascade amounts to replace the "Fokker-Planck" diffusion terms by general "master equation" terms. Up to the nonlinear gradient term  $P^{-1}(\frac{\partial P}{\partial x})^2$ , equation (4) is exactly a well known KPP equation [23]. It can be transformed into a linear one by setting  $H(x, s, \chi) = -\ln P(x, s, \chi)$ . By considering traveling front type solutions [22, 23], i.e.,

$$H(x,s,\chi) = F\left(x - (\beta - \lambda^2/2)s\right)$$
(5)

where  $\beta - \lambda^2/2$  is the front "velocity", one gets the differential equation:

$$-\beta F'(x) = \frac{\lambda^2}{2} F''(x) + (1+\chi)F(x)$$
(6)

If one seeks for exponential solutions  $F(x) = e^{qx}$ , we obtain the "dispersion relationship":

$$\beta(q) = -\frac{1+\chi}{q} - \lambda^2 \frac{q}{2} \tag{7}$$

A velocity selection criterion can be obtained using a standard stability analysis [22, 23]. Provided mild conditions on the initial condition, the selected velocity can be shown to correspond to the maximum of  $\beta(q)$ , i.e., to a value of  $q_*$ :

$$q_{\star} = \sqrt{\frac{2(1+\chi)}{\lambda^2}} \tag{8}$$

For a continuous log-Normal cascade, we have thus established that, if N is the total sample length, there exists some value  $\alpha = \beta(q_{\star}) - \lambda^2/2$  such that, when  $N \to +\infty$  (i.e.,  $s \to +\infty$ ), the law of the maximum of the normalized volatility  $\tau^{\alpha} M(t,\tau)$  is a Frechet law<sup>2</sup> with a tail exponent  $\mu = q_{\star}$  that depends on  $\chi$  (Eq. (8)).

In order to interpret these asymptotic results more precisely and notably to describe the shape of the tail of the observed cdf of M on a finite sample, an additional analysis of finite size effects would be necessary. Such a analysis can be partially done one the ground of multifractal analysis [22]. Using this formalism, one can predict that (i) the estimated tail exponent of the volatility is  $\mu = q_{\star}$  as given by Eq. (8) but also that (ii) the scaling range over which the scaling regime extends is of magnitude order  $(T/\tau)^{\sqrt{2\lambda^2}}$ . That simply means that, for small enough observation scale, the volatility statistics of log-Normal cascades, may empirically appear as power-law distributed with a rather small exponent and over a wide range of values. It is important to notice that when  $L \simeq T$  is fixed,  $\chi \simeq 0$ , and the predicted value of the exponent does not depends on the observation scale  $\tau \ll T$ . Increasing the sample size by considering finer return horizons  $\tau$  does not change the tail estimation. This kind of "stability" properties over time aggregation has been observed empirically. All these results can be generalized to arbitrary log-infinitely divisible multifractal statistics like e.g. log-Poisson or log-Gamma [22]. In the later case,  $\omega$  is Gamma distributed, i.e., of probability density proportional to  $\omega^{\lambda^2 \beta^2} e^{-\beta \omega}$ , Eq. (8) becomes:

$$q_{\star} = \beta \left[ 1 - e^{1 + W_{-1} \left( -e^{-1 - \frac{(1+\chi)}{\lambda^2 \beta^2}} \right) + \frac{(1+\chi)}{\lambda^2 \beta^2}} \right] \underset{\beta \to +\infty}{\sim} \sqrt{\frac{2(1+\chi)}{\lambda^2}} - 4 \frac{(1+\chi)}{3\beta\lambda^2} + \dots$$
(9)

where  $W_{-1}$  is the second branch of the Lambert W function. Let us remark that when  $\chi \to \infty$ , one recovers the unconditional tail exponent for log-Gamma  $e^{\omega}$  cascade weights,  $q_{\star} \to \beta$ . Unlike the log-Normal case, the theoretical tail exponent can be rather small, but as illustrated in the next section, still large as compared to the estimated one.

<sup>&</sup>lt;sup>1</sup> In fact, a detailed computation [23, 25] shows that the non-linear term is trickier than the one of Eq. (4) but the linear analysis, sufficient for the level of description adopted in this paper, leads to the same results

<sup>&</sup>lt;sup>2</sup> The Frechet law is of the form  $\exp(-x^{-\mu})$ . The parameter  $\mu$  is the tail exponent since, in the limit  $x \to \infty$ ,  $F(x) \sim 1 - x^{-\mu}$ 



FIG. 1: Rank-Frequency (Zipf) plots in log-log representation for a log-Normal cascade (top) and a log-Gamma cascade (bottom). The slope of the right linear part provides an estimate of the tail exponent  $\mu$ . Dashed lines indicate analytical expectations (see text).

# III. APPLICATION TO EMPIRICAL OBSERVATIONS

Let us compare the previous theoretical analysis to empirical observations. As briefly recalled in the introduction, multifractal models for volatility fluctuations have been considered by many authors in the recent literature. Even if the precise nature of the multifractal statistics that fits better market returns is still matter of discussion, the multifractal scaling exponents and correlations have been considered by many empirical studies (see e.g. [9, 10, 12, 13]). One of the key points raised in refs. [10, 26] is that these volatility correlations are found empirically very close to the "log-correlations" of continuous multifractal cascades (Eq. (1)). It notably appears from these studies that the integral scale T is very large with a typical value of one to few years. Since heavy tails of asset return cdf are generally studied on time scales ranging from few minutes to days [2, 27] with sample length of few years, it can be considered that  $\chi \simeq 0$ . If one refers to previous analysis, this notably implies that the observed tail exponent does not depend on the considered return horizon. Such a "stability" property has been confirmed empirically and the famous "inverse cubic" law for return fluctuations, according to which the return tail exponent is around 3, 4, has been observed on few minutes as well as daily returns [5, 27]. Let us note that, because the volatility  $M(t, \tau)$  is simply a mean square return, the expected tail exponent for the volatility is half the return exponent, i.e.,  $\mu \in [1.5, 2]$ .

It is noteworthy that the theoretical tail exponents of the log-Normal or the log-Gamma volatility models themselves (resp., in the case the limit  $l \to 0^+$  is not reached,  $\mu = \infty$  and  $\mu = \beta$ ) would be obtained in the "asymptotic" regime where most of the data used for the estimation were decorrelated : in the case where  $\chi \to \infty$  in Eq. (2). In the case of financial time series, "most" of the data comes from scales smaller than the decorrelation scale T ( $\chi \simeq 0$ ), and with the above order of magnitude of  $\lambda^2$ , Eqs. (8), and (9) gives respectively  $\mu \simeq 3.15$  and  $\mu \simeq 2$  for the log-Normal and log-Gamma ( $\beta = 4$ ) cascades. In the log-Gamma case, the value is very close to previously mentioned empirical values. Therefore, given the time scales involved in the empirical data, one is far from the "asymptotic" regime where the theoretical tail exponent of the model can be estimated. Though the "inverse cubic" law is well fitted by our model, it does not necessarily correspond to the tail exponent of the 1-point volatility distribution. From a practical point of view, the estimation of this theoretical exponent is almost impossible. Indeed, in order to perform such estimation, one would need to use data over length scales such that  $\chi$  satisfies  $\chi \gg 1$ . In the best case, if we consider that T = 1 year and  $\tau = T/250$  (i.e., daily data), the value  $\chi = 1$  is already unreachable since it corresponds to a time series length of the order of 250 years !

In Fig. 1, the rank ordering of a log-Normal and a log-Gamma (with  $\beta = 4$ ) cascade processes with an intermittency coefficient  $\lambda^2 = 0.2$  and at scale  $\tau/T \simeq 10^{-4}$  are plotted in doubly logarithmic scale (such plots are often referred to



FIG. 2: Rank-Frequency plot of CAC40 daily volatility estimates (•) as compared to similar plot for a log-Gamma continuous cascade with T = 253 days,  $\lambda^2 = 0.2$  and  $\beta = 4$  (thin line). The log-Gamma plot has been slightly shifted for illustration purpose. A least square fit of the extreme tail provides an estimation  $\mu = 2.1 \pm 0.2$  for both plots. In the inset, one can see that a 'logQ-logQ' plot (*bullet*) of CAC40 volatility distribution versus log-Gamma volatility model is very close to the diagonal (continuous line).



FIG. 3: Estimated tail exponent as a function of estimated intermittency coefficient for a basket of future contracts over 10 years. The dashed line represents the theoretical prediction (8) with  $\beta = 4$ .

as "rank-frequency" plots or "Zipf" plots [28]). In this representation, the linear rightmost part of each distribution is associated with the extreme power-law part of the distribution. A least square fit of the extreme points provides respectively the values  $\mu = 2.9 \pm 0.2$  in the log-Normal case and  $\mu = 1.9 \pm 0.2$  in the log-Gamma case (where the error corresponds to the observed slope variation when the number of extreme points varies from 10 to  $10^2$ ), in good agreement with the exponent values predicted by Eqs. (8,9), reported as dashed lines in the figures. One can also observe that the scaling range associated with the log-Gamma cascade is wider than for the log-Normal measure. This can be analytically proved using the multifractal formalism [22]. Let us mention that these plots are "typical" just given for illustrative purpose and we refer the reader to ref. [22] for a study of tail exponent estimators, like Hill or Pickands, for cascade processes.

In order to provide a qualitative comparison to real volatility distribution, in Fig. 2 we have reported a rankfrequency plot of estimated daily volatilities associated with the 40 stock values composing to the French CAC 40 index. The data are daily 'open' 'high' 'low' 'close' quotes over a mean period of 10 years. The daily volatilities are estimated using the widely used Garman-Klass method [29] and each volatility sample mean has been normalized to 1. For comparison the Zipf plot associated with a multifractal log-Gamma measure with T = 1 year,  $\lambda^2 = 0.2$  and  $\beta = 4$ has been also reported. One can see that both curves behave very similarly with a power-law tail exponent  $\mu \simeq 2$ . Let us note that for small volatilities values, the behavior of CAC40 volatility pdf is slightly different from the log-Gamma cascade probably because of the hight frequency noise in the Garman-Klass volatility estimates. A precise comparison of cascade models to return distributions of various markets, over different time horizons is out of the scope of this paper and will be considered in a forthcoming study [25]. This figure illustrates very well that the observed tail exponent behavior, is for a large part, dependent on the cascading nature of the volatility and not only on the volatility unconditional probability unconditional distribution.

In order to study the possible dependence of the tail exponent as a function of the intermittency coefficient  $\lambda^2$ , we have studied a set of 33 future contracts from 1994 to 2004. This set constitute a quite heterogeneous basket of assets since it includes interest rates, FX rates, Stock index rates, energy, metal and agricultural commodities. In fig. 3

we have plotted the estimated tail exponent as a function of the estimated intermittency coefficient. The plot looks very erratic because the noise in estimated coefficient is large as respect to the observed domain of variation of the parameters. However it seems that there is a trend according to which  $\mu$  is a decreasing function of  $\lambda^2$  as predicted by Eqs. (8) and (9). In dashed line, we have superimposed the analytical prediction associated with a log-Gamma cascade (with  $\beta = 4$ ).

## IV. CONCLUSION AND PROSPECTS

In this paper, we have discussed, within the paradigm of continuous cascades, some problems related to estimation of the statistics associated with extreme returns. In the case when the statistical sample involves a sampling scale much smaller than the correlation time-scale (the integral scale T) and, at the same time, has a length which is of the order of this correlation time-scale, it has been shown that the observed extreme statistics are far from the theoretical statistics obtained for independent observations. In other words, the tail exponent estimators lead to highly biased (underestimated) values. Moreover, we have also emphasized that such property is stable as respect to time aggregation: the estimated tail exponent does not depend on the time scale.

The previous statistical conditions are satisfied by usual samples of financial time series where volatility correlation time is one or few years, the observation scale ranges from minutes to few hours and the total sampling window is also few years. One therefore expects that, to a large extent, the observed inverse cubic law of return pdf, is certainly not exclusively originating from the unconditional small scale volatility statistics but is also biased by the volatility longrange correlations <sup>3</sup>. Thus, though the "effective" risk (i.e., the estimated tail exponent) is certainly well described by the observed cubic law, it does not correspond to the "theoretical" asymptotic risk (i.e., the theoretical tail exponent) of the underlying model. One should not try to fit the parameters of the model in order to match the theoretical tail exponent with the estimated tail exponent. In particular, rejecting a model on the ground of the extreme value behavior can be misleading.

In a future work, we will try to use intraday data to test more precisely the predictions of our approach. Moreover, since the results presented here are mainly of asymptotic nature, it is also interesting to study finite size corrections to scaling. Finally, we plan to extend to study a such "non-ergodic" effects to other observables than extreme values, like e.g. finite order moments or correlations functions associated with return variations.

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 $<sup>^{3}</sup>$  Let us note that this situation is somehow analogous to the one observed in the physics of disordered systems at low temperature where the pertinent fluctuations are those of the "frozen" disorder which correlations can be of long-range nature [23].

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