

Random cascade models in the limit of infinite integral scale as the exponential of a non stationary $1/f$ noise. Application to volatility fluctuations in stock markets

Jean-François Muzy* and Rachel Baïle
SPE UMR 6134 CNRS, Université de Corse, Quartier Grossetti, 20250 Corte, France

Emmanuel Bacry
CMAP UMR 7641 CNRS, Ecole Polytechnique, 91128 Palaiseau, France
(Dated: April 23, 2012)

In this paper we consider the problem of the existence of some large correlation (integral) scale in random cascade models. We propose a new model that possesses multifractal properties without involving any integral scale. This model relies on a non stationary log-normal process which properties, over any finite time interval, are very close to continuous cascade models. These latter models are notably well known to reproduce faithfully the main stylized fact of financial time series but the integral scale where the cascade is initiated is hard to interpret. Moreover the reported empirical values of this large scale turn out to be closely correlated to the overall length of the sample. As illustrated by the example of Dow-Jones index, this feature is precisely predicted by our model.

PACS numbers: 05.45.Df, 05.40.-a, 02.50.Ey, 05.45.Tp, 89.65.Gh

For several decades, random cascade models have been at the heart of a wide number of studies in mathematics as well as in applied sciences. They were introduced to account for the intermittency phenomenon in fully developed turbulence and are involved every time one observes a multifractal behavior in the variations of statistical properties of some field across different scales. In a loose mathematical sense, a stochastic measure $M(t) = M([0, t])$ [1] is called multifractal if the moments of its increments $M(t+\tau) - M(t) = M([t, t+\tau])$ (assumed to be stationary) display some multiscaling properties, i.e.,

$$\mathbb{E}[M([t, t+\tau])^q] = \mathbb{E}[M(\tau)^q] \sim C_q \tau^{\zeta(q)}, \quad (1)$$

where $\zeta(q)$ is a non linear function of the moment order q . In order to quantify the multifractality one often defines the so-called *intermittency coefficient* λ^2 as the curvature of $\zeta(q)$ around $q = 0$, $\lambda^2 = -\zeta''(0)$. The scaling behavior of Eq. (1) is generally interpreted in the limit of small time scales $\tau \rightarrow 0$. Accordingly, if one computes the kurtosis behavior, $\mathcal{F}(\tau) = \frac{\mathbb{E}[M(\tau)^4]}{\mathbb{E}[M(\tau)^2]^2} \sim \tau^{\zeta(4)-2\zeta(2)}$, one directly sees that, because $\mathcal{F}(\tau) \geq 1$, one must have $\zeta(4) - 2\zeta(2) < 0$. This argument can be generalized (thanks to Hölder inequality) to prove that $\zeta(q)$ is concave (and then $\lambda^2 > 0$). It also implies that Eq. (1) cannot hold for arbitrary large scales τ and there exists an *integral scale* T beyond which multiscaling ceases to be valid. Multifractal scaling is generally associated with the existence of a random cascade by which small scale structures are constructed from the splitting of larger ones and multiplication by a random factor. One clearly sees that such a scenario necessarily involves a largest scale T where the cascade is initiated. In most applications where cascade models appear to be pertinent, one can identify such a scale: in turbulence for instance, it

corresponds to the injection scale, i.e., the time/space scale where kinetic energy is injected into the flow [2].

Under the impetus of early studies of Mandelbrot and his collaborators [3], the notions of multifractals and random cascades have been proposed to account for the volatility dynamics in many studies of financial time series (see e.g. [4–8]). Volatility is one of the most important risk measures in finance since it corresponds to the conditional variance associated with price fluctuations at any time t [9]. The continuous random cascades [10] and in particular the MRW model, provide a parsimonious class of random processes that reproduces very well most of stylized facts characterizing the price return fluctuations [6, 9]. Within this framework, various empirical estimations reported so far indicate that the value of T can vary from few months [4, 5] to several years [11]. Even if it is well admitted that a precise estimation of T can be hardly achieved [11, 12], one can wonder why one observes such a large range in the estimated integral scale values. Beyond the problem of the determination of T , a challenging question remains to understand the *meaning* of the integral scale in finance. Unlike turbulence, there is no natural large scale that would obviously appear to be associated with some “source of volatility”.

We propose in this paper a random cascade model without any integral scale built as the exponential of a non stationary $1/f$ noise. The model is such that every single trajectory, for each finite time interval, can hardly be distinguished from a multifractal process where the integral scale is precisely the time interval size. Our approach is based on an extension of the construction proposed in Refs [10, 13]. We introduce a new random measure (representing e.g. the volatility) whose logarithm is a non-stationary gaussian process that can be interpreted as a fractional Brownian motion in the limit of Hurst parameter $H \rightarrow 0$. We show that this process is well defined

in the sense of distributions and cannot be distinguished from a continuous cascade model (as the MRW process defined in [6]) over any finite time interval.

Let us recall that a discrete random cascade measure $dM(t)$ is constructed by starting with an interval of length T where $dM(t)$ is uniformly spread (meaning that the density is constant) and by splitting this interval in two equal parts: On each sub-interval, the density is multiplied by (positive) i.i.d. random factors W and this step is repeated recursively towards smaller scales ad infinitum. More recently, continuous cascade constructions have been proposed [6, 10, 13, 14] as stationary, continuously scale invariant, alternatives to discrete constructions. These models can be viewed as a “densification” of the discrete case where the multiplication along the dyadic tree associated with successive fragmentation steps, $dM = \prod_i W_i = e^{\sum_i \ln(W_i)}$, is replaced by the exponential of some integral (instead of a discrete sum) of a white noise $d\xi$ (instead of $\xi_i = \ln(W_i)$) over a cone-like domain in the time-scale plane [10, 13] (instead of the tree node set). The process $\omega_{l,T}(t) = \ln \frac{dM_{l,T}(t)}{dt}$ is defined as

$$\omega_{l,T}(t) = \mu_{l,T} + \int_{(u,s) \in C_{l,T}(t)} d\xi(u,s) \quad (2)$$

where $\mu_{l,T}$ is a constant such that $\mathbb{E}[e^{\omega_{l,T}(t)}] = 1$, $d\xi(u,s)$ is a Gaussian white noise of variance $\lambda^2 s^{-2} dt ds$ [15], and $C_{l,T}(t)$ is the cone-like domain of $\mathbb{R}^+ \times \mathbb{R}_*^+$:

$$(u,s) \in C_{l,T}(t) \iff \{s \geq l, t - \min(s,T) \leq u \leq t\} \quad (3)$$

The covariance of $\omega_{l,T}(t_1)$ and $\omega_{l,T}(t_2)$ is the area of $C_{l,T}(t_1) \cap C_{l,T}(t_2)$. It is easy to check that it decreases as a logarithmic function:

$$\text{Cov}[\omega_{l,T}(t), \omega_{l,T}(t + \tau)] = \lambda^2 \ln\left(\frac{T}{\tau}\right) \quad (4)$$

and vanishes for $\tau \geq T$. This feature has been shown to directly reflect the tree structure of discrete random cascades (see Refs [5, 16]). The final log-normal multifractal measure dM_T is then obtained as the weak limit of $dM_{l,T}$ when $l \rightarrow 0$, i.e., $M_T(t) = \lim_{l \rightarrow 0} \int_0^t dM_{l,T}(t) = \lim_{l \rightarrow 0} \int_0^t e^{\omega_{l,T}(u)} du$. All the scaling properties of $M_T(t)$ can be shown to result from the logarithmic nature of the auto-covariance of $\omega_{l,T}$.

The main scope of this paper is to consider a variant of previous construction in order to get rid of any integral scale. For that purpose, let us define a process $\omega_l(t)$ as in Eq. (2), but the integration is performed over a *time-dependent* cone-like domain in a time-scale plane, where, at time t , the integral scale parameter T in Eq. (3) is precisely t : $C_l(t) = C_{l,t}(t)$. This construction is illustrated in Fig. 1. By choosing a time-dependent appropriate mean value of $\omega_l(t)$, one can use a martingale argument as in Ref. [10] to establish the existence of a

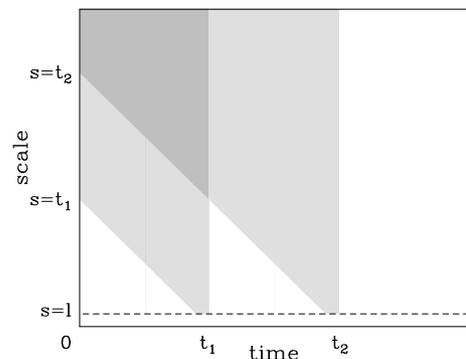


FIG. 1. Construction of the non-stationary $\omega_l(t)$ process as the integral of a white noise over a cone-like domain $C_l(t)$ in the time-scale plane. The covariance of $\omega_l(t_1)$ and $\omega_l(t_2)$ is the area of the intersection $C_l(t_1) \cap C_l(t_2)$

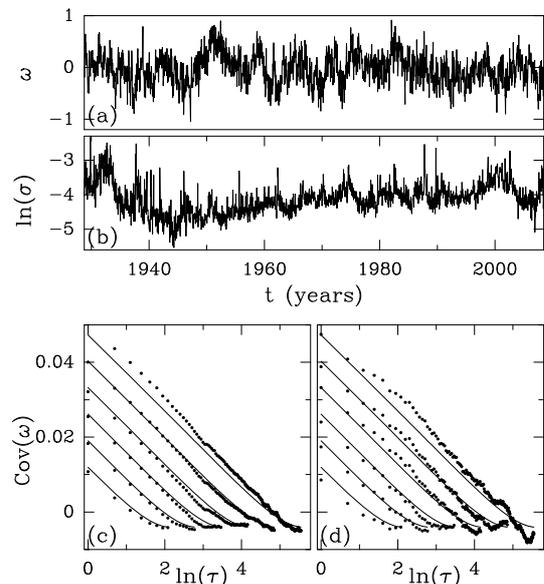


FIG. 2. (a) Sample path of $\omega_l(t)$ of length $2 \cdot 10^4$ where the numerical construction has been performed by sampling in both space and scale the cone-like sets $C_l(t)$ defining $\omega_l(t)$. (b) Magnitude estimated as $\ln(\sigma(t))$ where $\sigma(t)$ is the daily range (difference between highest and lowest daily return values) associated with the Dow-Jones time series from 1929 to 2007. In (c) and (d) are reported the magnitude auto-covariance estimation for different sample sizes in semi-logarithmic scale $\Delta t = 16, 32, 64, 128, 256$ and 512 (from bottom to top). (c) Estimation from the model sample path in (a). (d) Estimation from the Dow-Jones daily data in (b). The solid lines represent the theoretical predictions from Eq. (7).

non trivial limit process $M(t) = \lim_{l \rightarrow 0} \int_0^t e^{\omega_l(u)} du$. In Fig. 2(a) we have plotted a sample of $\omega_l(t)$ generated at rate $\delta t = l = 1$ over $2 \cdot 10^4$ points. As one can see, the non-stationary nature of $\omega_l(t)$ is not obvious but one can observe departures from the mean value that can extend over arbitrary long time intervals. We can compute the auto-covariance of $\omega_l(t)$ that corresponds to the C_l inter-

section areas (Fig. 1). For $t_1 \leq t_2 = t_1 + \tau$, its expression reads (for $\tau > l$):

$$\text{Cov}[\omega_l(t_1), \omega_l(t_2)] = \lambda^2 \ln\left(\frac{t_2}{\tau}\right). \quad (5)$$

Although $\omega_l(t)$ is clearly a non stationary gaussian process, its auto-covariance has striking similarities with the stationary situation (Eq. (4)) where the integral scale T has been replaced by the current time t (or $\max(t_1, t_2)$). Remark that this non-stationarity is reminiscent of an aging behavior as observed in off-equilibrium relaxing systems [17] where the ‘‘age’’ of the process t controls the characteristic correlation length. The increments of $\omega_l(t)$, $\delta_h \omega_l(t) = \omega_l(t+h) - \omega_l(t)$ ($h > l$) has a time dependent variance so it is not stationary but can be shown to have a power-spectrum such that $P_{\delta_h \omega}(f) \sim |f|$ when $f \ll h^{-1}$. Since $P_\omega(f) \sim f^{-2} P_{\delta_h \omega}(f)$, it results that $\lim_{l \rightarrow 0} \omega_l(t)$ has a $1/f$ power-spectrum and can be interpreted as the limit $H \rightarrow 0$ of a fractional Brownian motion (fBm) $B_H(t)$ of Hurst parameter H [18]. This analogy can also be seen from the self-similarity property of $\omega_{l \rightarrow 0}$. Indeed, from the auto-covariance expression (5) and the definition of $M(t)$, one can establish the following invariance relationships for $\omega_l(t)$ and $M(t)$:

$$\omega_{rl}(rt) =_{\text{law}} \omega_l(t) \Rightarrow M(rt) =_{\text{law}} rM(t). \quad (6)$$

The left equality extends to $H = 0$ the fBm self-similarity $B_H(rt) =_{\text{law}} r^H B_H(t)$. The right equality can be interpreted as an ‘‘infinite integral scale’’ in the framework of a standard multifractal self-similarity where one has $M_{rT}(rt) =_{\text{law}} rM(t)$ and then $M(t) \equiv M_{T \rightarrow \infty}(t)$. Notice that $\omega_{l \rightarrow 0}(t)$ has the drawbacks of both fractional Gaussian noise and fractional Brownian motion since it exists only in the sense of distributions and it is a non-stationary process.

Let us remark that Eq. (6) leads to the trivial scaling $\mathbb{E}[M(\tau)^q] = C_q \tau^q$ and, in the sense of Eq. (1), it thus appears that $M(t)$ is not a multifractal process. However, one must carefully interpret this scaling law since $M(t)$ has no stationary increments. In that respect, there is no reason that the moments $\mathbb{E}[M(\tau)^q]$ and $\mathbb{E}[(M(t+\tau) - M(t))^q]$ behave in the same way. Let us make the explicit computation for $q = 2$. In that case, $\mathbb{E}[(M(t+\tau) - M(t))^2]$ reads:

$$\int_t^{t+\tau} \int_t^{t+\tau} \left[\frac{\max(u, v)}{|u-v|} \right]^{\lambda^2} dudv = t^2 \int_0^{\frac{\tau}{t}} \int_0^{\frac{\tau}{t}} \left[\frac{1 + \max(u, v)}{|u-v|} \right]^{\lambda^2} dudv$$

If one supposes that $\frac{\tau}{t} \ll 1$, then in the last integral the term $\max(u, v) \ll 1$ can be neglected and, using the change of variables $u' = ut/\tau$ and $v' = vt/\tau$, we obtain $\mathbb{E}[M([t, t+\tau])^2] \simeq C_2(t) \tau^{2-\lambda^2}$ with $C_2(t) \sim t^{-\lambda^2}$. This equation shows that in the limit $\tau \ll t$, the mean square of the increments of $M(t)$ is scaling as the increment of the multifractal measure $M_T(t)$ where t plays precisely the role of the integral scale T . This behavior can be directly established from Eq. (5): Indeed, let us

consider two times t_1, t_2 in some interval $[t_0, t_0 + \Delta t]$. If $\Delta t \ll t_0$, then to the first order in $t_0/\Delta t$, we have $\text{Cov}[\omega(t_1), \omega(t_2)] = \lambda^2 \ln(t_0/|t_1 - t_2|)$, i.e. the same auto-covariance of the process $\omega_{l,T}$ used to build an exact multifractal random measure with $T = t_0$. This means that, over any interval $[t_0, t_0 + \Delta t]$, the non stationary process $M(t)$ cannot be distinguished from a (stationary) multifractal random measure $M_{t_0}(t)$ of integral scale $T = t_0$ to the first order in $t_0/\Delta t$.

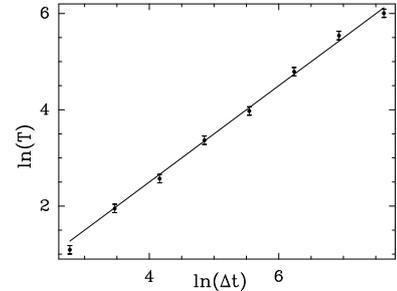


FIG. 3. Estimated integral scale T as a function of the sample size Δt (in log-log coordinates) for the Dow-Jones daily time series from 1929 to 2007. The solid line represents the value $e^{-3/2} \Delta t$ one expects theoretically. The reported error bars correspond to standard deviation of the empirical mean value estimated from the observed dispersion over all sub-intervals.

Let us now address some statistical estimation issues. The problem of the estimation of the auto-covariance of ω and in particular that of the estimation of the integral scale T , has already been considered by various authors [11, 12]. Let us suppose that one studies a (log-normal) multifractal random measure $M_T(t)$ over an interval $[t_0, t_0 + \Delta t]$ with $\Delta t < T$. Then, as shown in [11], from the self-similarity properties of $M_T(t)$, one has, for all $r < \Delta t/T < 1$, $M_T(t) =_{\text{law}} r^{-1} M_{rT}(rt) =_{\text{law}} r^{-1} W_r M_{rT}(t)$, where W_r is a log-normal random variable of variance $-\lambda^2 \ln(r)$. On a single realization, W_r is fixed, so the previous equality clearly means that one cannot distinguish any of the MRW measures $M_T(t)$ with $T \geq \Delta t$ over an interval $[t_0, t_0 + \Delta t]$. Estimating the integral scale on a single realization of $M_T(t)$ over an interval of length $\Delta t < T$ is thus impossible. One can wonder which value an empirical estimation leads to. Empirically, as advocated e.g., in Ref. [11], the correlation properties of $\omega(t)$ can be estimated using a proxy (called the ‘‘magnitude process’’) $\Omega_h(t) = \ln \delta_h M(t) \simeq \omega_h(t)$. In Ref. [11] it has been shown that (to the first order in λ^2) the auto-covariance estimator of Ω_h over a window of size Δt reads:

$$\widehat{\gamma}_{\Delta t}(\tau) \simeq \lambda^2 \ln\left(\frac{e^{-3/2} \Delta t}{\tau}\right) \left(1 - \frac{\tau}{\Delta t}\right). \quad (7)$$

This equation means that, over a sample of size Δt , the estimated auto-covariance of the magnitude associated with a multifractal process of integral scale $T > \Delta t$ is the

auto-covariance of a multifractal process of integral scale $e^{-3/2}\Delta t$. Since over every interval $[t_0, t_0 + \Delta t]$, $M(t) =_{law} M_{t_0}(t)$, we can conclude that, as soon as $t_0 > \Delta t$, the estimated auto-covariance of $\Omega_h(t)$ will be provided by Eq. (7). This is illustrated in Fig. 2(c) where we have reported the estimation of the magnitude auto-covariance for various sample length. More precisely, for each size $\Delta t = 16, 32, \dots, 512$, a single sample of overall size L (as e.g. $L = 2 \cdot 10^4$ in Fig. 2(a)) is divided in $L/\Delta t$ intervals of length Δt and the reported estimator $\widehat{\gamma}_{\Delta t}(\tau)$ is the average of the obtained estimator over each interval. One can see that the theoretical prediction in Eq. (7) (solid lines) is in good agreement with the observations (●) and one clearly observes an apparent integral scale that varies as $e^{-3/2}\Delta t$.

As recalled in the introduction, various authors have recently suggested that most of stylized facts characterizing the volatility associated with asset prices in financial markets can be accounted by multifractal measure. The model introduced in this paper allows one to explain the large discrepancies of the reported integral scale values as a consequence of the non-stationary nature of log-volatility. In Fig. 2(b) is plotted the estimated values of the daily log-volatility (as measured by the daily return range) of the Dow-Jones index over several decades (from 1929 to 2007). Very much like the model (Fig. 2(a)), one can observe excursions away from the mean value lasting for several years. In Fig. 2(d) we report the results of the same magnitude auto-covariance estimation experiment we conducted for the model. For all considered sample lengths Δt , one clearly sees that the multifractal shape $\lambda^2 \ln(T/\tau)$ reproduces very well the estimated auto-covariance with a constant intermittency coefficient $\lambda^2 \simeq 0.01$ but with an apparent integral scale that depends on Δt as very well reproduced by the model predictions (solid lines). This is confirmed in Fig. 3 where we have plotted (in log-log scale) the estimated integral scale as a function of the sample size Δt . One can see that the analytical prediction $T(\Delta t) = e^{-3/2}\Delta t$ (solid line) is in a very good agreement with the empirical data.

To conclude we have introduced a new stochastic measure as the exponential of a non-stationary gaussian $1/f$ noise. We have shown that, over any finite time interval, provided the considered time t is large enough, this model can be hardly distinguished from multifractal random cascade with an integral scale that is equal to the sample length. Our approach can be very appealing to model all phenomena where multiscaling properties are observed without the existence of any obvious large “correlation” (or “injection”) scale in space or time. For example, in finance, the agreement of the model predictions with the observed behavior of log-volatility correlation in the Dow-Jones index is striking and suggests a peculiar (aging) non-stationary nature of volatility fluctuations.

In a forthcoming extended version of this paper, we will provide all mathematical details, perform a more extensive empirical study over different market data and discuss its relationship with former related approaches as e.g. those published in Refs [19, 20].

* muzy@univ-corse.fr

- [1] We only account in this paper for multifractal “measures”, i.e. non decreasing functions $M(t) = M[0, t]$. The extension to multifractal processes $X(t)$ can be performed by considering $M(t)$ as the “stochastic time” of some self-similar process $B(t)$ and defining $X(t) = B(M(t))$.
- [2] U. Frisch, *Turbulence: The legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
- [3] A. Fisher, L. Calvet, and B. B. Mandelbrot, “Multifractality of dem/\$ rates,” (1999), preprint, Cowles Foundation Discussion paper 1165.
- [4] S. Ghasghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge, *Nature* **381**, 767 (1996).
- [5] A. Arneodo, J. F. Muzy, and D. Sornette, *Eur. Phys. J. B* **2**, 277 (1998).
- [6] J. F. Muzy, J. Delour, and E. Bacry, *Eur. J. Phys. B* **17**, 537 (2000).
- [7] T. Lux, *Quantitative finance* **1**, 632 (2001).
- [8] L. Calvet and A. Fisher, *Multifractal volatility* (Academic Press, Burlington, MA, 2008).
- [9] J. P. Bouchaud and M. Potters, *Theory of Financial Risk and Derivative Pricing* (Cambridge University Press, Cambridge, 2003).
- [10] E. Bacry and J. F. Muzy, *Comm. in Math. Phys.* **236**, 449 (2003).
- [11] E. Bacry, A. Kozhemyak, and J. F. Muzy, *Quantitative Finance* (2012), in press.
- [12] O. Løvstetten and M. Rypdal, *ArXiv e-prints* (2011), arXiv:1112.0105 [physics.data-an].
- [13] J. F. Muzy and E. Bacry, *Phys. Rev. E* **66**, 056121 (2002).
- [14] F. Schmitt and D. Marsan, *European Physical Journal B* **20**, 3 (2001).
- [15] For the sake of simplicity, we will consider in this paper exclusively log-normal random cascades, i.e. Gaussian $d\xi(u, s)$ (meaning the $\zeta(q)$ in Eq. (1) is parabolic). All our results can be easily extended to arbitrary log infinitely divisible laws within the framework introduced in Refs [10, 13].
- [16] A. Arneodo, E. Bacry, S. Manneville, and J. F. Muzy, *Phys. Rev. Lett.* **80**, 708 (1998).
- [17] J. K. J.P. Bouchaud, L.F. Cugliandolo and M. Mézard, in *Spin glasses and gandom fields*, Series on Directions in Condensed Matter Physics, Vol. 12, edited by A. Youg (Word Scientific, 1998) p. 161.
- [18] M. S. Taqqu and G. Samorodnisky, *Stable non-Gaussian random processes* (Chapman & Hall, New-York, 1994).
- [19] A. Saichev and D. Sornette, *Phys. Rev. E* **74**, 011111 (2006).
- [20] J. Duchon, R. Robert, and V. Vargas, *Mathematical Finance* **22**, 83 (2012).