

Modelling financial time series using Multifractal Random Walks

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Abstract

Multifractal Random Walks (MRW) correspond to simple solvable “stochastic volatility” processes. Moreover, they provide a simple interpretation of multifractal scaling laws and multiplicative cascade process paradigms in terms of volatility correlations. We show that they are able to reproduce most of recent empirical findings concerning financial time series: no correlation between price variations, long-range volatility correlations and multifractal statistics.

Key words: Multifractals, long-range correlations, stochastic volatility,
multiplicative cascades

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1 Introduction

Multifractal processes and multiplicative cascades have been widely used in many contexts to account for the time scale dependence of the statistical properties of a time-series. Recent empirical findings [3,10,7,16] suggest that in finance, this framework is likely to be pertinent. The recently introduced Multifractal Random Walks (MRW) [4] are multifractal processes that can be seen as simple “stochastic volatility” models (with stationary increments) whose statistical properties can be precisely controlled across the time scales using very few parameters. In that respect, they reproduce many features that characterize market price changes [13] including the decorrelation of the price increments, the long-range correlation of the volatility and the way the probability density function (pdf) of the price increments changes across time-scales, going from quasi Gaussian distributions at rather large time scales

to fat tail distributions at fine scales. In a recent work [14], Muzy et. al. have elaborated a “multivariate multifractal” framework that accounts for the time scale dependence of the mutual statistical properties of several time-series. Though initially introduced for modelling single asset variations, the MRW models can be naturally extended in order to fit this new multivariate framework [5]. In this paper we will focus on single asset modelling.

The goal of this paper is to show, using real data, that eventhough MRW’s involve only 3 parameters, they allow one to capture very precisely both the multifractal statistics and the correlation structure of single assets. The paper is organized as follows. In section 2, after recalling briefly the main notations and definitions involved in the “classical” multifractal framework, we introduce the MRW model defined in [4], recall its main properties and give explicit analytical expressions of the q -order moments of its increments and of the volatility correlation structure. In section 3, we show that MRW models are able to reproduce very precisely many statistical quantities of financial time-series. They are shown to reproduce (i) the multifractal statistics (i.e., the scaling exponents), (ii) the values of the q -order moments of the increments for a wide range of q ’s and (ii) the correlation structure of the volatility both at a given time-scale and in between two different time-scales. Conclusions and prospects are reported in section 4.

2 The Multifractal Random Walk (MRW) model

2.1 Multifractal processes

A multifractal process is a process wich has some scale invariance properties. These properties are generally characterized by the exponents ζ_q which govern the power law scaling of the absolute moments of its fluctuations, i.e.,

$$M(q, l) = K_q l^{\zeta_q}, \quad (1)$$

where

$$M(q, l) = E(|\delta_l X(t)|^q) = E(|X(t+l) - X(t)|^q), \quad (2)$$

where $X(t)$ is supposed to be a stochastisc process with stationary increments. Some very popular stochastic processes are the so-called *self-similar processes* [18]. They are defined as processes $X(t)$ which have stationary increments and which verify (in law)

$$\delta_l X(t) \stackrel{law}{=} (l/L)^H \delta_L X(t), \quad \forall l, L > 0.$$

For these processes, one easily gets $\zeta_q = qH$, i.e., the ζ_q spectrum is a linear function of q . Widely used examples of such processes are (fractional) Brownian motions (fBm) or Levy walks.

However, many empirical studies have shown that the ζ_q spectrum of return fluctuations is a non linear convex function. Let us note that, using a simple argument, it is easy to show that if ζ_q is a non-linear convex function the scaling behavior (1) cannot hold for all scales l but only for scales smaller than an arbitrary large scale L that is generally referred to as the *integral scale*. A very common approach originally proposed by several authors in the field of fully developed turbulence [15,17,11,9,8], has been to describe such processes in the scale domain, describing the *cascading* process that rules how the fluctuations evolves when going from coarse to fine scales. Basically, it amounts in stating that the fluctuations at the integral scale L are linked to the ones at a smaller scale $l < L$ using the cascading rule

$$\delta_l X(t) =^{law} W_{l/L} \delta_L X(t), \quad \text{for fixed } t, \quad (3)$$

where $W_{l/L}$ is a log infinitely divisible stochastic variable which depends only on the ratio l/L . A straightforward computation [8] then shows that the pdf $P_l(\delta X)$ of $\delta_l X$ changes when varying the time-scale l according to the rule

$$P_l(\delta X) = \int G_{l/L}(u) e^{-u} P_L(e^{-u} \delta X) du, \quad (4)$$

where the *self-similarity kernel* $G_{l/L}$ is the pdf of $\ln W_{l/L}$. Since $W_{l/L}$ is a log infinitely divisible variable, the Fourier transform of $G_{l/L}$ is of the form

$$\hat{G}_{l/L}(k) = \hat{G}^{\ln l/L}(k). \quad (5)$$

From that equation, one easily gets the expression of the ζ_q spectrum

$$\zeta_q = \ln \hat{G}(-iq). \quad (6)$$

In this framework, the simplest non-linear case is the so-called log-normal model that corresponds to a parabolic ζ_q and a Gaussian kernel.

2.2 Cascade models and magnitude correlations

Multiplicative cascading processes are examples of processes satisfying the cascading rule (3) for discrete scales $l_n = 2^{-n}L$. The cascade paradigm is that

the local variation of the process $\delta_{l_n} X$ at scale l_n is obtained from the variation at scale L as

$$\delta_{l_n} X(t) = \left(\prod_{i=1}^n W_i \right) \delta_L X(t) \quad (7)$$

where W_i are i.i.d. random positive factors. Realizations of such processes can be constructed using orthonormal wavelet bases as discussed in Ref. [2]. If one defines the *magnitude* $\omega(t, l)$ at time t and scale l as the logarithm of “local volatility” [3]:

$$\omega(t, l) = \frac{1}{2} \ln(|\delta_l X(t)|^2), \quad (8)$$

then the previous cascade equation becomes a simple random walk equation, at fixed time t , versus the logarithm of scales:

$$\omega(t, l_{n+1}) = \omega(t, l_n) + \ln(W_{n+1}) .$$

The correlation structure implied by multifractal cascades has already been addressed in previous works [3,2,1]. It has been shown that $\delta_l X(t)^2$ is long-range correlated. More precisely, one can show that it generally leads to magnitude correlation functions that behave like

$$C_\omega(\tau, l_1, l_2) = \text{Cov}(\omega(t, l_1), \omega(t + \tau, l_2)) \simeq -\lambda^2 \ln(\tau/L), \quad L > \tau \gg \max(l_1, l_2). \quad (9)$$

Let us note that this behavior has been shown to provide good fits of the empirical estimates of the correlation functions from financial time-series [3].

As explained in Ref. [4], the problem with cascade processes is that they involve representations (e.g., orthonormal wavelet bases) that are constructed on a discrete set of scales (e.g. dyadic scales $l_n = 2^{-n}$) and in turn cannot be invariant under continuous scale dilations. To our knowledge, the MRW’s are the only known multifractal processes with continuous dilation invariance, i.e., processes that satisfy Eq. (4) for a continuous values of l .

2.3 Introducing the MRW model

An MRW process $X(t)$ is the limit process (when the time discretization step Δt goes to 0) of a standard random walk $X_{\Delta t}[k]$ with a stochastic variance (volatility), i.e.,

$$X(t) = \lim_{\Delta t \rightarrow 0} X_{\Delta t}(t),$$

with

$$X_{\Delta t}(t) = \sum_{k=1}^{t/\Delta t} \epsilon_{\Delta t}[k] e^{\omega_{\Delta t}[k]},$$

where $e^{\omega_{\Delta t}[k]}$ is the stochastic volatility and $\epsilon_{\Delta t}$ a gaussian white noise of variance $\sigma^2 \Delta t$ and which is independent of $\omega_{\Delta t}$. The choice for the process $\omega_{\Delta t}$ is simply dictated by the fact that we want the scaling (1) to be exact for all time scales $l \leq L$. Some long but straightforward computations [4] show that this is achieved if $\omega_{\Delta t}$ is a stationary Gaussian process such that $E(\omega_{\Delta t}[k]) = -\text{Var}(\omega_{\Delta t}[k])$ and whose covariance is

$$\text{Cov}(\omega_{\Delta t}[k], \omega_{\Delta t}[l]) = \lambda^2 \ln \rho_{\Delta t}[|k - l|]$$

where

$$\rho_{\Delta t}[k] = \begin{cases} \frac{L}{(|k|+1)\Delta t} & \text{for } |k| \leq L/\Delta t - 1 \\ 1 & \text{otherwise} \end{cases}$$

Let us note that it corresponds to a log-normal volatility which is correlated up to a time lag L and that it is mimicing Eq (9).

One can then prove [4] the multifractal scaling property

$$M(q, l) = K_q l^{\zeta_q}, \quad \forall l \leq L, \quad (10)$$

where K_q (for q even) is

$$K_q = L^{q/2} \sigma^q (q-1)!! \int_0^1 du_1 \dots \int_0^1 du_{q/2} \prod_{i < j} |u_i - u_j|^{-4\lambda^2}, \quad (11)$$

and

$$\zeta_q = (q - q(q-2)\lambda^2)/2. \quad (12)$$

Since ζ_q is a parabolic function, it indicates that the self-similarity kernel $G_{l/L}$ which links the pdf's at different time scales (Eq. (4)) is Gaussian. The parameter λ^2 which governs the non-linearity of ζ_q is called the *intermittency* factor since it controls how far from a self-similar behavior (i.e., ζ_q linear) the process is. In the limit $\lambda^2 = 0$, ζ_q becomes linear, the process $X(t)$ is a Brownian motion and $W_{l/L}$ corresponds to a deterministic value.

Let us note that, one can prove [6] that, in the case of MRW processes, the equality in law in Eq. (3) is not only true for fixed time t but basically also for the processes themselves (i.e., when t is varying). In that case $W_{l/L}$ is a random variable which does not depend on time.

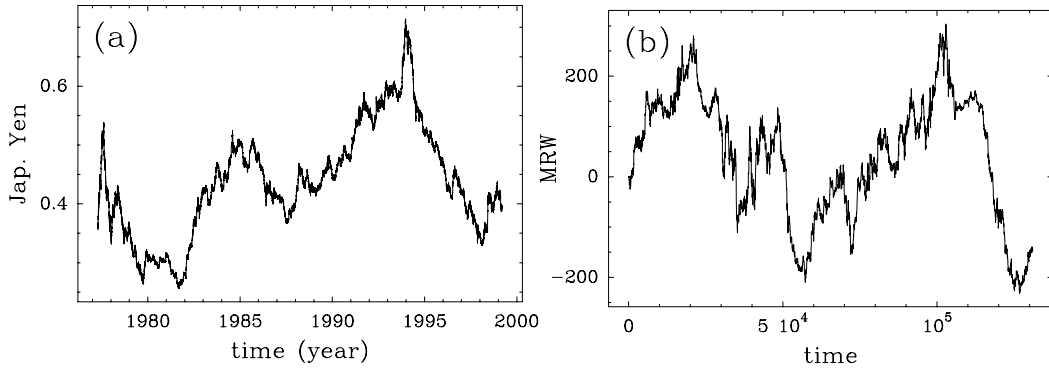


Fig. 1. **Modelling intraday Japanese Yen futures using MRW.** The MRW is used to model the de-seasonalized logarithm of the time-series displayed in (a). The parameters have been estimated to $\sigma^2 \simeq 4 \cdot 10^{-5} \text{ year}^{-1}$, $\lambda^2 \simeq 0.02$ and $L \simeq 1 \text{ year}$. (a) Plot of the original index time-series : Japanese Yen futures from March 77 to February 99 (intraday tick by tick data). (b) Plot of a sample time series of length 2^{17} of the MRW model.

An explicit analytical formula for K_q can be obtained for even q values since the multiple integral in Eq. (11) can be evaluated using the celebrated Selberg integral formula [19]. One then obtains:

$$K_q = L^{q/2} \sigma^q (q-1)!! \prod_{k=0}^{q/2-1} \frac{\Gamma(1-2\lambda^2 k)^2 \Gamma(1-2\lambda^2(k+1))}{\Gamma(2-2\lambda^2(q/2+k-1)) \Gamma(1-2\lambda^2)}. \quad (13)$$

Let us note that this last equation seems to indicate that the critical value q^* above which the moments of the MRW are infinite statisfies

$$M(q, l) = +\infty \iff q \geq q^* = 2 + \frac{1}{2\lambda^2}. \quad (14)$$

Moreover, one can show [4] that the magnitude correlation $C_\omega(\tau, l_1, l_2)$ behaves like in Equation (9), i.e.,

$$C_\omega(\tau, l_1, l_2) \simeq -\lambda^2 \ln \left(\frac{\tau}{L} \right), \quad L > \tau \gg \max(l_1, l_2). \quad (15)$$

3 Modelling return fluctuations using MRW

MRW processes can be used to model the return fluctuations of a financial time-series [13]. In this case, the price $S(t)$ of the asset will be modelled by $e^{X(t)}$ where $X(t)$ is an MRW. For this purpose 3 parameters need to be estimated : the variance σ^2 , the integral scale L and the intermittency parameter λ^2 of the MRW $X(t)$. The variance σ^2 can be estimated using the simple relation

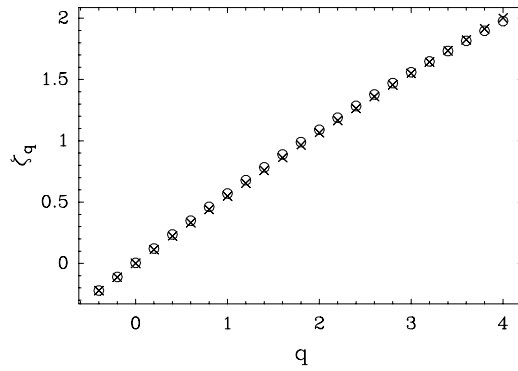


Fig. 2. **Multifractal exponents** ζ_q spectrum estimations for the Yen futures fluctuations (\circ) and for the MRW model prediction (\times) (Eq. (12)).

$Var(X(t)) = \sigma^2 t$. Both, the decorrelation scale L and the parameter λ can be obtained from the expression (15) of the magnitude correlation. Let us note that λ^2 can be also estimated independantly from the ζ_q spectrum (Eq. (12)). The consistency between these two completely different estimators of λ is a very good test for the validity of the model.

We have estimated the parameters σ^2 , λ^2 and L for the time-series of the japanese Yen futures from March 1977 to February 1999 (tick by tick intra-day data). The plots of this financial time-series and of a realization of the corresponding MRW model are shown in figure 1.

3.1 Fitting the multifractal exponents ζ_q

As shown in figure 2, the MRW model reproduces very precisely the parabolic ζ_q spectrum. Let us recall that the ζ_q exponents describe, through the self-similarity kernel $G_{l/L}$ and Eqs (6), (4), how the return fluctuation pdf evolves when going from one time scale to another. Thus this figure shows that the self-similarity kernel $G_{l/L}$ (resp. the random variable $W_{l/L}$) is very close to a Gaussian function (resp. log-normal random variable).

3.2 Fitting the moments $M(q, l)$

In the previous section we have shown that the scaling behavior of the moments $M(q, l)$ of the Yen time-series are well fitted using the MRW model. However, it does not say anything about whether the values themselves of $M(q, l)$ are well fitted or not. Figure 3 displays $\ln M(q, l)$ as a function of q for three different values of l ($l = 1$ day, 5 days and 10 days). It appears clearly that the numerical results obtained on the Yen time-series match closely the theoretical prediction (Eq. (13)) obtained for the MRW model.

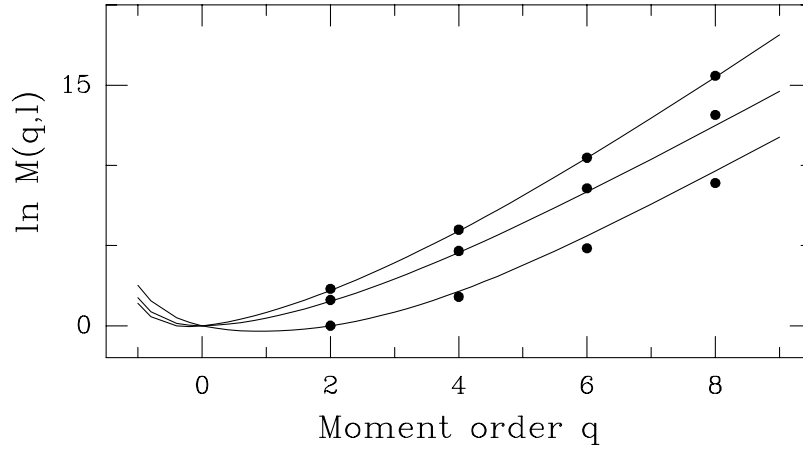


Fig. 3. **q -order moments $M(q, l)$ of $\delta_l X$.** $\ln M(q, l)$ (Eq. (2)) is displayed versus q for (from bottom to top) $l = 1, 5$ and 10 days. $\ln M(q, l)$ has been computed for the Yen futures fluctuations (\bullet) which daily fluctuation variance has been arbitrarily normalized to 1. The theoretical prediction (Eq. (13)) obtained for the MRW model corresponds to the solid line.

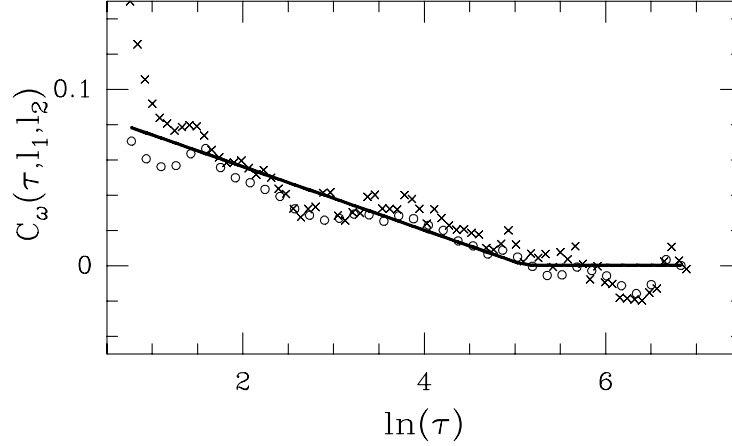


Fig. 4. **Correlation function C_ω of the magnitude:** The correlation function $C_\omega(\tau, l_1, l_2)$ versus $\ln(\tau)$. From Eq. (15), we expect these plots to be straight lines as long as $L > \tau \gg \max(l_1, l_2)$. $C_\omega(\tau, l_1, l_2)$ for the Yen futures fluctuations is displayed for $l_1 = l_2 = 1$ day (\circ) and for $l_1 = 1$ day and $l_2 = 5$ days (\times). The solid line corresponds to the prediction $\lambda^2 \ln(L/\tau)$

3.3 Fitting the correlation structure of the magnitude $C_\omega(\tau, l_1, l_2)$

In this section we have computed the magnitude correlation function $C_\omega(\tau, l_1, l_2)$ on the Yen time-series. The computation has been made both at the same scale ($l_1 = l_2 = 1$ day) and at two different scales ($l_1 = 1$ day and $l_2 = 5$ days). As explained in section 2.3, we expect that, as long as $L > \tau \gg \max(l_1, l_2)$ we expect C_ω to be proportional to $\ln(L/\tau)$ (Eq. (15)). Moreover the slope should be the intermittency factor $\lambda^2 \simeq 0.02$. As illustrated in figure 4, the numerical

experiments are in very good agreement with the theoretical prediction.

4 Conclusion

In this paper, we have shown that Multifractal Random Walks (MRW) can be used for modelling the time-fluctuations of a single-asset. It is able to reproduce the main observed characteristics of financial time-series: no correlation between price variations, long-range volatility correlations, linear and non-linear correlation between assets. Moreover, it reproduces precisely the q -order moments of the series (for a wide range of q 's) and the volatility correlation structure in between different time-scales. All of these features can be controlled using only 3 parameters : the intermittency factor λ^2 which both controls the scale invariance properties and the volatility correlation, the integral scale L which controls the volatility decorrelation scale and the variance σ^2 of the fluctuations.

Some preliminary works have been done on extending this framework to a multivariate framework [14,5]. The so-obtained Multivariate MRW (MMRW) is likely to capture the whole return joint law of a basket of assets at all time horizons. We are currently performing some extensive numerical experiments on multivariate financial data and started to use MMRW for (historical) volatility and value at risk prediction as well as for portfolio management.

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