

RECTANGULAR RANDOM MATRICES, ENTROPY, AND FISHER'S INFORMATION

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ABSTRACT. We prove that independent rectangular random matrices, when embedded in an algebra of larger square matrices, are asymptotically free with amalgamation over a commutative finite dimensional subalgebra \mathcal{D} (under an hypothesis of unitary invariance). Then we consider elements of a W^* -probability space containing \mathcal{D} , which have kernel and range projection in \mathcal{D} . We associate them a free entropy constructed with micro-states given by rectangular matrices. We also associate them a free Fisher's information with a conjugate variables approach. Both approaches give rise to optimization problems whose solutions involve freeness with amalgamation over \mathcal{D} . It could possibly be a first proposition for the study of sets of operators between different Hilbert spaces with the tools of free probability. As an application, we prove a result of freeness with amalgamation between the two parts of the polar decomposition of R -diagonal elements with non trivial kernel.

KEYWORDS: *Free probability, random matrices, free entropy, free Fisher's information, Marchenko-Pastur distribution*

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INTRODUCTION

In a previous paper [2], we considered an independent family of rectangular random matrices with different sizes, say $n \times p$, $p \times n$, $n \times n$ and $p \times p$. We embedded them, as blocks, in $(n + p) \times (n + p)$ matrices by the following rules

$$(0.1) \quad M \rightarrow \begin{cases} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} & \text{if } M \text{ is } n \times n, & \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} & \text{if } M \text{ is } n \times p, \\ \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} & \text{if } M \text{ is } p \times p, & \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} & \text{if } M \text{ is } p \times n, \end{cases}$$

and we proved that under an assumption of invariance under actions of unitary groups and of convergence of singular laws (i.e. uniform distribution on eigenvalues of the absolute value), the embedded matrices are asymptotically free with

amalgamation on the two-dimensional commutative subalgebra \mathcal{D} generated by the projectors

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix}.$$

Asymptotically refers to the limit when $n, p \rightarrow \infty$ in a ratio having a non negative limit. In fact, we considered not only two sizes n, p , but a finite family $q_1(n), \dots, q_d(n)$ of sizes, and the large matrices were represented as $d \times d$ block matrices.

In this paper, we prove a similar result with different technics (which allows us to remove the hypothesis of convergence of singular laws). Then we consider a W^* -probability space (\mathcal{A}, φ) endowed with a finite dimensional commutative subalgebra \mathcal{D} . Note that such a situation can arise if one considers operators between different spaces, say H_1, H_2 , and embeds them in $B(H_1 \oplus H_2)$ as it was made for matrices in (0.1). We define a microstate free entropy for N -tuples (a_1, \dots, a_N) of elements of \mathcal{A} which have kernel and range projections in \mathcal{D} : it is the asymptotic logarithm of the volume of N -tuples of rectangular matrices whose joint distribution (under the state defined by the trace) is closed to the joint distribution of (a_1, \dots, a_N) in (\mathcal{A}, φ) .

This free entropy is subadditive, and we prove that it is additive only on families which are free with amalgamation over \mathcal{D} . This is one of the properties that has made us consider this free entropy possibly relevant to study sets of operators between different Hilbert spaces with the tools of free probability. These results are proved using some change of variable formulae we establish here. These formulae still remain valid for Voiculescu's free entropy for non hermitian operators, as it is defined in section 1.2 of [32] or p. 279 of [10].

Another optimization problem has given rise to an interesting analogy. In the previous paper [2], for each $\lambda \in (0, 1)$, we defined a free convolution $\mu_{\boxplus \lambda \nu}$ of symmetric probability measures as the distribution in $(q\mathcal{A}q, \frac{1}{\varphi(q)}\varphi)$ of $a + b$, where a, b are free with amalgamation over \mathcal{D} , have kernel projection $\leq p = 1 - q \in \mathcal{D}$ and range projection $\leq q$ such that $\frac{\varphi(q)}{\varphi(p)} = \lambda$, and have symmetrized distributions μ, ν in $(q\mathcal{A}q, \frac{1}{\varphi(q)}\varphi)$. We established, in [3], and deepened, in [1, 4], a correspondence (like the Bercovici-Pata bijection) between \boxplus_λ -infinitely divisible distributions and $*$ -infinitely divisible distributions. In this correspondence, the analogue of Gaussian distributions are symmetrizations of Marchenko-Pastur distributions. In this paper, we prove that among the set of elements a with kernel projection $\leq p$ and range projection $\leq q$ and such that $\varphi(aa^*) \leq \varphi(q)$, the elements which maximize free entropy are the elements a such that the distribution of aa^* in $(q\mathcal{A}q, \frac{1}{\varphi(q)}\varphi)$ is a Marchenko-Pastur distribution.

We also construct a free Fisher's information with the conjugate variables approach for elements which have kernel and range projections in \mathcal{D} . We have a Cramér-Rao inequality, where Marchenko-Pastur distributions appear again as

the distributions which realize equality, and a superadditivity result where freeness with amalgamation over \mathcal{D} is equivalent to additivity (when quantities are finite).

The main relevance, according to the author, of these problems of optimization, is the *legitimization* of these notions. Indeed, the analogous problems, for the classical entropy and information on the one hand and for the entropy and the information defined by Voiculescu on the other hand, have been solved (see [9], [10], [15], [16], [14]), and the solutions were actually the analogues of the solutions given here. This supports the idea that the notions proposed here are the right ones to apply the tools and the ideas of free probability theory to the study of operators between different Hilbert spaces. Moreover, the solutions of optimization problems for entropy and information under certain constraints are, in a sense, the *generic* objects which realize these constraints.

In section 1 and 2, we define the objects we are going to use and we recall definitions and basic properties of operator valued cumulants.

In section 3, we prove that under certain hypothesis, freeness with respect to the state φ implies freeness with amalgamation over the finite dimensional commutative algebra \mathcal{D} . As an application, we prove a result about polar decomposition of R -diagonal elements with non trivial kernel: the partial isometry and the positive part are free with amalgamation over the algebra generated by the kernel projection.

In section 4, we prove asymptotic freeness with amalgamation over \mathcal{D} of rectangular independent random matrices (as a consequence of results of the previous section). This result is used section 5, where we define our microstates free entropy and solve the optimization problems we talked about above, using some change of variable formulae we establish in the same section. Similarly, in section 6, we construct our free Fisher's information with the conjugate variables approach and solve some optimization problems.

1. DEFINITIONS

In this section, we will define the spaces and the notions. For all d integer, we denote by $[d]$ the set $\{1, \dots, d\}$.

Consider a tracial $*$ -noncommutative probability space (\mathcal{A}, φ) endowed with a family (p_1, \dots, p_d) of self-adjoint non zero projectors (i.e. $\forall i, p_i^2 = p_i$) which are pairwise orthogonal (i.e. $\forall i \neq j, p_i p_j = 0$), and such that $p_1 + \dots + p_d = 1$. Any element x of \mathcal{A} can then be represented

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & & \vdots \\ x_{d1} & \cdots & x_{dd} \end{bmatrix},$$

where $\forall i, j, x_{ij} = p_i x p_j$. This notation is compatible with the product and the involution.

Let us define, for all $i, j \in [d]$, $\mathcal{A}_{i,j} = p_i \mathcal{A} p_j$ (the comma between i and j will often be omitted). We call *simple elements* the non zero elements of the union of the \mathcal{A}_{ij} 's ($i, j \in [d]$). We define $\varphi_i := \frac{1}{\rho_i} \varphi|_{\mathcal{A}_{ii}}$, with $\rho_i := \varphi(p_i)$. Note that, since φ is a trace, every φ_i is a trace, but for $i, j \in [d]$, $a \in \mathcal{A}_{ij}$, $b \in \mathcal{A}_{ji}$, one has

$$(1.1) \quad \rho_i \varphi_i(ab) = \rho_j \varphi_j(ba).$$

Note also that the linear span \mathcal{D} of $\{p_1, \dots, p_d\}$ is a $*$ -algebra, which will be identified to the set of $d \times d$ diagonal complex matrices by

$$\sum_{i=1}^d \lambda_i p_i \simeq \text{diag}(\lambda_1, \dots, \lambda_d).$$

The application E , which maps $x \in \mathcal{A}$ to $\text{diag}(\varphi_1(x_{11}), \dots, \varphi_d(x_{dd}))$, is then a conditional expectation from \mathcal{A} to \mathcal{D} :

$$\forall (d, a, d') \in \mathcal{D} \times \mathcal{A} \times \mathcal{D}, E(dad') = d E(a) d'.$$

A family $(\mathcal{A}_i)_{i \in I}$ of subalgebras of \mathcal{A} which all contain \mathcal{D} is said to be *free with amalgamation over \mathcal{D}* if for all n , $i_1 \neq \dots \neq i_n \in I$, for all $x(1) \in \mathcal{A}_{i_1} \cap \ker E, \dots, x(n) \in \mathcal{A}_{i_n} \cap \ker E$, one has

$$(1.2) \quad E(x(1)) \cdots x(n) = 0.$$

A family $(\chi_i)_{i \in I}$ of subsets of \mathcal{A} is said to be *free with amalgamation over \mathcal{D}* if there exists free with amalgamation over \mathcal{D} subalgebras $(\mathcal{A}_i)_{i \in I}$ (which all contain \mathcal{D}) such that for all i , $\chi_i \subset \mathcal{A}_i$.

The \mathcal{D} -distribution of a family $(a_i)_{i \in I}$ of elements of \mathcal{A} is the application which maps a word $X_{i_0}^{\epsilon_0} d_1 X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n} d_n$ in X_i, X_i^* ($i \in I$) and elements d_1, \dots of \mathcal{D} to $E(a_{i_0}^{\epsilon_0} d_1 a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n} d_n)$.

It is easy to see that the \mathcal{D} -distribution of a free with amalgamation over \mathcal{D} family depends only on the individual \mathcal{D} -distributions.

Consider a sequence $(\mathcal{A}_n, \varphi_n)$ of tracial $*$ -noncommutative probability spaces such that for all n , \mathcal{D} can be identified with a $*$ -subalgebra of \mathcal{A}_n (the identification is not supposed to preserve the state). The *convergence in \mathcal{D} -distribution* of a sequence $(a_i(n))_{i \in I}$ of families of elements of the \mathcal{A}_n 's to a family $(a_i)_{i \in I}$ of \mathcal{A} is the pointwise convergence of the sequence of \mathcal{D} -distributions. In this case, if $I = \cup_{s \in \mathcal{S}} I_s$ is a partition of I , then the family of subsets $(\{a_i(n); i \in I_s\})_{s \in \mathcal{S}}$ is said to be *asymptotically free with amalgamation over \mathcal{D}* if the family of subsets $(\{a_i; i \in I_s\})_{s \in \mathcal{S}}$ is free with amalgamation over \mathcal{D} .

It is easy to see that the \mathcal{D} -distribution of a free with amalgamation over \mathcal{D} family depends only on the individual \mathcal{D} -distributions.

2. CUMULANTS

The theory of cumulants in a \mathcal{D} -probability space (i.e. in an algebra endowed with a conditional expectation on a subalgebra \mathcal{D}) has been developed in [24]. In this section, we will begin by giving the main lines of this theory, and then we will investigate the special case of the situation we presented in the previous section.

2.1. GENERAL THEORY OF CUMULANTS IN A \mathcal{D} -PROBABILITY SPACE. In this section, we consider an algebra \mathcal{A} , a subalgebra \mathcal{D} of \mathcal{A} , and a conditional expectation E from \mathcal{A} to \mathcal{D} .

Let us begin with algebraic definitions. A \mathcal{D} -bimodule is a vector space M over \mathbb{C} such that the algebra \mathcal{D} acts on M on the right and on the left. The tensor product $M \otimes_{\mathcal{D}} N$ of two \mathcal{D} -bimodules M, N is their tensor product as \mathbb{C} -vector spaces, where for all $(m, d, n) \in M \times \mathcal{D} \times N$, $(m.d) \otimes n$ and $m \otimes (d.n)$ are identified. $M \otimes_{\mathcal{D}} N$ is endowed with a structure of \mathcal{D} -bimodule by $d_1.(m \otimes n).d_2 = (d_1.m) \otimes (n.d_2)$. This allows us to define, for n positive integer, $\mathcal{A}^{\otimes_{\mathcal{D}} n} = \underbrace{\mathcal{A} \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \mathcal{A}}_{n \text{ times}}$.

Consider a sequence $(f_n)_{n \geq 1}$ of maps, each f_n being a \mathcal{D} -bimodule morphism between $\mathcal{A}^{\otimes_{\mathcal{D}} n}$ and \mathcal{D} . For n positive integer and $\pi \in \text{NC}(n)$ (noncrossing partition of $[n]$), we define the \mathcal{D} -bimodule morphism f_π between $\mathcal{A}^{\otimes_{\mathcal{D}} n}$ and \mathcal{D} in the following way: if $\pi = 1_n$ is the one-block partition, $f_\pi = f_n$. In the other case, a block V of π is an interval $[k, l]$. If $k = 1$ (resp. $l = n$), then $f_\pi(a_1 \otimes \cdots \otimes a_n) = f_{l-k+1}(a_1 \otimes \cdots \otimes a_l) f_{\pi \setminus \{V\}}(a_{l+1} \otimes \cdots \otimes a_n)$ (resp. $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1}) f_{l-k+1}(a_k \otimes \cdots \otimes a_n)$). In the other case, one has $1 < k \leq l < n$. Then $f_\pi(a_1 \otimes \cdots \otimes a_n)$ is defined to be $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1}) f_{l-k+1}(a_k \otimes \cdots \otimes a_l) a_{l+1} \otimes \cdots \otimes a_n$ or $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1}) f_{l-k+1}(a_k \otimes \cdots \otimes a_l) a_{l+1} \otimes \cdots \otimes a_n$, both are the same by definition of $\otimes_{\mathcal{D}}$.

For example, if $\pi = \{\{1, 6, 8\}, \{2, 5\}, \{3, 4\}, \{7\}, \{9\}\}$, then

$$f_\pi(a_1 \otimes \cdots \otimes a_9) = f_3(a_1 f_2(a_2 f_2(a_3 \otimes a_4) \otimes a_5) \otimes a_6 f_1(a_7) \otimes a_8) f_1(a_9).$$

Let us define, for all $n \geq 1$, the \mathcal{D} -bimodule morphism E_n between $\mathcal{A}^{\otimes_{\mathcal{D}} n}$ and \mathcal{D} which maps $a_1 \otimes \cdots \otimes a_n$ to $E(a_1 \cdots a_n)$. Then one can define the sequence $(c_n)_{n \geq 1}$ of maps, each c_n being a \mathcal{D} -bimodule morphism between $\mathcal{A}^{\otimes_{\mathcal{D}} n}$ and \mathcal{D} , by one of the following four equivalent formulae:

$$(2.1) \quad \forall n, \forall \pi \in \text{NC}(n), \quad E_\pi = \sum_{\sigma \leq \pi} c_\sigma,$$

$$(2.2) \quad \forall n, \quad E_n = \sum_{\sigma \in \text{NC}(n)} c_\sigma$$

$$(2.3) \quad \forall n, \forall \pi \in \text{NC}(n), \quad c_\pi = \sum_{\sigma \leq \pi} \mu(\sigma, \pi) E_\sigma,$$

$$(2.4) \quad \forall n, \quad c_n = \sum_{\sigma \in \text{NC}(n)} \mu(\sigma, 1_n) E_\sigma,$$

where μ is the Möbius function [20, 14] of the lattice $\text{NC}(n)$ endowed with the refinement order.

The following result is a consequence of Proposition 3.3.3 of [24], used with the formula of cumulants with products as entries (Theorem 2 of [26]), which can be generalized to \mathcal{D} -probability spaces.

THEOREM 2.1. *A family $(\chi_i)_{i \in I}$ of subsets of \mathcal{A} is free with amalgamation over \mathcal{D} if and only if for all $n \geq 2$, for all non constant $i \in I^n$, for all $a_1 \in \chi_{i_1}, \dots, a_n \in \chi_{i_n}$, one has $c_n(a_1 \otimes \dots \otimes a_n) = 0$.*

Note that this theorem is a little improvement of Theorem 1 of [26].

2.2. THE SPECIAL CASE WHERE $\mathcal{D} = \text{Span}(p_1, \dots, p_d)$. For the rest of the text, we consider again, without introducing them, the same objects as in section 1. By linearity of the cumulant functions, we will work only with simple elements (i.e. non zero elements of the union of the \mathcal{A}_{ij} 's, $1 \leq i, j \leq d$).

(a) First, for all $i, j, k, l \in [d]$ such that $j \neq k$, one has $\mathcal{A}_{ij} \otimes_{\mathcal{D}} \mathcal{A}_{kl} = \{0\}$ (because $p_j p_k = 0$). So we will only have to compute the cumulant functions on subspaces of the type $\mathcal{A}_{i_0 i_1} \otimes_{\mathcal{D}} \mathcal{A}_{i_1 i_2} \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} \mathcal{A}_{i_{n-1} i_n}$, with $i_0, i_1, \dots, i_n \in [d]$.

(b) Moreover, on such a subspace, c_n takes values in $\mathcal{A}_{i_0 i_n}$, because it is a \mathcal{D} -bimodule morphism. So, if $i_0 \neq i_n$, since $\mathcal{D} \cap \mathcal{A}_{i_0 i_n} = \{0\}$, c_n is null on $\mathcal{A}_{i_0 i_1} \otimes_{\mathcal{D}} \mathcal{A}_{i_1 i_2} \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} \mathcal{A}_{i_{n-1} i_n}$. So it is easily proved by induction that for $\pi \in \text{NC}(n)$, for all $i_0, i_1, \dots, i_n \in [d]$, c_π is null on $\mathcal{A}_{i_0 i_1} \otimes_{\mathcal{D}} \mathcal{A}_{i_1 i_2} \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} \mathcal{A}_{i_{n-1} i_n}$ whenever a block $\{k_1 < \dots < k_m\}$ of π is such that $i_{k_1-1} \neq i_{k_m}$.

(c) Hence the function c_π factorizes on the complex vector space $\mathcal{A}_{i_0 i_1} \otimes_{\mathcal{D}} \mathcal{A}_{i_1 i_2} \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} \mathcal{A}_{i_{n-1} i_n}$ in the following way: for $(a_1, \dots, a_n) \in \mathcal{A}_{i_0 i_1} \times \dots \times \mathcal{A}_{i_{n-1} i_n}$,

$$(2.5) \quad c_\pi(a_1 \otimes \dots \otimes a_n) = \left(\prod_{\substack{V \in \pi \\ V = \{k_1 < \dots < k_m\}}} c_m^{(i_{k_m})}(a_{k_1} \otimes \dots \otimes a_{k_m}) \right) \cdot p_{i_n},$$

where for all $m, c_m^{(1)}, \dots, c_m^{(d)}$ are the linear forms on the complex vector space $\mathcal{A}^{\otimes_{\mathcal{D}} m}$ defined by

$$\forall x \in \mathcal{A}^{\otimes_{\mathcal{D}} m}, c_m(x) = \begin{bmatrix} c_m^{(1)}(x) & & \\ & \ddots & \\ & & c_m^{(d)}(x) \end{bmatrix}.$$

Formula (2.5) can be written in the following way: for $(a_1, \dots, a_n) \in \mathcal{A}_{i_0 i_1} \times \dots \times \mathcal{A}_{i_{n-1} i_n}$,

$$(2.6) \quad c_\pi(a_1 \otimes \dots \otimes a_n) = \prod_{\substack{V \in \pi \\ V = \{k_1 < \dots < k_m\}}} \eta_{i_n, i_{k_m}} \circ c_m(a_{k_1} \otimes \dots \otimes a_{k_m}),$$

where for all $i, j \in [d]$, $\eta_{i,j}$ is the involution of \mathcal{D} which permutes the i -th and the j -th columns in the representation of elements of \mathcal{D} as $d \times d$ complex matrices.

REMARK 2.2. In (b), (c), we only used the fact that for all n , c_n is a \mathcal{D} -bimodule morphism, so everything stays true if one replaces c_n by E_n and c_π by E_π .

(d) Now it remains only to investigate the relation between the functions $c_n^{(1)}, \dots, c_n^{(d)}$. We will prove, by induction on n , a formula analogous to (1.1). Consider $(a_1, \dots, a_n) \in \mathcal{A}_{i_0 i_1} \times \dots \times \mathcal{A}_{i_{n-1} i_n}$, with $i_0 = i_n$. Then one has

$$(2.7) \quad \rho_{i_0} c_n^{(i_0)}(a_1 \otimes \dots \otimes a_n) = \rho_{i_1} c_n^{(i_1)}(a_2 \otimes \dots \otimes a_n \otimes a_1).$$

For $n = 1$, it is clear. Now suppose the result proved to the ranks $1, \dots, n-1$, and consider $(a_1, \dots, a_n) \in \mathcal{A}_{i_0 i_1} \times \dots \times \mathcal{A}_{i_{n-1} i_n}$, with $i_0 = i_n$. One has, by formulae (2.2), (2.5),

$$c_n^{(i_0)}(a_1 \otimes \dots \otimes a_n) = \underbrace{\varphi_{i_0}(a_1 \dots a_n)}_X - \underbrace{\sum_{\substack{\pi \in \text{NC}(n) \\ \pi < 1_n}} \prod_{\substack{V \in \pi \\ V = \{k_1 < \dots < k_m\}}} c_m^{(i_{k_m})}(a_{k_1} \otimes \dots \otimes a_{k_m})}_Y,$$

and

$$\underbrace{\varphi_{i_1}(a_2 \dots a_n a_1)}_{X'} - \underbrace{\sum_{\substack{\pi \in \text{NC}(n) \\ \pi < 1_n}} \prod_{\substack{V \in \pi \\ V = \{k_1 < \dots < k_m\}}} c_m^{(i_{\sigma(k_m)})}(a_{\sigma(k_1)} \otimes \dots \otimes a_{\sigma(k_m)})}_{Y'},$$

where σ is the cycle $(12 \dots n)$ of $[n]$.

Since $\rho_{i_0} X = \rho_{i_1} X'$ (by formula (1.1)), it suffices to prove that

$$\rho_{i_0} Y = \rho_{i_1} Y'.$$

To do that, it suffices to propose a bijective correspondence $\pi \mapsto \tilde{\pi}$ from $\text{NC}(n) - \{1_n\}$ to $\text{NC}(n) - \{1_n\}$ such that for all $\pi \in \text{NC}(n) - \{1_n\}$,

$$\rho_{i_0} \times \prod_{\substack{V \in \pi \\ V = \{k_1 < \dots < k_m\}}} c_m^{(i_{k_m})}(a_{k_1} \otimes \dots \otimes a_{k_m}) = \rho_{i_1} \times \prod_{\substack{V \in \tilde{\pi} \\ V = \{k_1 < \dots < k_m\}}} c_m^{(i_{\sigma(k_m)})}(a_{\sigma(k_1)} \otimes \dots \otimes a_{\sigma(k_m)}).$$

By induction hypothesis, the correspondence which maps $\pi \in \text{NC}(n) - \{1_n\}$ to $\tilde{\pi}$ defined by

$$k \tilde{\pi} l \Leftrightarrow \sigma(k) \pi \sigma(l)$$

is convenient.

The following theorem has been proved in the section called *Rectangular Gaussian distribution and Marchenko-Pastur distribution* of [3].

THEOREM 2.3. *For $k, l \in [d]$ such that $\rho_k \leq \rho_l$, $b \in \mathcal{A}_{k,l}$ satisfies, for all positive integer n ,*

$$c_{2n}^{(k)}(b \otimes b^* \otimes \cdots \otimes b^*) = \frac{\rho_l}{\rho_k} \delta_{n,1}$$

if and only if the moments of bb^ in $(\mathcal{A}_{kk}, \varphi_k)$ is the Marchenko-Pastur distribution with parameter $\frac{\rho_l}{\rho_k}$ (defined p. 101 of [10]).*

3. FREENESS WITH RESPECT TO φ VERSUS FREENESS WITH AMALGAMATION OVER \mathcal{D}

3.1. \mathcal{D} -CENTRAL LIMIT THEOREMS. On sets of matrices, $\|\cdot\|$ will denote the operator norm associated to the canonical hermitian norms. A self-adjoint element X of \mathcal{A} is said to be *\mathcal{D} -semicircular with covariance φ* if it satisfies

- (i) $c_1(X) = 0$
- (ii) $\forall d \in \mathcal{D}, c_2(Xd \otimes X) = \varphi(d)$,
- (iii) $\forall k \geq 3, \forall d_1, \dots, d_k \in \mathcal{D}, c_k(Xd_1 \otimes \cdots \otimes Xd_k) = 0$

Note that it determines the \mathcal{D} -distribution of X .

THEOREM 3.1 (\mathcal{D} -central limit theorem). *Consider a family $(X_i)_{i \geq 1}$ of self-adjoint elements of \mathcal{A} which satisfy*

- (a) X_1, X_2, \dots are free with amalgamation over \mathcal{D} ,
- (b) $\forall i, \forall d \in \mathcal{D}, \mathbb{E}(X_i) = 0, \mathbb{E}(X_i d X_i) = \varphi(d)$,
- (c) $\forall k, \forall d_1, \dots, d_k \in \mathcal{D}, \sup_{i \geq 1} \|\mathbb{E}(X_i d_1 \cdots X_i d_k)\| < \infty$.

Then $Y_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ converges in \mathcal{D} -distribution to a \mathcal{D} -semicircular element with covariance φ .

This theorem is very closed to many well-known results of free probability theory (e.g. Theorem 4.2.4 of [24]).

We prove now a kind of multidimensional \mathcal{D} -central limit Theorem, analogous to Theorem 2.1 of [29]:

THEOREM 3.2. *Consider a family $(T_j)_{j \in \mathbb{N}}$ of self-adjoint elements of \mathcal{A} and a $*$ -subalgebra \mathcal{B} of \mathcal{A} containing \mathcal{D} , such that*

$$(H1) \forall m, \forall B_1, \dots, B_m \in \mathcal{B}, \sup_{i_1, \dots, i_m \in \mathbb{N}} \|\mathbb{E}(T_{i_1} B_1 T_{i_2} \cdots T_{i_m} B_m)\| < \infty,$$

(H2) for $m \geq 1$, for $B_0, \dots, B_m \in \mathcal{B}$, for $\alpha : [m] \rightarrow \mathbb{N}$, one has

(a) $\mathbb{E}(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = 0$ if an element of \mathbb{N} has exactly one antecedent by α ,

(b) $\mathbb{E}(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = \varphi(B_r) \mathbb{E}(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(r-1)} B_{r-1} B_{r+1} \cdots T_{\alpha(m)} B_m)$ if no element of \mathbb{N} has strictly more than two antecedents by α and $\alpha(r) = \alpha(r+1)$, with $1 \leq r < m$,

(c) $E(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = 0$ if no element of \mathbb{N} has strictly more than two antecedents by α and for all $1 \leq r < m$, $\alpha(r) \neq \alpha(r+1)$.

Consider $\beta : \mathbb{N}^2 \rightarrow \mathbb{N}$ injective, and define $X_{m,n} = \frac{1}{\sqrt{n}} \sum_{j=1}^n T_{\beta(m,j)}$. Then for all m , $X_{m,n}$ converge in distribution, when $n \rightarrow \infty$, to a \mathcal{D} -semicircular element with covariance φ , and the family of subsets $(B, (\{X_{m,n}\})_{m \in \mathbb{N}})$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$.

Moreover, if

- (a') $(B, (\{T_j\})_{j \in \mathbb{N}})$ is free with amalgamation over \mathcal{D} ,
- (b') $\forall j, \forall d \in \mathcal{D}, E(T_j) = 0, E(T_j d T_j) = \varphi(d)$,
- (c') $\forall m, \forall d_1, \dots, d_m \in \mathcal{D}, \sup_{j \in \mathbb{N}} \|E(T_j d_1 \cdots T_j d_m)\| < \infty$,

then (H1) and (H2) are satisfied.

Proof. We shall proceed as in the proof of theorem 2.1 of [29]. First we prove that [(a'),(b'),(c')] implies [(H1),(H2)]. Then we prove that it suffices to prove the result replacing [(H1),(H2)] by [(a'),(b'),(c')], and at last we prove the result in this particular case. For $x \in \mathcal{A}$, we define $\overset{\circ}{x} := x - E(x)$.

Step I. Suppose that the T_j 's and \mathcal{D} satisfy [(a'),(b'),(c')].

The proof of the fact that (a') and (c') together imply (H1) is along the same lines as the proof of 1° of the Step I of the proof of Theorem 2.1 of [29], so we leave it to the reader.

Consider $m \geq 1, B_0, \dots, B_m \in \mathcal{B}, \alpha : [m] \rightarrow \mathbb{N}$.

(H2).(a) follows from (a'), (b'), and the following easy result:

$$(3.1) \quad \forall a, b, c \in \mathcal{A}, \{a\}, \{b, c\} \text{ free with amalgamation over } \mathcal{D} \Rightarrow E(bac) = E(bE(a)c).$$

Suppose no element of \mathbb{N} has strictly more than two antecedents by α and $\alpha(r) = \alpha(r+1)$, with $1 \leq r < m$.

Let us prove that

$$E(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = \varphi(B_r) E(\underbrace{B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(r-1)} B_{r-1}}_{:=A} \underbrace{B_{r+1} \cdots T_{\alpha(m)} B_m}_{:=B}).$$

Suppose first that $B_r \in \mathcal{D}$. Then $\{Y_{\alpha(r)} B_r Y_{\alpha(r+1)}\}, \{A, B\}$ are free with amalgamation over \mathcal{D} , so, by (3.1),

$$E(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = E(A E(Y_{\alpha(r)} B_r Y_{\alpha(r+1)}) B).$$

But by (b'), $E(Y_{\alpha(r)} B_r Y_{\alpha(r+1)}) = \varphi(B_r)$, which allows us to conclude.

So, by linearity, we can now suppose that $E(B_r) = 0$. In this case, $\varphi(B_r) = 0$, so it suffices to prove that $E(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = 0$. It follows from (a') and (1.2),

applied to all terms of the right hand side of:

$$\begin{aligned} \mathbb{E}(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) &= \mathbb{E}(\overset{\circ}{A} Y_{\alpha(r)} B_r Y_{\alpha(r+1)} \overset{\circ}{B}) + \mathbb{E}(A) \mathbb{E}(Y_{\alpha(r)} B_r Y_{\alpha(r+1)} \overset{\circ}{B}) \\ &\quad + \mathbb{E}(\overset{\circ}{A} Y_{\alpha(r)} B_r Y_{\alpha(r+1)}) \mathbb{E}(B) \\ &\quad + \mathbb{E}(A) \mathbb{E}(Y_{\alpha(r)} B_r Y_{\alpha(r+1)}) \mathbb{E}(B). \end{aligned}$$

Suppose that no element of \mathbb{N} has strictly more than two antecedents by α and that for all $1 \leq r < m$, $\alpha(r) \neq \alpha(r+1)$. By linearity, $\mathbb{E}(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m)$ is equal to

$$\sum_{P \subset \{0, \dots, m\}} \mathbb{E} \left((1_P(0) \overset{\circ}{B}_0 + 1_{P^c}(0) \mathbb{E}(B_0)) T_{\alpha(0)} \cdots T_{\alpha(m)} (1_P(m) \overset{\circ}{B}_0 + 1_{P^c}(m) \mathbb{E}(B_0)) \right),$$

where for $P \subset \{0, \dots, m\}$, 1_P (resp. 1_{P^c}) denotes the characteristic function of P (resp. of its complementary). It follows from (a') and (1.2), applied to all terms of the sum, that $\mathbb{E}(B_0 T_{\alpha(1)} B_1 \cdots T_{\alpha(m)} B_m) = 0$.

Step II. After having eventually extended \mathcal{A} , consider a free with amalgamation over \mathcal{D} family $(x_m)_{m \geq 1}$ of \mathcal{D} -semicircular elements of \mathcal{A} with covariance φ , which is also free with amalgamation over \mathcal{D} with \mathcal{B} . Let us show that in order to prove that for all $r \geq 1$, $B_0, \dots, B_r \in \mathcal{B}$, $m : [r] \rightarrow \mathbb{N}$,

$$\mathbb{E}(B_0 X_{m(1),n} B_1 \cdots B_{r-1} X_{m(r),n} B_r) \xrightarrow{n \rightarrow \infty} \mathbb{E}(B_0 x_{m(1)} B_1 \cdots B_{r-1} x_{m(r)} B_r),$$

it suffices to prove it in the particular case where [(a'),(b'),(c')] are satisfied.

So consider $r \geq 1$, $B_0, \dots, B_r \in \mathcal{B}$, and $m : [r] \rightarrow \mathbb{N}$. Define, for $n \geq 1$, the set $P_n = m([r]) \times [n]$, and define, for $I = (p_1, \dots, p_r) \in P_n^r$,

$$II_I = B_0 T_{\beta(p_1)} B_1 \cdots B_{r-1} T_{\beta(p_r)} B_r.$$

Then by linearity, there exists a family $(C_I)_I$ of elements of $\{0, 1\}$, indexed by $I \in P_n^r$, such that we have:

$$\mathbb{E}(B_0 X_{m(1),n} B_1 \cdots B_{r-1} X_{m(r),n} B_r) = \frac{1}{n^{\frac{r}{2}}} \sum_{I \in P_n^r} C_I \mathbb{E}(II_I).$$

By (H2).(a), if $\mathbb{E}(II_I) \neq 0$, then no element of P_n appears exactly once in I . Let $R_{n,r}$ be the set of elements I of P_n^r such that no element of P_n appears exactly once in I and an element of P_n appears at least three times in I . Its cardinality is less than $|P_n| \times |P_n|^{(r-3)/2} r! = o(n^{\frac{r}{2}})$, so, since by (H1) there exists $M > 0$ such that for all n , $I \in P_n^r$, $\|\mathbb{E}(II_I)\| \leq M$, one has

$$\frac{1}{n^{\frac{r}{2}}} \sum_{I \in R_{n,r}} \|C_I \mathbb{E}(II_I)\| \xrightarrow{n \rightarrow \infty} 0.$$

So

$$(3.2) \quad \lim_{n \rightarrow \infty} \mathbb{E}(B_0 X_{m(1),n} B_1 \cdots B_{r-1} X_{m(r),n} B_r)$$

exists if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{r}{2}}} \sum_{\substack{I \in P_n^r \text{ such that each element} \\ \text{of } P_r \text{ appears exactly 0 or 2 times in } I}} C_I E(\Pi_I),$$

exists, and in this case, the limits are the same.

But the computation of $E(\Pi_I)$, for elements I of P_n^r such as those considered in the previous sum, is completely determined by (H2). So the limit (3.2) will be the same (and exist in the same time) if one replaces the T_j 's by another family which satisfies [(H1),(H2)]. In particular, by Step I, one can suppose that [(a'),(b'),(c')] are satisfied.

Step III. Suppose now that [(a'),(b'),(c')] are satisfied. The previous theorem allows us to claim that for all m , $X_{m,n}$ converges in \mathcal{D} -distribution, as $n \rightarrow \infty$, to a \mathcal{D} -semicircular with covariance φ . Moreover, for all n , the family of subsets $(\mathcal{B}, (\{X_{m,n}\})_{m \in \mathbb{N}})$ is free with amalgamation over \mathcal{D} , so the theorem is proved. ■

The main theorem of this section is the following one. Recall that a family $(\mathcal{A}_i)_{i \in I}$ of subalgebras of \mathcal{A} is said to be *free* if for all n , $i_1 \neq \dots \neq i_n \in I$, for all $x(1) \in \mathcal{A}_{i_1} \cap \ker \varphi, \dots, x(n) \in \mathcal{A}_{i_n} \cap \ker \varphi$, one has

$$(3.3) \quad \varphi(x(1) \cdots x(n)) = 0.$$

A family $(\chi_i)_{i \in I}$ of subsets of \mathcal{A} is said to be *free* if there exists free subalgebras $(\mathcal{A}_i)_{i \in I}$ such that for all i , $\chi_i \subset \mathcal{A}_i$. In order to avoid confusion between freeness and freeness with amalgamation over \mathcal{D} , freeness will be called φ -freeness. We use the notion of φ -distribution of a family $(a_i)_{i \in I}$ of elements of \mathcal{A} : it is the application which maps a word $X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n}$ in X_i, X_i^* ($i \in I$) to $\varphi(a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n})$. It is easy to see that the φ -distribution of a φ -free family depends only on the individual φ -distributions, and that the \mathcal{D} -distribution of a family which contains \mathcal{D} is determined by its φ -distribution. At last, recall that φ -semicircular elements are elements whose moments are given by the moments of the semicircle distribution with center 0 and radius 2.

THEOREM 3.3. *Consider, in \mathcal{A} , a family $(y(s))_{s \in \mathbb{N}}$ of φ -semicircular elements, and a subalgebra \mathcal{B} of \mathcal{A} which contains \mathcal{D} such that the family $(\mathcal{B}, (\{y(s)\})_{s \in \mathbb{N}})$ is φ -free. Then the family $(\mathcal{B}, (\{y(s)\})_{s \in \mathbb{N}})$ is also free with amalgamation over \mathcal{D} , and the \mathcal{D} -distribution of $y(s)$'s is the \mathcal{D} -semicircular distribution with covariance φ .*

Proof. Consider $\beta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ injective. By stability of φ -semicircular distribution under free convolution, it is clear that for all $n \geq 1$, the family

$$\left(\mathcal{B}, \frac{1}{\sqrt{n}} \sum_{j=1}^n y(\beta(m, j)) \right)_{m \geq 0}$$

has the same φ -distribution (and hence \mathcal{D} -distribution, because contains \mathcal{D}) as $(\mathcal{B}, (\{y(s)\})_{s \in \mathbb{N}})$. So it suffices to prove that $(\mathcal{B}, (\{y(s)\})_{s \in \mathbb{N}})$ satisfies (H1) and (H2).

(H1) is due to the fact that for $m \geq 1$ and $b_1, \dots, b_m \in \mathcal{B}$ fixed, for all $s_1, \dots, s_m \in \mathbb{N}$,

$$\begin{aligned} \|\mathbb{E}(y(s_1)b_1 \cdots y(s_m)b_m)\| &= \left\| \sum_{k=1}^d \varphi_k(p_k y(s_1)b_1 \cdots y(s_m)b_m p_k) \cdot p_k \right\| \\ &= \max_{1 \leq k \leq d} \frac{1}{\rho_k} |\varphi(p_k y(s_1)b_1 \cdots y(s_m)b_m p_k)|, \end{aligned}$$

which only depends on the partition π of $[m]$ which links two elements i, j if and only if $s_i = s_j$.

To prove (H2)(a),(b),(c), since for all $x \in \mathcal{A}$,

$$\mathbb{E}(x) = \sum_{k=1}^d \frac{1}{\rho_k} \varphi(p_k x p_k) \cdot p_k$$

and the algebra \mathcal{B} contains all p_i 's, it suffices to prove it with \mathbb{E} replaced by φ . Then it follows from the last assertion of Theorem 2.1 of [29]. ■

REMARKS ABOUT THE PREVIOUS THEOREM. (1) Let \mathcal{C} be a subalgebra of \mathcal{A} which is φ -free with \mathcal{D} . It is easy to see that for all $x \in \mathcal{C}$, $\mathbb{E}(x) = \varphi(x) \cdot 1$, hence for all $n \geq 1$, $a_1, \dots, a_n \in \mathcal{C}$,

$$c_n(a_1 \otimes \cdots \otimes a_n) = \mathfrak{K}_n(a_1, \dots, a_n) \cdot 1,$$

where \mathfrak{K}_n is the n -th φ -cumulant function. So a φ -free family of subalgebras of \mathcal{C} is also \mathbb{E} -free: φ -freeness implies vanishing of mixed φ -cumulants, which implies \mathbb{E} -freeness, by theorem 2.1. It is not enough to prove our result, because the algebra \mathcal{C} cannot in the same time be φ -free with \mathcal{D} and contain \mathcal{D} , hence cannot contain \mathcal{B} .

(2) This theorem recalls Theorem 3.5 of [18]. But to prove our result using this theorem, it would be necessary to compute \mathcal{B} -cumulant functions.

3.2. POLAR DECOMPOSITION R -DIAGONAL ELEMENTS WITH NON TRIVIAL KERNEL. In the following, we shall use *polar decomposition* of non invertible elements of von Neumann algebras (for example, in the following section, non invertible matrices). Recall that the polar decomposition of an element x of a von Neumann algebra consists in writing $x = uh$, where $h \geq 0$ such that $\ker h = \ker x$, and u is a partial isometry with initial space the orthogonal of $\ker x$ and with final space the closure of the image of x (see the appendix of [7] or the section 0.1 of [27]).

R -diagonal elements have been introduced by Nica and Speicher in [13]. In this section, we consider a W^* -noncommutative probability space (\mathcal{M}, τ) . In 1.9 of [13], R -diagonal elements of (\mathcal{M}, τ) were characterized as the elements x which can be written $x = uh$, where u is a Haar unitary (i.e. u is unitary, and for all $n \in \mathbb{Z} - \{0\}$, $\tau(u^n) = 0$), and h is a positive element τ -free with u . If $x \in \mathcal{M}$ is R -diagonal and if x has a null kernel, then with the previous notations, uh is the polar decomposition of x . In the case where x has a non trivial kernel, the polar decomposition of x is $(up)h$, where p is the projection on the orthogonal of

$\ker(x)$. In this section, we shall prove that up, h are free with amalgamation over the algebra $\text{Span}\{p, 1 - p\}$.

We first have to prove a preliminary result:

PROPOSITION 3.4. *Consider the space (\mathcal{A}, φ) introduced in section 1, suppose moreover that (\mathcal{A}, φ) is a W^* -probability space. Consider, in \mathcal{A} , a family $(y(s))_{s \in \mathbb{N}}$ of normal elements. Consider also a subalgebra \mathcal{B} of \mathcal{A} which contains \mathcal{D} such that the family $(\mathcal{B}, (\{y(s)\}_{s \in \mathbb{N}}))$ is φ -free. Then the family $(\mathcal{B}, (\{y(s)\}_{s \in \mathbb{N}}))$ is also free with amalgamation over \mathcal{D} .*

Proof. Let $(\mathcal{N}, \tilde{\tau})$ be a W^* -probability space which is generated, as a W^* -algebra, by a family $x(s)$ ($s \in \mathbb{N}$) of τ -semicircular elements and an algebra $\tilde{\mathcal{B}}$ isomorphic to \mathcal{B} by a map $x \rightarrow \tilde{x}$, such that the family $(\tilde{\mathcal{B}}, (\{x(s)\}_{s \in \mathbb{N}}))$ is $\tilde{\tau}$ -free. The distributions of the $x(s)$'s are non-atomic, so for each $s \in \mathbb{N}$, there exists a Borel function f_s on the real line such that $f_s(x(s))$ has the same distribution as $y(s)$. Note that the $\tilde{\tau}$ -freeness (resp. freeness with amalgamation over $\tilde{\mathcal{D}}$) of a family $(\mathcal{A}_i)_{i \in I}$ of $*$ -subalgebras of \mathcal{N} (resp. of $*$ -subalgebras of \mathcal{N} which all contain $\tilde{\mathcal{D}}$) is equivalent to the φ -freeness (resp. the freeness with amalgamation over $\tilde{\mathcal{D}}$) of the family $(\mathcal{A}'_i)_{i \in I}$ of von Neumann algebras they generate. So, by theorem 3.3 and by the fact that for all s , $f_s(x(s)) \in \{x(s)\}''$, the family $(\tilde{\mathcal{B}}, (\{f_s(x(s))\}_{s \in \mathbb{N}}))$ is free with amalgamation over \mathcal{D} . But the map

$$\begin{aligned} \mathcal{B} \cup \{y(s); s \in \mathbb{N}\} &\rightarrow \tilde{\mathcal{B}} \cup \{f_s(x(s)); s \in \mathbb{N}\} \\ z &\mapsto \begin{cases} \tilde{z} & \text{if } z \in \mathcal{B} \\ f_s(x(s)) & \text{if } z = y(s) \end{cases} \end{aligned}$$

extends clearly to a W^* -probability spaces isomorphism, hence the family

$$(\mathcal{B}, (\{y(s)\}_{s \in \mathbb{N}}))$$

is also free with amalgamation over \mathcal{D} . ■

COROLLARY 3.5. *Consider a W^* -noncommutative probability space (\mathcal{M}, τ) , and an R -diagonal element x of \mathcal{M} with non trivial kernel. Let p_1 be the projection on $\ker x$, and $p_2 = 1 - p_1$. Then the polar decomposition $x = vh$ of x is such that*

- v, h are free with amalgamation over $\mathcal{D} := \text{Span}\{p_1, p_2\}$,
- v has the \mathcal{D} -distribution of up_2 , where u is a Haar unitary τ -free with p_2 ,
- the \mathcal{D} -distribution of h is defined by the fact that $h^2 = x^*x$ and $p_2hp_2 = h$.

Moreover, the projection on the final subspace of v is τ -free with h .

Note that this result could also have been deduced from lemma 2.6 of [22], but the proof of this lemma is incomplete, and a complete proof of the lemma would take as long as what we use to prove this corollary.

REMARK 3.6. Elements with τ -distributions such as the one of v are called an (α, α) -Haar partial isometries in Remark 1.9 3° of [17]

Proof. By 1.9 of [13], x can be written $x = uh$, where u is a Haar unitary τ -free with h , and the polar decomposition of x is $(up_2)h$, where p_2 is the projection on the orthogonal of $\ker x = \ker h$. Thus, with the notation $\mathcal{D} := \text{Span}\{1 - p_2, p_2\}$, it suffices to prove the freeness with amalgamation over \mathcal{D} of up_2 and h , which follows from the freeness with amalgamation over \mathcal{D} of u and h , which follows from proposition 3.4. The projection on the final subspace of up_2 is up_2u^* is τ -free with h by lemma 3.7 of [8]. ■

REMARK 3.7. In the same way, we can prove the following: let q_1 be the projector on $\overline{\text{Ran}(x)}$ and $q_2 = 1 - q_1$, then the polar decomposition of x is wh , where

- w, h are free with amalgamation over $\mathcal{D} := \text{Span}\{q_1, q_2\}$,
- w has the \mathcal{D} -distribution of q_1u , where u is a Haar unitary τ -free with q_1 ,
- the \mathcal{D} -distribution of h is defined by the fact that $h^2 = x^*x$ and $p_2hp_2 = h$.

Moreover, the projection on $\ker w$ is τ -free with \mathcal{D} .

4. ASYMPTOTIC FREENESS WITH AMALGAMATION OVER \mathcal{D} OF RECTANGULAR RANDOM MATRICES

Since in the present section, we will prove asymptotic freeness with amalgamation over \mathcal{D} of random matrices in an analogous way to the proofs of [29] and [32], we shall frequently refer to those papers.

Consider, for $n \geq 1$, $q_1(n), \dots, q_d(n)$ positive integers with sum n such that $\frac{q_1(n)}{n} \xrightarrow[n \rightarrow \infty]{} \rho_1, \dots, \frac{q_d(n)}{n} \xrightarrow[n \rightarrow \infty]{} \rho_d$ (recall that $\rho_1 = \varphi(p_1), \dots, \rho_d = \varphi(p_d)$). Then for all n , \mathcal{D} can be identified with a $*$ -subalgebra of the algebra \mathfrak{M}_n of complex $n \times n$ matrices by

$$\forall \lambda_1, \dots, \lambda_d \in \mathbb{C}, \quad \text{diag}(\lambda_1, \dots, \lambda_d) \simeq \begin{bmatrix} \lambda_1 I_{q_1(n)} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_d I_{q_d(n)} \end{bmatrix}$$

The image of each p_k will be denoted by $p_k(n)$. tr will denote the normalized trace on \mathfrak{M}_n , while Tr will denote the trace. $e(i, j; n)$ will denote the matrix-units of \mathfrak{M}_n .

We shall refer to M_n as a set of $n \times n$ random matrices (over a probability space not mentioned here), while the elements of \mathfrak{M}_n (which is a subalgebra of M_n) will be called constant matrices. M_n is endowed with the state $\mathbb{E}(\text{tr}(\cdot))$, and the identification of \mathcal{D} with a $*$ -subalgebra of M_n allows us to speak of convergence in \mathcal{D} -distribution of random matrices.

The following result is an immediate corollary of Theorem 2.2 of [32] and of theorem 3.3.

THEOREM 4.1. *Let, for $s \geq 0, n \geq 1, Y(s, n) = \sum_{1 \leq i, j \leq n} a(i, j; n, s) e(i, j; n)$ be a random matrix. Assume that $a(i, j; n, s) = \overline{a(j, i; n, s)}$ and that*

$$\{\Re a(i, j; n, s); 1 \leq i \leq j \leq n, s \in \mathbb{N}\} \cup \{\Im a(i, j; n, s); 1 \leq i < j \leq n, s \in \mathbb{N}\}$$

are independent Gaussian random variables, which are $(0, (2n)^{-1})$ if $i < j$ and $(0, n^{-1})$ if $i = j$. Let further $(B(j, n))_{j \in \mathbb{N}}$ be a family of elements of \mathfrak{M}_n , stable under multiplication and adjunction, which contains $p_1(n), \dots, p_d(n)$, such that for all j , the sequence $(\|E(B(j, n))\|)_n$ is bounded, and which converges in \mathcal{D} -distribution.

Then the family $(\{B(j, n); j \in \mathbb{N}\}, \{Y(s, n)\}_s)$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$, and the limit \mathcal{D} -distribution of each $Y(s, n)$ is the \mathcal{D} -semicircular \mathcal{D} -distribution with covariance φ .

COMPARISON WITH THEOREM 4.1 OF [21]. Since the $B(j, n)$'s are diagonal and since convergence in \mathcal{D} -distribution is less restrictive than convergence for $\|\cdot\|_\infty$, which is the one used by Shlyakhtenko in [21], this result cannot be deduced from Theorem 4.1 of [21].

In order to modelize asymptotic \mathcal{D} -distribution of non hermitian gaussian random matrices, let us introduce the \mathcal{D} -circular distribution with covariance φ . It is the \mathcal{D} -distribution of an element of \mathcal{A} which can be written $c = \frac{a+ia'}{\sqrt{2}}$, with a, a' \mathcal{D} -semicircular elements with covariance φ , which are free with amalgamation over \mathcal{D} . Note that the \mathcal{D} -distribution of c can be defined by the following rules:

- (i) $c_1(c) = c_1(c^*) = 0$,
- (ii) $\forall d \in \mathcal{D}, c_2(cd \otimes c) = c_2(c^*d \otimes c^*) = 0, c_2(cd \otimes c^*) = c_2(c^*d \otimes c) = \varphi(d)$,
- (iii) $\forall k \geq 3, \varepsilon_1, \dots, \varepsilon_k \in \{., *\}, d_1, \dots, d_k \in \mathcal{D}, c_k(c^{\varepsilon_1}d_1 \otimes \dots \otimes c^{\varepsilon_k}d_k) = 0$.

COROLLARY 4.2. *The hypothesis are the same as the one of the previous theorem, except that the random matrices are not self-adjoint anymore, and their law is defined by the fact that*

$$\{\Re a(i, j; n, s); 1 \leq i, j \leq n, s \in \mathbb{N}\} \cup \{\Im a(i, j; n, s); 1 \leq i, j \leq n, s \in \mathbb{N}\}$$

are independent gaussian random variables, which are $(0, (2n)^{-1})$. Then the family $(\{B(j, n); j \in \mathbb{N}\}, \{Y(s, n)\}_s)$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$, and the limit \mathcal{D} -distribution of each $Y(s, n)$ is the \mathcal{D} -circular distribution with covariance φ .

Proof. It suffices to notice that if Y, Y' are independent random matrices as in the hypothesis of the previous theorem, then $\frac{Y+iY'}{\sqrt{2}}$ has the distribution of the ones of the hypothesis of the corollary. ■

The previous corollary allows us to modelize asymptotic collective behavior of independent rectangular gaussian random matrices with different sizes: consider, for $s \geq 0, k, l \in [d], n \geq 1, M(s, k, l, n)$ a random matrix of size $q_k(n) \times q_l(n)$, with independent complex gaussian entries. In order to have a non trivial limit for asymptotic singular values of the $M(s, k, l, n)$'s (the *singular values* of a $q \times q'$

matrix M are the eigenvalues of MM^* if $q \leq q'$, and of M^*M if $q \geq q'$, it is well known, by results about Wishart matrices (see, e.g., [10],[19]) that the variance of the entries must have the order of $q_k(n)$, i.e. of n . So we will suppose that the real and imaginary parts of the entries of the $M(s, k, l, n)$'s are independent $N(0, (2n)^{-1})$. To give the asymptotic behavior of these matrices amounts to give the asymptotic normalized traces of words of the type:

$$(4.1) \quad M(s_1, k_1, l_1, n)^{\varepsilon_1} M(s_2, k_2, l_2, n)^{\varepsilon_2} \cdots M(s_m, k_m, l_m, n)^{\varepsilon_m},$$

where $m \geq 1$, $s_1, \dots, s_m \in \mathbb{N}$, $\varepsilon_1, \dots, \varepsilon_m \in \{., *\}$, $k_1, l_1, \dots, k_m, l_m \in [d]$ such that the product is possible and gives a square matrix.

In order to avoid problems of definition of the products, let us embed all these matrices in $n \times n$ matrices: for all (s, k, l, n) , $M(s, k, l, n)$ will be replaced by $X(s, k, l, n) := p_k(n)Y(s, k, l, n)p_l(n)$, where $Y(s, k, l, n)$ is a random matrix as in the hypothesis of the previous corollary. Then if the product (4.1) is not defined, the product

$$(4.2) \quad X(s_1, k_1, l_1, n)^{\varepsilon_1} X(s_2, k_2, l_2, n)^{\varepsilon_2} \cdots X(s_m, k_m, l_m, n)^{\varepsilon_m}$$

is zero. In the other case, the product (4.2) is a simple element of M_n (simple refers to the definition given in section 1), whose only non zero block is (4.1). If moreover, (4.1) is a square matrix, its normalized trace is the only non zero coordinate of

$$E [(X(s_1, k_1, l_1, n)^{\varepsilon_1} X(s_2, k_2, l_2, n)^{\varepsilon_2} \cdots X(s_m, k_m, l_m, n)^{\varepsilon_m})].$$

So the following corollary gives an answer to the question of the asymptotic collective behavior of independent rectangular Gaussian random matrices with different sizes.

COROLLARY 4.3. *Let, for $s \geq 0, k, l \in [d], n \geq 1$,*

$$X(s, k, l, n) = p_k(n) \left[\sum_{1 \leq i, j \leq n} a(i, j; n, k, l, s) e(i, j; n) \right] p_l(n)$$

be a random matrix. Assume that

$$\begin{aligned} & \{\Re a(i, j; n, k, l, s); i, j \in [n], k, l \in [d], s \in \mathbb{N}\} \\ & \cup \{\Im a(i, j; n, k, l, s); i, j \in [n], k, l \in [d], s \in \mathbb{N}\} \end{aligned}$$

are independent Gaussian random variables, which are $(0, (2n)^{-1})$. Let further

$$(B(j, n))_{j \in \mathbb{N}}$$

be a family elements of \mathfrak{M}_n , which satisfies the same assumptions as in the hypothesis of Theorem 4.1.

Then the family $(\{B(j, n); j \in \mathbb{N}\}, (\{X(s, k, l, n)\})_{s, k, l})$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$.

Proof. It is an immediate consequence of the previous corollary and of the fact that freeness with amalgamation over \mathcal{D} is preserved by multiplication by elements of \mathcal{D} . ■

For $n \geq 1$, the set of matrices U of $p_k(n)\mathfrak{M}_n p_k(n)$ such that $UU^* = U^*U = p_k(n)$ will be denoted by $\mathbb{U}_k(n)$. It is a compact group, isometric to the group of $q_k(n) \times q_k(n)$ unitary matrices. By lemma 4.3.10 p. 160 of [10]), the partial isometry of the polar decomposition of $X(s, k, k, n)$ is uniform on $\mathbb{U}_k(n)$ (i.e. distributed according to the Haar measure).

PROPOSITION 4.4. *Let, for $n \geq 1$, $V(s, k, n)$ ($s \in \mathbb{N}, k \in [d]$), be a family of independent random matrices, such that for all s, k , $V(s, k, n)$ is uniform on $\mathbb{U}_k(n)$. Let further $(B(j, n))_{j \in \mathbb{N}}$ be a family elements of \mathfrak{M}_n which satisfies the same assumptions as in the previous results. Then the family $(\{B(j, n); j \in \mathbb{N}\}, \{V(s, k, n)\}_{s, k})$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$.*

Note that two elements of respectively $\mathcal{A}_{kk'}$, $\mathcal{A}_{ll'}$, with $k \neq l$ (or more generally of $\mathcal{A}_{kk'}$, $\mathcal{A}_{ll'}$, with $\{k, k'\} \cap \{l, l'\} = \emptyset$) are always free with amalgamation over \mathcal{D} , and that elements of \mathcal{A}_{kk} are free with amalgamation over \mathcal{D} if and only if they are free in the compressed space $(\mathcal{A}_{kk}, \varphi_k)$. So without the set of constant matrices, Proposition 4.4 would be an easy consequence of Theorem 3.8 of [29]. That being said, the proof of This proposition is very closed to the one of Theorem 3.8 of [29].

The proof of the proposition relies on the following lemma. We endow M_n with the norms $|M|_p = \left(\mathbb{E} \operatorname{tr}(MM^*)^{r/2}\right)^{\frac{1}{r}}$. They fulfill Hölder inequalities (see [12]).

LEMMA 4.5. *Let, for $n \geq 1$, $(M(i, n))_{i \in I}$ be a family of $n \times n$ random matrices, and $(B(j, n))_{j \in \mathbb{N}}$ be a family elements of \mathfrak{M}_n , which satisfies the same assumptions as in the hypothesis of Theorem 4.1. Suppose moreover that for all $i \in I, r \geq 1$, the sequence $|M(i, n)|_r$ is bounded. Suppose that for all $\delta > 0$ and $n \geq 1$, there exists a family $(M(i, n, \delta))_{i \in I}$ of random $n \times n$ matrices such that*

- (i) *the family $(\{B(j, n); j \in \mathbb{N}\}, \{M(i, n, \delta)\}_{i \in I})$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$,*
- (ii) *for all $i \in I, r \geq 1$, $\overline{\lim}_{n \rightarrow \infty} |M(i, n, \delta) - M(i, n)|_r := C(i, r, \delta)$ is such that $C(i, r, \delta) \xrightarrow{\delta \rightarrow 0} 0$.*

Then the family $(\{B(j, n); j \in \mathbb{N}\}, \{M(i, n)\}_{i \in I})$ is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$.

Proof. Note first that if a sequence (D_n) in \mathcal{D} is such that for all positive δ , there is a sequence $(D_n(\delta))$ in \mathcal{D} which converges and such that $\overline{\lim}_{n \rightarrow \infty} \|D_n(\delta) - D_n\| := C(\delta)$ tends to zero as δ tends to zero, then (D_n) is Cauchy, and hence converges. So, by Hölder inequalities, the family $((B(j, n))_{j \in \mathbb{N}}, (M(i, n))_{i \in I})$ has limit \mathcal{D} -distribution as $n \rightarrow \infty$. Moreover, Hölder inequalities imply too that the

later limit \mathcal{D} -distribution is the limit, for convergence in \mathcal{D} -distribution, as δ tends to zero, of the limit \mathcal{D} -distribution of $((B(j, n))_{j \in \mathbb{N}}, (M(i, n, \delta))_{i \in I})$ as $n \rightarrow \infty$. But the set of \mathcal{D} -distributions of families $((b_j)_{j \in \mathbb{N}}, (m_i)_{i \in I})$ such that the family $(\{b_j; j \in \mathbb{N}\}, (\{m_i\})_{i \in I})$ is free with amalgamation over \mathcal{D} is obviously closed, so the lemma is proved. ■

Let us now give the proof of proposition 4.4.

Proof. Consider independent random matrices $X(s, k, n)$ ($s \in \mathbb{N}, k \in [d], n \in \mathbb{N}$), such that for all s, k, n , $X(s, k, n)$ has the same distribution as $X(s, k, k, n)$ of the previous corollary. Then, as noted before, one can suppose that for all s, k, n , $V(s, k, n)$ is the partial isometry of the polar decomposition of $X(s, k, k, n)$. In this proof, we shall use a particular functional calculus with the matrices

$$X(s, k, n)^* X(s, k, n).$$

Note that these matrices are simple elements of M_n (simple refers to the definition given in section 1): they belong to $p_k(n)M_n p_k(n)$. Here we shall “erase” the action of a function f on the orthogonal of the image of the projector $p_k(n)$. This means that $f(X(s, k, n)^* X(s, k, n))$ will mean $p_k(n)f(X(s, k, n)^* X(s, k, n))p_k(n)$. So we can write:

$$V(s, k, n) = \lim_{\varepsilon \rightarrow 0^+} X(s, k, n)(\varepsilon + X(s, k, n)^* X(s, k, n))^{-1/2}.$$

Step I. As stated in the Step I of the proof of Theorem 3.8 of [29], there exists $C > 0$ such that for all continuous bounded function $f : [0, \infty) \rightarrow (0, \infty)$, for all polynomial P , all s, k , and all $r \geq 1$, the limsup of

$$\overline{\lim}_{n \rightarrow \infty} |X(s, k, n)P(X(s, k, n)^* X(s, k, n)) - X(s, k, n)f(X(s, k, n)^* X(s, k, n))|_r$$

is not larger than $C \sup_{0 \leq t \leq C} |P(t) - f(t)|$.

Step II. Consider $\varepsilon > 0$, and let

$$Y(s, k, n, \varepsilon) = X(s, k, n)(\varepsilon + X(s, k, n)^* X(s, k, n))^{-1/2}.$$

We claim that the family

$$(\{B(j, n); j \in \mathbb{N}\}, (\{Y(s, k, n, \varepsilon)\})_{s, k})$$

is asymptotically free with amalgamation over \mathcal{D} as $n \rightarrow \infty$. It is an easy application of the lemma, using, for all positive δ , the random matrices

$$X(s, k, n)P_\delta(X(s, k, n)^* X(s, k, n)),$$

where P_δ is a polynomial such that $\sup_{0 \leq t \leq C} |P_\delta(t) - (\varepsilon + t)^{-1/2}| \leq \delta$. Let us prove that the hypothesis of the lemma are satisfied. For s, k, r , the boundness of the sequence $(|Y(s, k, n, \varepsilon)|_r)_n$ comes from the boundness of the function $t \mapsto (t/(\varepsilon + t))^{1/2}$ on the positive half line, (i) is due to the previous corollary, and (ii) follows from step I.

Step III. The conclusion is another application of the lemma, where the $Y(s, k, n, \varepsilon)$'s will play the roll of the $M(i, n, \delta)$'s (and ε the roll of δ). Let us, again,

prove that the hypothesis of the lemma are satisfied. For s, k, r , the sequence $(|V(s, k, n)|_r)_n$ is bounded because matrices are in $\mathbb{U}_k(n)$, and (i) follows from step II. Let us prove (ii). We have

$$\begin{aligned} & |Y(s, k, n, \varepsilon) - V(s, k, n)|_r = \\ & |V(s, k, n)(X(s, k, n)^* X(s, k, n))^{1/2}(\varepsilon + X(s, k, n)^* X(s, k, n))^{-1/2} - V(s, k, n)| = \\ & |(X(s, k, n)^* X(s, k, n))^{1/2}(\varepsilon + X(s, k, n)^* X(s, k, n))^{-1/2} - p_k(n)|_r \\ & \text{But } 0 \leq t \leq \varepsilon^{1/2} \quad \Rightarrow \quad 0 \leq t^{1/2}/(\varepsilon + t)^{1/2} \leq 1, \\ & \text{and } t \geq \varepsilon^{1/2} \quad \Rightarrow \quad 1 - \frac{\varepsilon^{1/2}}{2(1 + \varepsilon^{1/2})} \leq t^{1/2}/(\varepsilon + t)^{1/2} \leq 1. \end{aligned}$$

So, if $Q(n)$ denotes the spectral projection of $X(s, k, n)^* X(s, k, n)$ for $[0, \varepsilon^{1/2}]$,

$$|V(s, k, n) - Y(s, k, n, \varepsilon)|_r \leq \mathbb{E}(\text{tr}(p_k(n)Q(n)p_k(n))) + \frac{\varepsilon^{1/2}}{2(1 + \varepsilon^{1/2})},$$

so (ii) is checked, since it is known (see [28] for a precise result) that there exists a constant C' such that

$$\overline{\lim}_{n \rightarrow \infty} \mathbb{E}(\text{tr}(p_k(n)Q(n)p_k(n))) \leq C' \varepsilon^{1/2}.$$

■

Note that, as to Remark 2.3 of [32], it is obvious that the previous proposition also holds for subsequences of the natural numbers. It allows us to prove the following corollary. Its proof is along the same lines as the one of Corollary 2.6 of [32]: it relies on the fact that the topology of convergence in \mathcal{D} -distribution, for countable families, has countable bases of neighborhoods, and hence if for all n , $X(i, n)$ ($i \in I$ countable) is a family of random matrices with norms uniformly bounded, one can extract a subsequence $k(1) < k(2) < \dots$ such that the family $(X(i, k(n)))_{i \in I}$ converges in \mathcal{D} distribution.

COROLLARY 4.6. *Let, for $n \geq 1$, $V(s, k, n)$ ($s \in \mathbb{N}, k \in [d]$), be a family of independent random matrices, such that for all s, k , $V(s, k, n)$ is uniform on $\mathbb{U}_k(n)$ and let $F(\mathbb{N} \times [d]) \ni g \mapsto V^g(n) \in M_n$ be the semigroup morphism which maps the (s, k) -th generator to $V(s, k, n)$. Then, given $N \in \mathbb{N}, R > 0$, and $g_0, \dots, g_N \in F(\mathbb{N} \times [d]) - \{e\}$, the supremum, over families B_1, \dots, B_N of elements of \mathfrak{M}_n such that for all $k \in [N]$, $\|B_k\| \leq R$ and $\mathbb{E}(B_k) = 0$, of*

$$|\text{tr}(V^{g_0}(n)B_1V^{g_1}(n)B_2 \dots V^{g_{N-1}}(n)B_NV^{g_N}(n))|$$

tends to zero as $n \rightarrow \infty$.

At last, in the same way, one can translate the proof of Theorem 2.7 of [32] to prove the following proposition:

PROPOSITION 4.7. *Let, for $n \geq 1$, $V(s, k, n)$ ($s \in \mathbb{N}, k \in [d]$), be a family of independent random matrices, such that for all s, k , $V(s, k, n)$ is uniform on $\mathbb{U}_k(n)$ and let $F(\mathbb{N} \times [d]) \ni g \mapsto V^g(n) \in M_n$ be the semigroup morphism which maps the (s, k) -th*

generator to $V(s, k, n)$. Fix $N \in \mathbb{N}$ and $R > 0$. Let, for each $n \in \mathbb{N}$, $B_1(n), \dots, B_N(n)$ be $n \times n$ constant matrices such that for all $k \in [N]$, $\|B_k(n)\| \leq R$ and $E(B_k(n)) = 0$. Then, given $g_0, \dots, g_N \in F(\mathbb{N} \times [d]) - \{e\}$ and $\varepsilon > 0$, the probability of the event

$$\{ \|E(V^{g_0}(n)B_1V^{g_1}(n)B_2 \cdots V^{g_{N-1}}(n)B_NV^{g_N}(n))\| \leq \varepsilon \}$$

tends to 1 as n goes to infinity.

5. ANALOGUE OF FREE ENTROPY OF SIMPLE ELEMENTS: THE MICROSTATES APPROACH

5.1. DEFINITIONS. For q, q' positive integers, we will denote by $\mathfrak{M}_{qq'}$ (\mathfrak{M}_q when $q = q'$) the set of $q \times q'$ complex matrices.

From now on, we suppose (\mathcal{A}, φ) to be a tracial W^* -probability space, endowed with a family of projectors such as presented in section 1: \mathcal{A} is endowed with a family p_1, \dots, p_d of self-adjoint pairwise orthogonal projectors with sum 1, and for all $k \in [d]$, ρ_k denotes $\varphi(p_k)$. If a is a simple element of \mathcal{A} , the unique $(k, l) \in [d]^2$ such that $a \in \mathcal{A}_{k,l}$ will be called the *type* of a .

In this section, we shall define the entropy of families of simple elements of \mathcal{A} as the asymptotic logarithm of the Lebesgue measure of sets of matrices with closed noncommutative moments. Thus we have to define, for all n , a set of matrices $\mathfrak{M}_n(a)$ where we shall choose the microstates associated to a simple element a . We let, for $k \in [d]$, $q_k(n)$ be the integer part of $\rho_k n$. With this definition of $q_1(n), \dots, q_d(n)$, we keep the notations introduced in the beginning of section 4. We define, for $(k, l) \in [d]^2$ and n positive integer,

$$\mathfrak{M}_n(k, l) = p_k(n)\mathfrak{M}_n p_l(n).$$

$\mathfrak{M}_n(k, l)$ is endowed with the Lebesgue measure arising from the Euclidean structure defined by $\langle M, N \rangle = \Re(\text{Tr } M^* N)$. The norm arising from this Euclidean structure will be denoted by $\|\cdot\|_2$, whereas $\|\cdot\|$ still denotes the operator norm associated to the canonical hermitian norm on \mathbb{C}^n . We denote in the same time, without distinction, by Λ the tensor product of these measures on any product of such spaces.

Consider $a_1, \dots, a_N \in \mathcal{A}$ simple elements with respective types

$$(k(1), l(1)), \dots, (k(N), l(N)).$$

Let us define, for n, r positive integers, ε, R positive numbers,

$$\Gamma_R(a_1, \dots, a_N; n, r, \varepsilon)$$

the set of families $(A_1, \dots, A_N) \in \mathfrak{M}_n(k(1), l(1)) \times \cdots \times \mathfrak{M}_n(k(N), l(N))$ such that for all $i = 1, \dots, N$, $\|A_i\| \leq R$ and for all $p \in \{1, \dots, r\}$, for all $i_1, \dots, i_p \in [N]$, for all $\varepsilon_1, \dots, \varepsilon_p \in \{*, \cdot\}$,

$$\|E(A_{i_1}^{\varepsilon_1} \cdots A_{i_p}^{\varepsilon_p}) - E(a_{i_1}^{\varepsilon_1} \cdots a_{i_p}^{\varepsilon_p})\| \leq \varepsilon.$$

Let us then define

$$\chi_R^{\mathcal{D}}(a_1, \dots, a_N; r, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log \Lambda(\Gamma_R(a_1, \dots, a_N; n, r, \varepsilon)) + L \log n + D,$$

where

$$L = \sum_{k,l=1}^d \rho_k \rho_l |\{i \in [N]; a_i \text{ has type } (k, l)\}|,$$

$$D = \sum_{k=1}^d \rho_k^2 \log \rho_k |\{i \in [N]; a_i \text{ has type } (k, k)\}|.$$

We define then

$$\chi_R^{\mathcal{D}}(a_1, \dots, a_N) = \inf_{r, \varepsilon} \chi_R^{\mathcal{D}}(a_1, \dots, a_N; r, \varepsilon),$$

and at last,

$$\chi^{\mathcal{D}}(a_1, \dots, a_N) = \sup_R \chi_R^{\mathcal{D}}(a_1, \dots, a_N).$$

5.2. PARTICULAR CASES, COMPARISON WITH ALREADY DEFINED QUANTITIES.

5.2.1. CASE WHERE ALL a_i 'S ARE OF THE SAME TYPE (k, k) . Then

$$\chi_R^{\mathcal{D}}(a_1, \dots, a_N; r, \varepsilon) = \rho_k^2 \chi_R^{\text{Voic}}(a_1, \dots, a_N; r, \varepsilon / \rho_k),$$

where $\chi_R^{\text{Voic}}(a_1, \dots, a_N; r, \varepsilon)$ stands for $\chi_R(a_1, \dots, a_N; r, \varepsilon)$ as it is defined in section 1.2 of [32] or p. 279 of [10], when considering a_1, \dots, a_N as non-selfadjoint elements of $(\mathcal{A}_{kk}, \varphi_k)$.

5.2.2. CASE WHERE $N = 1$ AND a_1 HAS TYPE (k, l) , WITH $k \neq l$, $q_k(n) = q_l(n)$ FOR ALL n . Then $\Gamma_R(a; n, r, \varepsilon)$ is the set of matrices $A \in \mathfrak{M}_n(k, l) \simeq \mathfrak{M}_{q_k(n)}$ such that $\|M\| \leq R$ and for all s positive integer such that $2s \leq r$, the s -th moment of the spectral law of AA^* is within ε with the s -th moment of the distribution of aa^* in $(\mathcal{A}_{kk}, \varphi_k)$. Thus $\chi^{\mathcal{D}}(a) = \rho_k^2 (\chi_+^{\text{Voic}}(aa^*) - \log \rho_k)$, where $\chi_+^{\text{Voic}}(aa^*)$ is defined p. 282 of [10], when considering aa^* as a positive element of $(\mathcal{A}_{kk}, \varphi_k)$.

Note that, if μ is the distribution of aa^* in $(\mathcal{A}_{kk}, \varphi_k)$, then

$$\chi_+^{\mathcal{D}}(aa^*) = \iint \log |x - y| d\mu(x) d\mu(y) + \log \pi + 3/2.$$

5.2.3. CASE WHERE $N = 1$ AND a_1 HAS TYPE (k, l) , WITH $k \neq l$. Then $\Gamma_R(a; n, r, \varepsilon)$ is the set of matrices $A \in \mathfrak{M}_n(k, l)$ such that $\|M\| \leq R$ and for all s positive integer such that $2s \leq r$, the s -th moment of the spectral law of AA^* is within ε with the s -th moment of the distribution of aa^* in $(\mathcal{A}_{kk}, \varphi_k)$ and the s -th moment of the spectral law of A^*A is within ε with the s -th moment of the distribution of a^*a in $(\mathcal{A}_{ll}, \varphi_l)$. Let us define

$$\mu = \begin{cases} \text{distribution of } aa^* \text{ in } (\mathcal{A}_{kk}, \varphi_k) & \text{if } \rho_k \leq \rho_l, \\ \text{distribution of } a^*a \text{ in } (\mathcal{A}_{ll}, \varphi_l) & \text{if } \rho_k > \rho_l. \end{cases}$$

Then we prove the following proposition (proof postponed in the appendix):

PROPOSITION 5.1. *If R^2 is more than the supremum of the support of μ ,*

$$(5.1) \quad \chi_R^{\mathcal{D}}(a) = \alpha^2 \Sigma(\mu) + (\beta - \alpha) \alpha \int \log x d\mu(x) + \alpha \beta (\log \frac{\pi}{\alpha} + 1) + \frac{\alpha^2}{4} - \alpha^2 \int_{\frac{\beta-\alpha}{\alpha}}^{\frac{\beta}{\alpha}} x \log x dx,$$

where $\Sigma(\mu) = \iint \log |x - y| d\mu(x) d\mu(y)$, $\alpha = \min\{\rho_k, \rho_l\}$ and $\beta = \max\{\rho_k, \rho_l\}$.

We can verify that when $\alpha = \beta$, this formula coincides with the formula of $\chi^{\mathcal{D}}(a)$ given by 5.2.2.

5.3. PRELIMINARY LEMMA. The following lemma is a very useful tool for integration on sets of rectangular matrices. It gives the ‘‘law’’ of the singular values of rectangular matrices distributed according to the Lebesgue measure. Its proof is postponed in the appendix. Consider $1 \leq q \leq q'$ integers. Denote by $\mathfrak{M}_{q,q'}$ the set of $q \times q'$ complex matrices. Denote by \mathcal{U}_q the group of $q \times q$ unitary matrices, and by T_q the torus of diagonal matrices of \mathcal{U}_q . As an homogeneous space, \mathcal{U}_q/T_q is endowed with a unique distribution invariant under the left action of \mathcal{U}_q , denoted by γ_q . Denote by $\mathcal{U}_{q,q'}$ the set of matrices v of $\mathfrak{M}_{q,q'}$ which satisfy $vv^* = I_q$ (i.e. whose lines are orthogonal with norm 1). As an homogeneous space (under the right action of $\mathcal{U}_{q'}$), $\mathcal{U}_{q,q'}$ is endowed with a unique distribution invariant under these actions, denoted by $\gamma_{q,q'}$. Note that $\gamma_{q,q'}$ is also invariant under the left action of \mathcal{U}_q .

LEMMA 5.2. *Define*

$$\mathbb{R}_{+,<}^q = \{x \in \mathbb{R}^q; 0 < x_1 < \dots < x_q\}.$$

Then the map

$$\begin{aligned} \Psi : \mathcal{U}_q/T_q \times \mathbb{R}_{+,<}^q \times \mathcal{U}_{q,q'} &\rightarrow \mathfrak{M}_{q,q'} \\ (uT_q, x, v) &\mapsto u \operatorname{diag}(x_1, \dots, x_q)^{1/2} u^* v \end{aligned}$$

is injective onto a set with complement of null Lebesgue measure. Moreover, the push-forward, by Ψ^{-1} , of the Lebesgue measure, is $\gamma_q \otimes \sigma_{q,q'} \otimes \gamma_{q,q'}$, where $\sigma_{q,q'}$ is the probability measure on $\mathbb{R}_{+,<}^q$ with density

$$(5.2) \quad \frac{\pi^{qq'}}{\prod_{j=1}^{q-1} j! \prod_{j=q'-q}^{q'-1} j!} \Delta(x)^2 \prod_{i=1}^q x_i^{q'-q}.$$

Remark. It will be more useful to use the following consequence of the lemma. Let us denote, for G compact group, $\operatorname{Haar}(G)$ the Haar probability measure on G . The measures γ_q and $\gamma_{q,q'}$ are push-forwards of $\operatorname{Haar}(\mathcal{U}_q)$, $\operatorname{Haar}(\mathcal{U}_{q'})$ by the respective maps $u \rightarrow uT_q$, $v \rightarrow Pv$, where P is the $q \times q'$ matrix with diagonal entries equal to 1, and other entries equal to 0. Then the map

$$\begin{aligned} \tilde{\Psi} : \mathcal{U}_q \times (\mathbb{R}^+)^q \times \mathcal{U}_{q'} &\rightarrow \mathfrak{M}_{q,q'} \\ (u, x, v) &\mapsto u \operatorname{diag}(x_1, \dots, x_q)^{1/2} u^* Pv \end{aligned}$$

is surjective and the push-forward of the measure $\text{Haar}(\mathcal{U}_q) \otimes \frac{1}{q!} \tilde{\sigma}_{q,q'} \otimes \text{Haar}(\mathcal{U}_{q'})$ by $\tilde{\Psi}$ is the Lebesgue measure, where $\tilde{\sigma}_{q,q'}$ is the measure on $(\mathbb{R}^+)^n$ with density given by formula (5.2).

5.4. CLASSICAL PROPERTIES OF ENTROPY. The following properties are analogous to properties of Voiculescu's entropy, the proofs are analogous too, and the straightforward adaptation will be left to the reader. For the proof of proposition 5.4, the classical change of variables formula used for square matrices needs to be replaced by the result given below lemma 5.2.

PROPOSITION 5.3. $\chi^{\mathcal{D}}$ is subadditive: for $1 \leq m < N$,

$$\chi^{\mathcal{D}}(a_1, \dots, a_N) \leq \chi^{\mathcal{D}}(a_1, \dots, a_m) + \chi^{\mathcal{D}}(a_{m+1}, \dots, a_N).$$

PROPOSITION 5.4. For $R_1 > R > \max\{\|a_1\|, \dots, \|a_N\|\}$, we have

$$\chi_{R_1}^{\mathcal{D}}(a_1, \dots, a_N) = \chi_R^{\mathcal{D}}(a_1, \dots, a_N).$$

PROPOSITION 5.5 (Upper semicontinuity). Consider, for $m \geq 1$, $a_{m,1}, \dots, a_{m,N}$ simple elements of \mathcal{A} such that

- for all i , $a_{m,i}$ has the same type as a_i ,
- the family $(a_{m,1}, \dots, a_{m,N})$ converges in \mathcal{D} -distribution to (a_1, \dots, a_N) ,
- for all i , the sequence $\|a_{m,i}\|$ is bounded.

Then $\chi^{\mathcal{D}}(a_1, \dots, a_N) \geq \limsup \chi^{\mathcal{D}}(a_{m,1}, \dots, a_{m,N})$.

For example, the proposition holds if for all i , $a_{m,i}$ has the same type as a_i and converges strongly to a_i .

PROPOSITION 5.6. Consider b_1, \dots, b_N simple elements such that for all $i \in [N]$, b_i has the type of a_i and $b_i - a_i \in \{a_1, \dots, a_{i-1}\}''$. Then $\chi^{\mathcal{D}}(a_1, \dots, a_N) = \chi^{\mathcal{D}}(b_1, \dots, b_N)$.

5.5. ENTROPY AND FREENESS WITH AMALGAMATION OVER \mathcal{D} . Adapting the section 5 of [30] and using proposition 4.7, we obtain the following result:

THEOREM 5.7. If the simple elements a_1, \dots, a_N are free with amalgamation over \mathcal{D} , then

$$\chi^{\mathcal{D}}(a_1, \dots, a_N) = \chi^{\mathcal{D}}(a_1) + \dots + \chi^{\mathcal{D}}(a_N).$$

5.6. CHANGE OF VARIABLE FORMULA. Consider $a_1, \dots, a_N \in \mathcal{A}$ simple elements with respective types $(k(1), l(1)), \dots, (k(N), l(N))$. In this section, for $i \in [N]$, in order to simplify expressions, we denote $\mathfrak{M}_n(k(i), l(i))$ by $\mathfrak{M}_n(a_i)$. Since we are going to work with adjoints of the a_i 's, we have to define, for $i \in [N]$, $\varepsilon \in \{*, 1\}$, $(k(i, \varepsilon), l(i, \varepsilon))$ to be the type of a_i^ε i.e.

$$(k(i, \varepsilon), l(i, \varepsilon)) = \begin{cases} (l(i), k(i)) & \text{if } \varepsilon = * \\ (k(i), l(i)) & \text{if } \varepsilon = 1 \end{cases}$$

Define \mathcal{F} to be the set of formal power series in the noncommutative variables $X_1, X_1^*, \dots, X_N, X_N^*$ endowed with the natural involution $F \rightarrow F^*$. Let us define, for $m \geq 0, i_1, \dots, i_m \in [N], \varepsilon_1, \dots, \varepsilon_m \in \{1, *\}$, $C_{i_1, \dots, i_m}^{\varepsilon_1, \dots, \varepsilon_m}$ the map from \mathcal{F} to \mathbb{C} which maps a series F to its coefficient in $X_{i_1}^{\varepsilon_1} \cdots X_{i_m}^{\varepsilon_m}$.

A *multi-radius of convergence* for $F \in \mathcal{F}$ is a family (R_1, \dots, R_N) of positive numbers such that

$$M(F; R_1, \dots, R_N) := \sum_{m \geq 0} \sum_{\substack{i_1, \dots, i_m \in [N] \\ \varepsilon_1, \dots, \varepsilon_m \in \{*, 1\}}} |C_{i_1, \dots, i_m}^{\varepsilon_1, \dots, \varepsilon_m}(F)| R_{i_1} \cdots R_{i_m} < \infty.$$

Define, for $i \in [N]$, the set \mathcal{F}_i of formal power series $F \in \mathcal{F}$ such that for all $m \geq 1$, for all $i_1, \dots, i_m, \varepsilon_1, \dots, \varepsilon_m$, such that $C_{i_1, \dots, i_m}^{\varepsilon_1, \dots, \varepsilon_m}(F) \neq 0$, we have

$$k(i) = k(i_1, \varepsilon_1), l(i_1, \varepsilon_1) = k(i_2, \varepsilon_2), \dots, l(i_{m-1}, \varepsilon_{m-1}) = k(i_m, \varepsilon_m), l(i_m, \varepsilon_m) = l(i).$$

Consider $F = (F^{(1)}, \dots, F^{(N)})$ in $\mathcal{F}_1 \times \cdots \times \mathcal{F}_N$. We suppose moreover that there exists (R_1, \dots, R_N) common multiradius of convergence of the $F^{(i)}$'s such that for all $i, \|a_i\| < R_i$.

Let also, for $n \geq 1$, F be the map defined on the Cartesian product, for $i \in [N]$, of the open ball of $(M_n(a_i), \|\cdot\|)$ with center zero and radius R_i , by

$$\begin{aligned} F : \prod_{i=1}^N B_{M_n(a_i)}(0, R_i) &\rightarrow \prod_{i=1}^N B_{M_n(a_i)}(0, R'_i) \\ (A_1, \dots, A_N) &\mapsto F(A_1, \dots, A_N) \end{aligned}$$

Then F is analytic, and with the natural identification between the set

$$\mathcal{L} \left(\prod_{i=1}^N M_n(a_i) \right)$$

of endomorphisms of $\prod_{i=1}^N M_n(a_i)$ and the Cartesian product

$$\prod_{1 \leq i, j \leq N} \mathcal{L}(M_n(a_j), M_n(a_i)),$$

the differential $DF(A)$ of F at $A = (A_1, \dots, A_N) \in \prod_{i=1}^N B_{M_n(a_i)}(0, R_i)$ has (i, j) -th block

$$\sum_{\substack{m \geq 1 \\ i_1, \dots, i_m \in [N] \\ \varepsilon_1, \dots, \varepsilon_m \in \{*, 1\}}} C_{i_1, \dots, i_m}^{\varepsilon_1, \dots, \varepsilon_m}(F^{(i)}) \left[\sum_{\substack{l \in [m] \\ i_l = j \\ \varepsilon_l = 1}} L(A_{i_1}^{\varepsilon_1} \cdots A_{i_{l-1}}^{\varepsilon_{l-1}}) \circ R(A_{i_{l+1}}^{\varepsilon_{l+1}} \cdots A_{i_m}^{\varepsilon_m}) \right]$$

$$+ \sum_{\substack{l \in [m] \\ i_l = j \\ \varepsilon_l = *}} L(A_{i_1}^{\varepsilon_1} \cdots A_{i_{l-1}}^{\varepsilon_{l-1}}) \circ R(A_{i_{l+1}}^{\varepsilon_{l+1}} \cdots A_{i_m}^{\varepsilon_m}) \circ \text{Adj} \Big],$$

where for A matrix, $L(A)$ (resp. $R(A)$) denotes the operator of left (resp. right) multiplication by A , and Adj denotes the operator of adjunction in \mathfrak{M}_n .

Now, we are going to compute the Jacobian of F at $A = (A_1, \dots, A_N)$. It is the absolute value of the determinant of the differential of F at (A_1, \dots, A_N) , that is

$$(\det DF(A)DF(A)^*)^{1/2} = \exp \frac{1}{2} \text{Tr} \log DF(A)DF(A)^*,$$

where the adjoint is taken when considering $DF(A)$ as an endomorphism of the space $\prod_{i=1}^N M_n(a_i)$ endowed with the product euclidean structure. Note that the identification between

$$\mathcal{L} \left(\prod_{i=1}^N M_n(a_i) \right) \text{ and } \prod_{1 \leq i, j \leq N} \mathcal{L} (M_n(a_j), M_n(a_i))$$

preserves the adjunction in the following way:

$$\left([M_{i,j}]_{i,j=1}^N \right)^* = [M_{j,i}^*]_{i,j=1}^N,$$

and composition in the following way

$$[L_{i,j}]_{i,j=1}^N \circ [M_{i,j}]_{i,j=1}^N = \left[\sum_{k=1}^n L_{i,k} \circ M_{k,j} \right]_{i,j=1}^N.$$

Let, for $n \geq 1$, \mathfrak{L}_n be the space of endomorphisms L of $\prod_{i=1}^N M_n(a_i)$ such that, for all $i, j \in [N]$, $L_{i,j}$ is a linear combination of linear maps of the type $L_A \circ L_B$ (with $A \in \mathfrak{M}_n(k(i), k(j))$ and $B \in \mathfrak{M}_n(l(j), l(i))$) and of maps of the type $L_A \circ L_B \circ \text{Adj}$ (with $A \in \mathfrak{M}_n(k(i), l(j))$ and $B \in \mathfrak{M}_n(k(j), l(i))$). \mathfrak{L}_n is a subalgebra of $\mathcal{L}(\prod_{i=1}^N M_n(a_i))$ closed under adjunction. Indeed, we have, for all X, Y, Z, T matrices with suitable sizes,

$$\begin{aligned} (L(X) \circ R(Y)) \circ (L(Z) \circ R(T)) &= L(XZ) \circ R(YT), \\ (L(X) \circ R(Y)) \circ (L(Z) \circ R(T) \circ \text{Adj}) &= L(XZ) \circ R(YT) \circ \text{Adj}, \\ (L(X) \circ R(Y) \circ \text{Adj}) \circ (L(Z) \circ R(T)) &= L(XT^*) \circ R(YZ^*) \circ \text{Adj}, \\ (L(X) \circ R(Y) \circ \text{Adj}) \circ (L(Z) \circ R(T) \circ \text{Adj}) &= L(XT^*) \circ R(YZ^*), \\ (L(X) \circ R(Y))^* &= L(X^*) \circ R(Y^*), \\ (L(X) \circ R(Y) \circ \text{Adj})^* &= L(Y) \circ R(X) \circ \text{Adj}. \end{aligned}$$

Thus in order to compute the Jacobian of F , it suffices to be able to compute the trace of a self-adjoint element of \mathfrak{L}_n . Note that the identification between

$$\mathcal{L} \left(\prod_{i=1}^N M_n(a_i) \right) \text{ and } \prod_{1 \leq i, j \leq N} \mathcal{L} (M_n(a_j), M_n(a_i))$$

preserves the trace in the following way:

$$\text{Tr} \left([M_{i,j}]_{i,j=1}^N \right) = \sum_{i=1}^N \text{Tr} M_{i,i}.$$

Moreover, if $[M_{i,j}]_{i,j=1}^N \in \mathfrak{L}_n$ is self-adjoint, then for all $i \in \{1, \dots, N\}$, $M_{i,i} = M_{i,i}^*$, and one can write

$$M_{i,i} = \sum_{\alpha} c_{\alpha} (L_{X_{\alpha}} \circ R_{Y_{\alpha}} + L_{X_{\alpha}^*} \circ R_{Y_{\alpha}^*}) + \sum_{\beta} c_{\beta} (L_{Z_{\beta}} \circ R_{T_{\beta}} \circ \text{Adj} + L_{T_{\beta}^*} \circ R_{Z_{\beta}^*} \circ \text{Adj}),$$

where α, β run in disjoint finite sets, and for all α, β , c_{α}, c_{β} are real,

$$X_{\alpha} \in p_{k(i)}(n) \mathfrak{M}_n p_{k(i)}(n), \quad Y_{\alpha} \in p_{l(i)}(n) \mathfrak{M}_n p_{l(i)}(n), \quad Z_{\beta}, T_{\beta} \in M_n(a_i).$$

Thus the trace of $M_{i,i}$ is

$$\sum_{\alpha} c_{\alpha} 2 \Re \left(\underbrace{\text{Tr} X_{\alpha} \text{Tr} Y_{\alpha} + \text{Tr} X_{\alpha}^* \text{Tr} Y_{\alpha}^*}_{\in \mathbb{R}} \right),$$

$$\text{i.e. } \sum_{\alpha} 2c_{\alpha} (\text{Tr} X_{\alpha} \text{Tr} Y_{\alpha} + \text{Tr} X_{\alpha}^* \text{Tr} Y_{\alpha}^*).$$

Thus, in order to compute the Jacobian of F at $A = (A_1, \dots, A_N)$, we have to introduce the following objects. Let $\mathfrak{S}_2 = \{e, \tau\}$ be the group of permutations of the set $\{1, 2\}$, and $\mathbb{C}[\mathfrak{S}_2] = \mathbb{C}e \oplus \mathbb{C}\tau$ be its convolution algebra. For $j \in [N]$, define the \mathbb{C} -linear map

$$D_j : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F} \otimes \mathbb{C}[\mathfrak{S}_2]$$

by

$$D_j(X_{i_1}^{\varepsilon_1} \dots X_{i_m}^{\varepsilon_m}) = \sum_{\substack{l \in [m] \\ i_l = j \\ \varepsilon_l = 1}} X_{i_1}^{\varepsilon_1} \dots X_{i_{l-1}}^{\varepsilon_{l-1}} \otimes X_{i_{l+1}}^{\varepsilon_{l+1}} \dots X_{i_m}^{\varepsilon_m} \otimes e + \sum_{\substack{l \in [m] \\ i_l = j \\ \varepsilon_l = *}} X_{i_1}^{\varepsilon_1} \dots X_{i_{l-1}}^{\varepsilon_{l-1}} \otimes X_{i_{l+1}}^{\varepsilon_{l+1}} \dots X_{i_m}^{\varepsilon_m} \otimes \tau.$$

Define then, for any $*$ -algebra \mathcal{M} (which will be \mathcal{F} , or \mathfrak{M}_n , or \mathcal{A}), the product and the adjunction on the linear space $\mathcal{M} \otimes \mathcal{M} \otimes \mathbb{C}[\mathfrak{S}_2]$ defined by the rules:

$$\begin{aligned} (X \otimes Y \otimes e) \times (Z \otimes T \otimes e) &= XZ \otimes YT \otimes e, \\ (X \otimes Y \otimes e) \times (Z \otimes T \otimes \tau) &= XZ \otimes YT \otimes \tau, \\ (X \otimes Y \otimes \tau) \times (Z \otimes T \otimes e) &= XT^* \otimes YZ^* \otimes \tau, \\ (X \otimes Y \otimes \tau) \times (Z \otimes T \otimes \tau) &= XT^* \otimes YZ^* \otimes e, \\ (X \otimes Y \otimes e)^* &= X^* \otimes Y^* \otimes e, \\ (X \otimes Y \otimes \tau)^* &= Y \otimes X \otimes \tau. \end{aligned}$$

If moreover, the algebra \mathcal{M} is endowed with a linear functional f (which will be Tr if $\mathcal{M} = \mathfrak{M}_n$, and φ if $\mathcal{M} = \mathcal{A}$), then we shall endow $\mathcal{M} \otimes \mathcal{M} \otimes \mathbb{C}[\mathfrak{S}_2]$ with the linear functional $2f \otimes f \otimes \delta_e$, where δ_e is the state on $\mathbb{C}[\mathfrak{S}_2]$ defined by $\delta_e(e) = 1$, $\delta_e(\tau) = 0$.

With these notations, if one uses the identifications, for $X, Y \in \mathfrak{M}_n$:

$$\begin{aligned} L(X) \circ R(Y) &\simeq X \otimes Y \otimes \delta_e \in \mathfrak{M}_n \otimes \mathfrak{M}_n \otimes \mathbb{C}[\mathfrak{S}_2], \\ L(X) \circ R(Y) \circ \text{Adj} &\simeq X \otimes Y \otimes \tau \in \mathfrak{M}_n \otimes \mathfrak{M}_n \otimes \mathbb{C}[\mathfrak{S}_2], \end{aligned}$$

and thus identifies \mathcal{L}_n with a subset of the finite-dimensional algebra $\mathfrak{M}_N \otimes (\mathfrak{M}_n \otimes \mathfrak{M}_n \otimes \mathbb{C}[\mathfrak{S}_2])$, the Jacobian of F at $A = (A_1, \dots, A_N)$ is

$$\exp \frac{1}{2} \text{Tr} \otimes \text{Tr} \otimes \text{Tr} \otimes \delta_e (\log(DF(A)DF^*(A))).$$

With these tools, adapting the proof of proposition 3.5 of [30], we have the following proposition. $\Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)}$ denotes the Kadison-Fuglede determinant of the linear functional $\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)$ on $\mathfrak{M}_N \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathbb{C}[\mathfrak{S}_2]$ (see section 2 of [8] for a brief introduction to Kadison-Fuglede determinant).

PROPOSITION 5.8. *Consider $F = (F^{(1)}, \dots, F^{(N)})$, $G = (G^{(1)}, \dots, G^{(N)})$ which both belong to $\mathcal{F}_1 \times \dots \times \mathcal{F}_N$, and such that for all $i \in [n]$,*

$$G^{(i)}(F^{(1)}, \dots, F^{(N)}) = X_i.$$

We suppose moreover that there exists (R_1, \dots, R_N) common multiradius of convergence of the $F^{(i)}$'s and (R'_1, \dots, R'_N) common multiradius of convergence of the $G^{(i)}$'s such that

- (i) *for all i , $\|a_i\| < R_i$,*
- (ii) *for all i , $M(F^{(i)}; R_1, \dots, R_N) < R'_i$.*

Then

$$\begin{aligned} &\chi^{\mathcal{D}}(F^{(1)}(a_1, \dots, a_N), \dots, F^{(N)}(a_1, \dots, a_N)) \\ &\geq \log \Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)} [D_j F^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N + \chi^{\mathcal{D}}(a_1, \dots, a_N). \end{aligned}$$

If moreover,

$$(M(G^{(1)}; R'_1, \dots, R'_N), \dots, M(G^{(N)}; R'_1, \dots, R'_N))$$

is a common multiradius of the $F^{(i)}$'s, then we have equality.

REMARK 5.9. Note that if instead of (ii) we have $M(F^{(i)}; \|a_1\|, \dots, \|a_N\|) < R'_i$ for all i , then one can reduce the R_i 's in order to have (i) and (ii).

REMARK 5.10. Note that by subsection 5.2.1, these formulae still remain valid for Voiculescu's free entropy for non hermitian operators, as it is defined in section 1.2 of [32] or p. 279 of [10].

As a corollary, we have the following result, whose proof is an adaptation of the proof of proposition 6.3.3 of [10].

COROLLARY 5.11. *Let P_1, \dots, P_N be polynomials in the noncommutative variables X_1, X_1^*, \dots, X_N^* such that for all $i \in [N]$, $a_i + P_i(a_1, \dots, a_N)$ has type (k, l) . Then for α sufficiently near 0,*

$$\begin{aligned} & \chi^{\mathcal{D}}(a_1 + \alpha P_1(a_1, \dots, a_N), \dots, a_N + \alpha P_N(a_1, \dots, a_N)) \\ &= \log \Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)} [D_j F_\alpha^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N + \chi^{\mathcal{D}}(a_1, \dots, a_N), \end{aligned}$$

where for all i , $F_\alpha^{(i)} = X_i + P(X_1, \dots, X_N)$.

5.7. FUNCTIONAL CALCULUS AND ENTROPY. For $x \in \mathcal{A}$, with spectral decomposition uh , and f real Borel function on $[0, \infty)$, bounded on the spectrum of h , let $f(x)$ denote $uf(h)$. If $f(0) = 0$ and f is positive on $(0, \infty)$, then the polar decomposition of $f(x)$ is $uf(h)$.

We begin with the following lemma, analogous to lemma 6.3.5 of [10].

LEMMA 5.12. *Consider $k, l \in [d]$ such that $k \leq l$, $a \in \mathcal{A}_{kl}$ such that the distribution μ of $(aa^*)^{1/2}$ in $(\mathcal{A}_{kk}, \varphi_k)$ satisfies*

$$\Sigma(\mu) > -\infty, \quad \int \log t d\mu(t) > -\infty,$$

and f a continuous increasing function on $[0, \infty)$, such that $f(0) = 0$ and f is positive on $(0, \infty)$. Then there exists a sequence (f_m) of smooth functions on $[0, \infty)$, such that for all m , $f_m(0) = 0$, f_m^l is positive on $[0, \infty)$,

$$\|f_m(a) - f(a)\| \xrightarrow{n \rightarrow \infty} 0, \quad \chi^{\mathcal{D}}(f_m(a)) \xrightarrow{n \rightarrow \infty} \chi^{\mathcal{D}}(f(a)).$$

Proof. Let us prove that there exists a sequence (f_m) of smooth functions on $[0, \infty)$, such that for all m , $f_m(0) = 0$, f_m^l is positive on $[0, \infty)$,

$$\|f_m(a) - f(a)\| \xrightarrow{n \rightarrow \infty} 0, \quad \chi^{\mathcal{D}}(f_m(a)) \xrightarrow{n \rightarrow \infty} \chi^{\mathcal{D}}(f(a)).$$

Then we will have, by upper semicontinuity: $\chi^{\mathcal{D}}(f(a)) \geq \limsup \chi^{\mathcal{D}}(f_m(a))$, and by (5.1), it will suffice to prove

$$\Sigma(f(\mu)) \leq \liminf \Sigma(f_m(\mu)), \quad \int \log(f(t)) d\mu(t) \leq \liminf \int \log(f_m(t)) d\mu(t),$$

where for all function g , $g(\mu)$ denotes the push-forward of μ by g .

Consider, for $m \geq 1$, $\delta(m) \in (0, 1/m)$ such that

$$\iint_{|t-s| < \delta(m)} \log |t-s| d\mu(t) d\mu(s) \geq -\frac{1}{m}, \quad \int_{[0, \delta(m)]} \log t d\mu(t) \geq -\frac{1}{m},$$

$$\iint_{|t-s| < \delta(m)} d\mu(t) d\mu(s) \leq \frac{1}{m \log m}, \quad \mu([0, \delta(m)]) \leq \frac{1}{m \log m}.$$

Let us extend f by the value 0 on the negative real numbers. Let, for $m \geq 1$, ϕ_m be a nonnegative smooth function with support in $[0, 1/m]$ such that $\int \phi_m(t) dt = 1$ and

$$|f * \phi_m(t) - f(t)| \leq \frac{\delta(m)}{m}$$

for all $t \in [0, S]$, where S is the maximum of the support of μ . Define $f_m(t) := \frac{t}{m} + f * \phi_m(t)$. Since f is increasing and $\phi_m \geq 0$, $f * \phi_m$ is increasing, hence $f'_m \geq 1/m$. Moreover, $f_m(0) = 0$ and f_m converges uniformly to f on $[0, S]$.

Similarly to p. 267 of [10], we can prove that for m large enough to satisfy such that

$$\forall s, t \in [0, S], |t-s| < \delta(m) \Rightarrow |f(t) - f(s)| < 1,$$

we have $\Sigma(f_m(\mu)) \geq \Sigma(f(\mu)) - \frac{2}{m}$. Moreover, for $t \in [0, \delta(m)]$, $f_m(t) \geq t/m$ and for $t \in [\delta(m), S]$,

$$f_m(t) \geq \frac{\delta(m)}{m} + f(t) - |f * \phi_m(t) - f(t)| \geq f(t),$$

so if $\delta(m) \leq 1$,

$$\begin{aligned} \int \log(f_m(t)) d\mu(t) &\geq \int_{[\delta(m), S]} \log(f(t)) d\mu(t) + \int_{[0, \delta(m)]} \log(T/m) d\mu(t) \\ &\geq \int_{[0, S]} \log(f(t)) d\mu(t) + \int_{[0, \delta(m)]} \log t d\mu(t) - \log(m) \mu([0, \delta(m)]) \\ &\geq \int \log(f(t)) d\mu(t) - \frac{2}{m}, \end{aligned}$$

what closes the proof. ■

Adapting the proof of proposition 6.3.6 of [10] (with the density of eigenvalues presented in lemma 5.2), we obtain the following proposition:

PROPOSITION 5.13. *Consider a family f_1, \dots, f_N of continuous increasing functions on $[0, \infty)$, with value 0 in 0, and positive on $(0, \infty)$. Consider a_1, \dots, a_N simple elements of \mathcal{A} such that for all i , $\chi^{\mathcal{D}}(a_i) > -\infty$. Then*

$$\chi^{\mathcal{D}}(f_1(a_1), \dots, f_N(a_N)) \geq \chi^{\mathcal{D}}(a_1, \dots, a_N) + \sum_{i=1}^N \chi^{\mathcal{D}}(f_i(a_i)) - \chi^{\mathcal{D}}(a_i).$$

Moreover, equality holds if the functions are strictly increasing.

5.8. MAXIMIZATION OF FREE ENTROPY.

5.8.1. ONE VARIABLE. The problem here is to maximize $\chi^{\mathcal{D}}(a)$, where a is taken among a set of simple elements of type (k, l) in \mathcal{A} . For such a , we have seen in 5.2.3 a formula which express $\chi^{\mathcal{D}}(a)$ as a function of μ (distribution of aa^* if $\rho_k \leq \rho_l$, distribution of a^*a in the other case), of ρ_k and of ρ_l . For example, let us suppose that $\rho_k \leq \rho_l$.

PROPOSITION 5.14. *Fix c positive. Then among elements a of type (k, l) such that $\varphi_k(aa^*) \leq c$, the maximizers of $\chi^{\mathcal{D}}$ are those for which the distribution of aa^* in $(\mathcal{A}_{kk}, \varphi_k)$ is the push-forward, by $t \rightarrow \frac{c\rho_k}{\rho_l}t$, of the Marchenko-Pastur distribution with parameter $\frac{\rho_l}{\rho_k}$.*

Proof. First of all, since for all a , for all $\lambda > 0$, $\chi^{\mathcal{D}}(\lambda a) = \chi^{\mathcal{D}}(a) + 2\rho_k\rho_l \log \lambda$, it suffices to prove it when $c = \frac{\rho_l}{\rho_k}$. According to proposition 5.1, $\chi^{\mathcal{D}}(a)$ is maximal if and only if

$$A(\mu) := \Sigma(\mu) + \left(\frac{\rho_l}{\rho_k} - 1 \right) \int \log x d\mu(x)$$

is maximal. Note that the condition $\varphi_k(aa^*) \leq c$ is equivalent to $\int x d\mu(x) \leq c$. But proposition 5.3.7 of [10] states that the functional

$$B(\mu) := \Sigma(\mu) + \left(\frac{\rho_l}{\rho_k} - 1 \right) \int \log x d\mu(x) - \int x d\mu(x)$$

is maximized, among probability measures on $[0, \infty)$, by the Marchenko-Pastur distribution μ_c with parameter c . So for all μ probability measure on $[0, \infty)$ such that $\int x d\mu(x) \leq c$,

$$\begin{aligned} A(\mu_c) - A(\mu) &= B(\mu_c) - B(\mu) + \int x d\mu_c(x) - \int x d\mu(x) \\ &= B(\mu_c) - B(\mu) + (c - \int x d\mu(x)) \\ &\geq 0, \end{aligned}$$

with equality if and only if $\mu = \mu_c$. ■

5.8.2. N VARIABLES. Fix $k \neq l \in [N]$ such that $\rho_k \leq \rho_l$. A consequence of sub-additivity, of theorem 5.7, and of the previous section is the fact that if c_1, \dots, c_N are positive numbers, among N -tuples (a_1, \dots, a_N) of type (k, l) elements which satisfy

$$\forall i \in [N], \quad \varphi_k(a_i a_i^*) = c_i,$$

the maximum of $\chi^{\mathcal{D}}(a_1, \dots, a_N)$ is realized on free with amalgamation over \mathcal{D} families (a_1, \dots, a_N) of elements such that the distribution of each $a_i a_i^*$ in $(\mathcal{A}_{kk}, \varphi_k)$ is the push-forward, by $t \rightarrow \frac{c_i \rho_k}{\rho_l} t$, of the Marchenko-Pastur distribution with parameter $\frac{\rho_l}{\rho_k}$. The following theorem states the reciprocal to this fact.

THEOREM 5.15. *If the maximum is realized on a family (a_1, \dots, a_N) , then the family is free with amalgamation over \mathcal{D} , and the distribution of each $a_i a_i^*$ in $(\mathcal{A}_{kk}, \varphi_k)$*

is the push-forward, by $t \rightarrow \frac{c_i \rho_k}{\rho_l} t$, of the Marchenko-Pastur distribution with parameter $\frac{\rho_l}{\rho_k}$.

Proof. Step I. First of all, since for all a_1, \dots, a_N of type (k, l) , for all positive numbers $\lambda_1, \dots, \lambda_N$,

$$(5.3) \quad \chi^{\mathcal{D}}(\lambda_1 a_1, \dots, \lambda_N a_N) = \chi^{\mathcal{D}}(a_1, \dots, a_N) + 2\rho_k \rho_l \log(\lambda_1 \cdots \lambda_N),$$

it suffices to prove it when each c_i is $\frac{\rho_l}{\rho_k}$.

Step II. By theorem 2.3, if b is an element of type (k, l) such that the distribution of bb^* in $(\mathcal{A}_{kk}, \varphi_k)$ is the Marchenko-Pastur distribution with parameter $\frac{\rho_l}{\rho_k}$, then for all n even integer,

$$c_n^{(k)}(b \otimes b^* \otimes \cdots \otimes b \otimes b^*) = \frac{\rho_l}{\rho_k} \delta_{n,2}.$$

Let us compute the φ -distribution of a free with amalgamation over \mathcal{D} family (b_1, \dots, b_N) of such elements. For all $i_1, \dots, i_{2r+1}, i_0 \in [N]$, we have

$$\begin{aligned} \varphi(b_{i_1} b_{i_2}^* \cdots b_{i_{2r+1}} b_{i_0} i_0^*) &= \rho_k \varphi_k(b_{i_1} b_{i_2}^* \cdots b_{i_{2k+1}} b_{i_0}^*) \\ &= \rho_k \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi_k(b_{i_1} \cdots b_{i_{2j}}^*) c_2^{(k)}(b_{i_0} \otimes \underbrace{\mathbb{E}(b_{2j+2}^* \cdots b_{2r+1})}_{\varphi_l(b_{2j+2}^* \cdots b_{2r+1}) \cdot \rho_l} b_{i_0}^*) \\ &= \rho_k \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi_k(b_{i_1} \cdots b_{i_{2j}}^*) c_2^{(k)}(b_{i_0} \otimes \varphi_l(b_{2j+2}^* \cdots b_{2r+1}) \cdot \underbrace{\rho_l b_{i_0}^*}_{b_{i_0}^*}) \\ &= \rho_k \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi_k(b_{i_1} \cdots b_{i_{2j}}^*) c_2^{(k)}(b_{i_0} \otimes \varphi_l(b_{2j+2}^* \cdots b_{2r+1}) \cdot b_{i_0}^*) \\ &= \rho_k \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi_k(b_{i_1} \cdots b_{i_{2j}}^*) \varphi_l(b_{2j+2}^* \cdots b_{2r+1}) c_2^{(k)}(b_{i_0} \otimes b_{i_0}^*) \\ &= \rho_l \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi_k(b_{i_1} \cdots b_{i_{2j}}^*) \varphi_l(b_{2j+2}^* \cdots b_{2r+1}) \end{aligned}$$

Step III. Now, consider a family (a_1, \dots, a_N) of elements of type (k, l) such that for all i , $\varphi_k(a_i a_i^*) = \frac{\rho_l}{\rho_k}$ and the maximum of the entropy is realized on (a_1, \dots, a_N) . Let us prove that the \mathcal{D} -distribution of the family is the one of (b_1, \dots, b_N) of step II. Since the elements are simple and for all x , $\varphi(x^*) = \overline{\varphi(x)}$, it suffices to prove that for all $r \geq 0$, for all $i_1, \dots, i_{2r+1}, i_0 \in [N]$, we have

$$\varphi(a_{i_1} a_{i_2}^* \cdots a_{i_{2r+1}} a_{i_0}^*) = \rho_l \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi_k(a_{i_1} \cdots a_{i_{2j}}^*) \varphi_l(a_{2j+2}^* \cdots a_{2r+1}).$$

Fix $\lambda \in \mathbb{C}$ and let us define the polynomial $P = \lambda X_{i_1} X_{i_2}^* \cdots X_{i_{2r+1}}$, and $d = P(a_1, \dots, a_N)$. We have, for α sufficiently near 0,

$$\chi^{\mathcal{D}}(a_1, \dots, a_N) \geq \chi^{\mathcal{D}}(a_1, \dots, a_{i-1}, \frac{\rho_l^{1/2}(a_{i_0} + \alpha d)}{(\rho_k \varphi_k((a_{i_0} + \alpha d)(a_{i_0} + \alpha d)^*))^{1/2}}, a_{i+1}, \dots, a_N).$$

Thus the derivative with respect to α , at $\alpha = 0$, of the difference between right-hand side and left-hand side of the previous equation is zero. According to (5.3) and to corollary 5.11, for α sufficiently near 0, the difference between right-hand side and left-hand side of the previous equation is equal to

$$\log \Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)} [D_j F_\alpha^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N - \rho_k \rho_l \log \frac{\rho_k \varphi_k((a_{i_0} + \alpha d)(a_{i_0} + \alpha d)^*)}{\rho_l},$$

where for all j , $F_\alpha^{(j)} = X_j + \alpha \delta_{i,j} P$. We have

$$\begin{aligned} & \log \Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)} [D_j F_\alpha^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N \\ &= \text{Tr} \otimes \varphi \otimes \varphi \otimes \delta_e \log \left([D_j F_\alpha^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N [D_i F_\alpha^{(j)}(a_1, \dots, a_N)^*]_{i,j=1}^N \right) \\ &= \text{Tr} \otimes \varphi \otimes \varphi \otimes \delta_e \log \left(I_N \otimes 1 \otimes 1 \otimes e + \alpha(A + A^*) + \alpha^2 A A^* \right), \end{aligned}$$

where A is the $N \times N$ matrix with (i, j) -th entry

$$\begin{cases} 0 & \text{if } i \neq i_0, \\ D_j P(a_1, \dots, a_N) & \text{if } i = i_0. \end{cases}$$

Thus

$$\begin{aligned} & \log \Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)} [D_j F_\alpha^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N = \\ & \alpha \varphi \otimes \varphi \otimes \delta_e (D_{i_0} P(a_1, \dots, a_N) + (D_{i_0} P(a_1, \dots, a_N))^*) + o(\alpha). \end{aligned}$$

Thus

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \Big|_{\alpha=0} \log \Delta_{\text{Tr} \otimes \varphi \otimes \varphi \otimes (2\delta_e)} [D_j F_\alpha^{(i)}(a_1, \dots, a_N)]_{i,j=1}^N = \\ & \varphi \otimes \varphi \otimes \delta_e (D_{i_0} P(a_1, \dots, a_N) + (D_{i_0} P(a_1, \dots, a_N))^*) = \\ & 2\Re \varphi \otimes \varphi \otimes \delta_e (D_{i_0} P(a_1, \dots, a_N)). \end{aligned}$$

Moreover, since $\varphi_k(a_{i_0} a_{i_0}^*) = \frac{\rho_l}{\rho_k}$,

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \Big|_{\alpha=0} \rho_k \rho_l \log \frac{\rho_k \varphi_k((a_{i_0} + \alpha d)(a_{i_0} + \alpha d)^*)}{\rho_l} = \\ & \rho_k^2 \varphi_k(a_{i_0} d^* + d a_{i_0}^*) = 2\rho_k^2 \Re \varphi_k(P(a_1, \dots, a_N) a_{i_0}^*). \end{aligned}$$

Thus $\varphi \otimes \varphi \otimes \delta_e (D_{i_0} P(a_1, \dots, a_N))$ and $\rho_k^2 \varphi_k(P(a_1, \dots, a_N) a_{i_0}^*)$ have the same real part. Recall that $P = \lambda X_{i_1} X_{i_2}^* \cdots X_{i_{2r+1}}$. What we did is true for any $\lambda \in \mathbb{C}$, so

$$\rho_k^2 \varphi_k(P(a_1, \dots, a_N) a_{i_0}^*) = \varphi \otimes \varphi \otimes \delta_e (D_{i_0} P(a_1, \dots, a_N)).$$

Now recall the definition of $D_{i_0}P$ and choose $\lambda = 1$. This gives

$$\rho_k^2 \varphi_k(a_{i_1} a_{i_2}^* \cdots a_{i_{2r+1}} a_{i_0}^*) = \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \varphi(a_{i_1} \cdots a_{i_{2j}}) \varphi(a_{2j+2}^* \cdots a_{2r+1}),$$

i.e.

$$\rho_k \varphi(a_{i_1} a_{i_2}^* \cdots a_{i_{2r+1}} a_{i_0}^*) = \sum_{\substack{0 \leq j \leq r \\ i_{2j+1} = i_0}} \rho_k \varphi_k(a_{i_1} \cdots a_{i_{2j}}) \rho_l \varphi_l(a_{2j+2}^* \cdots a_{2r+1}),$$

which closes the proof. ■

COROLLARY 5.16. *Consider a family (a_1, \dots, a_N) of elements of type (k, l) such that*

$$\chi^{\mathcal{D}}(a_1, \dots, a_N) = \chi^{\mathcal{D}}(a_1) + \cdots + \chi^{\mathcal{D}}(a_N) > -\infty.$$

Then the family is free with amalgamation over \mathcal{D} .

Proof. We use the notation defined in the beginning of section 5.7. Since for all i , $\chi^{\mathcal{D}}(a_i) > -\infty$, the distribution of $a_i a_i^*$ in $(\mathcal{A}_{kk}, \varphi_k)$ is non-atomic, hence the distribution of $|a_i|$ in $(\mathcal{A}_{kk}, \varphi_k)$ is also non-atomic. Hence we can find a continuous increasing function f_i on $[0, \infty)$, with value 0 in 0, and positive on $(0, \infty)$, such that the distribution of $f_i(a_i) f_i(a_i)^*$ is the Marchenko-Pastur distribution with parameter $\frac{\rho_l}{\rho_k}$. Then by proposition 5.13 and subadditivity,

$$\chi^{\mathcal{D}}(f_1(a_1), \dots, f_N(a_N)) = \chi^{\mathcal{D}}(f_1(a_1)) + \cdots + \chi^{\mathcal{D}}(f_N(a_N)),$$

hence by the previous theorem, $f_1(a_1), \dots, f_N(a_N)$ are free with amalgamation over \mathcal{D} , and so do a_1, \dots, a_N , because for all i , $a_i \in \{f_i(a_i)\}''$. ■

QUESTION. It would be interesting to have a characterization of R -diagonal elements with non trivial kernel involving the entropy defined in this paper. Inspired by the papers [9], [16], we ask the following question: given a compactly supported probability measure ν on \mathbb{R}^+ , what are the elements $a \in \mathcal{A}p_1$ such that aa^* has distribution ν in (\mathcal{A}_{22}, p_2) , and such that $\chi^{\mathcal{D}}(p_1 a p_2, p_2 a p_2, p_2 a^* p_1, p_2 a^* p_2)$ is maximal?

6. ANALOGUE OF FREE FISHER'S INFORMATION FOR SIMPLE ELEMENTS: THE MICROSTATE-FREE APPROACH

In this section, we present a notion of free Fisher's information for simple elements, constructed without using the microstates, like what was done by Voiculescu in [31] and by Shlyakhtenko in [22] for elements of a W^* -probability space. For a synthetic presentation of the free Fisher's information of elements of a W^* -probability space, see section 2 of [15].

6.1. DEFINITIONS. In this section, we suppose that (\mathcal{A}, φ) is a W^* -probability space, with φ faithful tracial state. $L^2(\mathcal{A})$ will denote the Hilbert space obtained by completing \mathcal{A} for the norm $\|a\|_2 = (\varphi(aa^*))^{\frac{1}{2}}$, $a \in \mathcal{A}$.

\mathcal{A} acts on $L^2(\mathcal{A})$ on the right and on the left, so one can define, for $k, l \in [d]$, $L^2(\mathcal{A})_{kl} = p_k L^2(\mathcal{A}) p_l$. We still have an identification between $L^2(\mathcal{A})$ and

$$\begin{bmatrix} L^2(\mathcal{A})_{11} & \cdots & L^2(\mathcal{A})_{1d} \\ \vdots & & \vdots \\ L^2(\mathcal{A})_{d1} & \cdots & L^2(\mathcal{A})_{dd} \end{bmatrix},$$

by the map $x \in L^2(\mathcal{A}) \rightarrow \begin{bmatrix} p_1 x p_1 & \cdots & p_1 x p_d \\ \vdots & & \vdots \\ p_d x p_1 & \cdots & p_d x p_d \end{bmatrix}$. We still call the non zero elements of $\bigcup_{k,l \in [d]} L^2(\mathcal{A})_{kl}$ the *simple elements* of $L^2(\mathcal{A})$. We define, for a non zero simple element of $L^2(\mathcal{A})$, $r(a)$ (r is for *row*) and $c(a)$ (c is for *column*) the unique numbers of $[d]$ such that

$$a \in L^2(\mathcal{A})_{r(a), c(a)}.$$

Moreover, $x \rightarrow x^*$ extends to $L^2(\mathcal{A})$, and for all $a, b \in \mathcal{A}$, $x \in L^2(\mathcal{A})$, we have $(axb)^* = b^* x^* a^*$. For all $k \in [d]$, the state φ_k on \mathcal{A}_{kk} extends to $L^2(\mathcal{A})_{kk}$, so the conditional expectation E extends to $L^2(\mathcal{A})$, and we still have

$$\forall d, d' \in \mathcal{D}, \forall x \in L^2(\mathcal{A}), E(dx d') = d E(x) d'.$$

In the same way, for $n \geq 1$ and $\pi \in \text{NC}(n)$, we can extend E_n , E_π , c_n and c_π to $L^2(\mathcal{A}) \otimes (\mathcal{A}^{\otimes n-1})$, and the relations (2.1), (2.2), (2.3), and (2.4) remain true.

A family $(a_i)_{i \in I}$ of elements of $L^2(\mathcal{A})$ is said to be a *self-adjoint family* if there exists an involution $*$ of I such that for all $i \in I$, $a_i^* = a_{*(i)}$.

A finite sequence (a_1, \dots, a_n) of simple elements of $L^2(\mathcal{A})$ is said to be a *square sequence* if for all $i \in [n]$, $c(a_i) = l(a_{i+1})$ (with $a_{n+1} := a_1$).

DEFINITION 6.1. Let $(a_i)_{i \in I}$ be a self-adjoint family of simple elements of \mathcal{A} . A family $(\xi_i)_{i \in I}$ of simple elements of $L^2(\mathcal{A})$ is said to *fulfill conjugate relations* for $(a_i)_{i \in I}$ if for all $i \in I$,

$$r(\xi_i) = c(a_i), \quad c(\xi_i) = r(a_i),$$

and if one of the following equivalent proposition is true :

- (i) for all $n \geq 0$, for all $i, i_1, \dots, i_n \in I$ such that $(\xi_i, a_{i_1}, \dots, a_{i_n})$ is a square sequence,

$$\varphi_{r(\xi_i)}(\xi_i a_{i_1} \cdots a_{i_n}) = \sum_{m=1}^n \delta_{i, i_m} \varphi_{c(\xi_i)}(a_{i_1} \cdots a_{i_{m-1}}) \varphi_{r(\xi_i)}(a_{i_{m+1}} \cdots a_{i_n}),$$

- (ii) - for all $i \in I$, $E(\xi_i) = 0$,
- for all $i, j \in I$, $E(\xi_i a_j) = \delta_{ij} p_{r(\xi_i)}$,

- for all $n \geq 2$, for all $i, i_1, \dots, i_n \in I$ such that $(\xi_i, a_{i_1}, \dots, a_{i_n})$ is a square sequence,

$$E(\xi_i a_{i_1} \cdots a_{i_n}) = \sum_{m=1}^n \delta_{i, i_m} \eta_{r(\xi_i), c(\xi_i)} \circ E(a_{i_1} \cdots a_{i_{m-1}}) E(a_{i_{m+1}} \cdots a_{i_n}),$$

(we recall that for all $k, l \in [d]$, $\eta_{k,l}$ is the involution of \mathcal{D} which permutes the k -th and the l -th columns in the representation of elements of \mathcal{D} as $d \times d$ diagonal matrices),

(iii) for all $n \geq 0$, for all $i, i_1, \dots, i_n \in I$ such that $(\xi_i, a_{i_1}, \dots, a_{i_n})$ is a square sequence,

$$c_{n+1}(\xi_i \otimes a_{i_1} \otimes \cdots \otimes a_{i_n}) = \delta_{n,1} \delta_{i, i_1} p_r(\xi_i).$$

Proof. Consider a family $(\xi_i)_{i \in I}$ of simple elements of $L^2(\mathcal{A})$ such that for all $i \in I$,

$$l_i := r(\xi_i) = c(a_i), \quad k_i := c(\xi_i) = r(a_i),$$

and let us prove the equivalence of (i), (ii), and (iii). The equivalence between (i) and (ii) is obvious by definition of E : for all $x \in L^2(\mathcal{A})$,

$$E(x) = \sum_{k=1}^d \varphi_k(p_k a p_k) \cdot p_k.$$

Now, suppose (ii) true, and let us prove (iii) by induction on n . Note that $c_1 = E_1$, and that for all $\xi \in L^2(\mathcal{A})$, $a \in \mathcal{A}$,

$$c_2(\xi \otimes a) = E(\xi a) - E(\xi) E(a),$$

so (iii) is proved for $n = 0, 1$. Now, suppose it proved for to all ranks $0, 1, \dots, n - 1$, with $n \geq 2$, and let us prove it to the rank n . Consider $i, i_1, \dots, i_n \in I$ such that $(\xi_i, a_{i_1}, \dots, a_{i_n})$ is a square sequence. We have

$$c_{n+1}(\xi_i \otimes a_{i_1} \otimes \cdots \otimes a_{i_n}) = E(\xi_i a_{i_1} \cdots a_{i_n}) - \sum_{\pi} c_{\pi}(\xi_i \otimes a_{i_1} \otimes \cdots \otimes a_{i_n}),$$

where the sum is taken over all noncrossing partitions π of $\{0, \dots, n\}$ which are $< 1_{\{0, \dots, n\}}$, and in which all blocks are associated to square subsequences of $(\xi_i, a_{i_1}, \dots, a_{i_n})$.

Consider such a partition π , and apply the factorization formula (2.6) to $c_{\pi}(\xi_i \otimes a_{i_1} \otimes \cdots \otimes a_{i_n})$. If it is not null, then the block of π containing 0 has only one other element, say m , and in this case, we have

$$c_{\pi}(\xi_i \otimes a_{i_1} \otimes \cdots \otimes a_{i_n}) = \prod_{\substack{V \in \pi_1 \\ V = \{t_1 < \cdots < t_r\}}} \eta_{l_i, l_r} \circ c_r(a_{i_{t_1}} \otimes \cdots \otimes a_{i_{t_r}}) \prod_{\substack{V \in \pi_2 \\ V = \{t_1 < \cdots < t_r\}}} \eta_{l_i, l_r} \circ c_r(a_{i_{t_1}} \otimes \cdots \otimes a_{i_{t_r}}),$$

where π_1 (resp. π_2) is the partition induced by π on $\{1, \dots, m - 1\}$ (resp. $\{m + 1, \dots, n\}$), i.e. to

$$\delta_{i, i_m} \eta_{l_i, k_i} \circ c_{\pi_1}(a_{i_1} \otimes \cdots \otimes a_{i_{m-1}}) c_{\pi_2}(a_{i_{m+1}} \otimes \cdots \otimes a_{i_n}),$$

Thus we have

$$\begin{aligned} & c_{n+1}(\xi_i \otimes a_{i_1} \otimes \cdots \otimes a_{i_n}) = \\ \mathbb{E}(\xi_i a_{i_1} \cdots a_{i_n}) & - \sum_{m=1}^n \sum_{\substack{\pi_1 \in \text{NC}(m-1) \\ \pi_2 \in \text{NC}(\{m+1, \dots, n\})}} \delta_{i, i_m} \eta_{l_i, k_i} \circ c_{\pi_1}(a_{i_1} \otimes \cdots \otimes a_{i_{m-1}}) c_{\pi_2}(a_{i_{m+1}} \otimes \cdots \otimes a_{i_n}) \\ & = \mathbb{E}(\xi_i a_{i_1} \cdots a_{i_n}) - \sum_{m=1}^n \delta_{i, i_m} \eta_{l_i, k_i} \circ \mathbb{E}(a_{i_1} \cdots a_{i_{m-1}}) \mathbb{E}(a_{i_{m+1}} \cdots a_{i_n}) = 0. \end{aligned}$$

The reciprocal implication is analogous. ■

DEFINITION 6.2. Let $(a_i)_{i \in I}$ be a self-adjoint family of simple elements of \mathcal{A} .

1. A family $(\xi_i)_{i \in I}$ of simple elements of $L^2(\mathcal{A})$ is said to be a *conjugate system* for $(a_i)_{i \in I}$ if it fulfills conjugate relations and in addition we have that

$$(6.1) \quad \forall i \in I, \quad \xi_i \in \overline{\text{Alg}(\{a_i; i \in I\} \cup \mathcal{D})}^{\|\cdot\|_2} \subset L^2(\mathcal{A}).$$

2. Let the \mathcal{D} -Fisher's information of $(a_i)_{i \in I}$ be

$$\Phi_r((a_i)_{i \in I}) = \begin{cases} \sum_{i \in I} \|\xi_i\|^2 & \text{if } (\xi_i)_{i \in I} \text{ is a conjugate system,} \\ \infty & \text{if there is no conjugate system.} \end{cases}$$

REMARK. 1. The algebra generated by $\{a_i; i \in I\} \cup \mathcal{D}$ is the set of elements of \mathcal{A} which have a $d \times d$ matrix representation of the type

$$\begin{bmatrix} P_{11}(a_i; i \in I) & \cdots & P_{1d}(a_i; i \in I) \\ \vdots & & \cdots \\ P_{d1}(a_i; i \in I) & \cdots & P_{dd}(a_i; i \in I) \end{bmatrix},$$

where for all $k, l \in [d]$, P_{kl} is a polynomial in the noncommutative variables $X_i; i \in I$, and $P_{kl}(a_i; i \in I) \in \mathcal{A}_{kl}$. So the conjugate relations can be viewed as a prescription for the inner products in $L^2(\mathcal{A})$ between ξ_i ($i \in I$) and elements of this subalgebra. It follows that the conjugate system for $(a_i)_{i \in I}$ is unique, if it exists. Note moreover that the existence of the conjugate system is equivalent to the existence of any family in $L^2(\mathcal{A})$ which fulfills the conjugate relations; indeed, if $(\xi_i)_{i \in I}$ fulfill the conjugate relations and if we set, for all $i \in I$, γ_i to be the projection of ξ_i onto $\overline{\text{Alg}(\{a_i; i \in I\} \cup \mathcal{D})}^{\|\cdot\|_2}$, then $(\gamma_i)_{i \in I}$ will also fulfill the conjugate relations, hence will give a conjugate system.

2. Consider an involution $*$ of I such that for all $i \in I$, $a_{*(i)} = a_i^*$. Consider a family $(\xi_i)_{i \in I}$ which fulfills conjugate relations. Then define, for all $i \in I$,

$$\tilde{\xi}_i = \frac{\rho_{c(a_i)}}{\rho_{r(a_i)}} \xi_{*(i)}^*.$$

Then $(\tilde{\zeta}_i)_{i \in I}$ fulfills conjugate relations, hence we have $\zeta_i = \tilde{\zeta}_i$ for all i if $(\zeta_i)_{i \in I}$ is a conjugate system. It can be written

$$(6.2) \quad \zeta_{*(i)} = \frac{\rho_{c(a_i)}}{\rho_{r(a_i)}} \tilde{\zeta}_i^*.$$

Proof. Let us prove 2. It suffices to prove that $(\tilde{\zeta}_i)_{i \in I}$ fulfills conjugate relations. Consider $n \geq 0$ and $i, i_1, \dots, i_n \in I$ such that $(\tilde{\zeta}_i, a_{i_1}, \dots, a_{i_n})$ is a square sequence. Then

$$\begin{aligned} c_{n+1}(\tilde{\zeta}_i \otimes a_{i_1} \otimes \dots \otimes a_{i_n}) &= \frac{\rho_{c(a_i)}}{\rho_{r(a_i)}} c_{n+1}(\tilde{\zeta}_{*(i)}^* \otimes a_{*(i_1)}^* \otimes \dots \otimes a_{*(i_n)}^*) \\ &= \frac{\rho_{c(a_i)}}{\rho_{r(a_i)}} (c_{n+1}(a_{*(i_n)} \otimes \dots \otimes a_{*(i_1)} \otimes \tilde{\zeta}_{*(i)}))^* \\ &= (\eta_{r(\tilde{\zeta}_i), c(\tilde{\zeta}_i)} \circ c_{n+1}(\tilde{\zeta}_{*(i)} \otimes a_{*(i_n)} \otimes \dots \otimes a_{*(i_1)}))^* \\ &= (\eta_{r(\tilde{\zeta}_i), c(\tilde{\zeta}_i)} \delta_{n,1} \delta_{*(i),*(i_1)} p_{r(\tilde{\zeta}_{*(i)})})^* \\ &= \delta_{i,i_1} p_{r(\tilde{\zeta}_i)}, \end{aligned}$$

which closes the proof. ■

3. A link with the already defined notions of Fisher's information can be made as follows: Consider $x \in \mathcal{A}_{kl}$ such that x^*x is invertible in \mathcal{A}_{ll} . Then a pair $(\zeta, \frac{\rho_k}{\rho_l} \zeta^*)$ fulfills conjugate relations for (x, x^*) if $\frac{\rho_k}{\rho_l} \zeta x (x^*x)^{-1}$ fulfills conjugate relations in the sense of [15] for x^*x in $(\mathcal{A}_{ll}, \varphi_l)$. The reciprocal is true if moreover, $\varphi_l(\zeta x (x^*x)^{-1})$ is null.

Proof. $(\zeta, \frac{\rho_k}{\rho_l} \zeta^*)$ fulfills conjugate relations for (x, x^*) if and only if for all $n \geq 0$,

$$\begin{cases} (A) & \varphi_l(\zeta x (x^*x)^n) = \sum_{i=0}^n \varphi_k((xx^*)^i) \varphi_k((x^*x)^{n-i}), \\ (B) & \frac{\rho_k}{\rho_l} \varphi_k(\zeta^* x^* (xx^*)^n) = \sum_{i=0}^n \varphi_l((x^*x)^i) \varphi_l((xx^*)^{n-i}), \end{cases}$$

where $(xx^*)^0$ stands for p_k and $(x^*x)^0$ stands for p_l .

But using $\varphi(\cdot^*) = \overline{\varphi(\cdot)}$, we have

$$(A) \Leftrightarrow \varphi_l((x^*x)^n x^* \zeta^*) = \sum_{i=0}^n \varphi_l((x^*x)^i) \varphi_k((xx^*)^{n-i}),$$

which is equivalent to (B) because $\varphi_l((x^*x)^n x^* \zeta^*) = \frac{\rho_k}{\rho_l} \varphi_k(\zeta^* x^* (xx^*)^n)$.

So $(\zeta, \frac{\rho_k}{\rho_l} \zeta^*)$ fulfills conjugate relations for (x, x^*) if and only if for all $n \geq 0$,

$$\varphi_l(\zeta x (x^*x)^n) = \sum_{i=0}^n \varphi_k((xx^*)^i) \varphi_k((x^*x)^{n-i}).$$

It is implied by the fact that $\frac{\rho_k}{\rho_l} \zeta x (x^*x)^{-1}$ fulfills conjugate relations in the sense of [15] for x^*x in $(\mathcal{A}_{ll}, \varphi_l)$, and the reciprocal is true if moreover, $\varphi_l(\zeta x (x^*x)^{-1})$ is null. ■

6.2. CRAMÉR-RAO INEQUALITY.

THEOREM 6.3. Consider a non null element a of \mathcal{A}_{kl} , with $k \neq l$ and $\rho_k \leq \rho_l$. Then

$$\varphi(aa^*)\Phi_r(a, a^*) \geq \rho_k^2 + \rho_l^2,$$

with equality if and only if there exists c positive number such that the moments of $c.aa^*$ in $(\mathcal{A}_{kk}, \varphi_k)$ are the moments of the Marchenko-pastur distribution with parameter $\frac{\rho_l}{\rho_k}$.

Proof. If (a, a^*) has no conjugate system, then it is obvious. If a conjugate system for (a, a^*) is $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*)$ (indeed, by (6.2), any conjugate system has this form), then

$$\Phi_r(a, a^*) = \left(1 + \frac{\rho_k^2}{\rho_l^2}\right) \varphi(\zeta\zeta^*),$$

so it suffices to prove that

$$\varphi(aa^*)\varphi(\zeta\zeta^*) \geq \rho_l^2$$

Note that we have $\varphi((\zeta^*)^*a) = \rho_l$, so the result follows from the Cauchy-Schwarz inequality. Moreover, we have equality if and only if there exists $c > 0$ such that $\zeta^* = ca$, which is equivalent to the fact that for all $n \geq 2$ even,

$$c_n^{(l)}(a^* \otimes a \otimes \cdots \otimes a) = \delta_{n,2} \frac{1}{c}$$

and

$$c_n^{(k)}(a \otimes a^* \otimes \cdots \otimes a^*) = \delta_{n,2} \frac{\rho_l}{\rho_k} \frac{1}{c},$$

which is equivalent, by theorem 2.3, to the fact that the moments of $c.aa^*$ in $(\mathcal{A}_{kk}, \varphi_k)$ are the moments of the Marchenko-pastur distribution with parameter $\frac{\rho_l}{\rho_k}$. ■

6.3. FISHER'S INFORMATION AND FREENESS.

THEOREM 6.4. Consider a non null elements x, y of \mathcal{A}_{kl} , with $k \neq l$. Then we have

$$(6.3) \quad \Phi_r(x, y, x^*, y^*) \geq \Phi_r(x, x^*) + \Phi_r(y, y^*),$$

and we have equality if x, y are free with amalgamation over \mathcal{D} . Moreover, if

$$\Phi_r(x, y, x^*, y^*) = \Phi_r(x, x^*) + \Phi_r(y, y^*) < \infty,$$

then x, y are free with amalgamation over \mathcal{D} .

Proof. - Let us prove that

$$\Phi_r(x, y, x^*, y^*) \geq \Phi_r(x, x^*) + \Phi_r(y, y^*).$$

If $\Phi_r(x, y, x^*, y^*) = \infty$, it is clear, and in the other case, let $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*, \zeta, \frac{\rho_k}{\rho_l}\zeta^*)$ be the conjugate system for (x, y, x^*, y^*) . Then $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*)$, (resp. $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*)$) satisfy conjugate relations for (x, x^*) (resp. for (y, y^*)), so the result is proved.

- Suppose that x, y are free with amalgamation over \mathcal{D} . If $\Phi(x, x^*) = \infty$ or $\Phi(y, y^*) = \infty$, then we have equality in (6.3). In the other case, let $(\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*)$, (resp. $(\zeta, \frac{\rho_k}{\rho_l} \zeta^*)$) be a conjugate system for (x, x^*) (resp. for (y, y^*)). It suffices to prove that $(\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*, \zeta, \frac{\rho_k}{\rho_l} \zeta^*)$ is a conjugate system for (x, y, x^*, y^*) . It is clear that

$$\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*, \zeta, \frac{\rho_k}{\rho_l} \zeta^* \in \overline{\text{Alg}(\{x, y, x^*, y^*\} \cup \mathcal{D})}^{\|\cdot\|_2},$$

so it suffices to prove that $(\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*, \zeta, \frac{\rho_k}{\rho_l} \zeta^*)$ fulfills conjugate relations for the family (x, y, x^*, y^*) . Let us prove condition (iii) of definition 6.1. Since $(\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*)$, (resp. $(\zeta, \frac{\rho_k}{\rho_l} \zeta^*)$) is a conjugate system for (x, x^*) (resp. for (y, y^*)), we have $c_1(\tilde{\zeta}) = c_1(\tilde{\zeta}^*) = c_1(\zeta) = c_1(\zeta^*) = 0$, and $c_2(\tilde{\zeta} \otimes x) = c_2(\tilde{\zeta} \otimes y) = p_l$, $c_2(\frac{\rho_k}{\rho_l} \tilde{\zeta}^* \otimes y^*) = c_2(\frac{\rho_k}{\rho_l} \tilde{\zeta}^* \otimes x^*) = p_k$. Since $\tilde{\zeta}, \tilde{\zeta}^* \in \overline{\text{Alg}(\{x, x^*\} \cup \mathcal{D})}^{\|\cdot\|_2}$, and $\zeta, \zeta^* \in \overline{\text{Alg}(\{y, y^*\} \cup \mathcal{D})}^{\|\cdot\|_2}$, by freeness with amalgamation over \mathcal{D} , we have

$$c_2(\tilde{\zeta} \otimes x) = c_2(\tilde{\zeta}^* \otimes x^*) = c_2(\tilde{\zeta} \otimes y) = c_2(\tilde{\zeta}^* \otimes y^*) = 0.$$

Consider now $n \geq 2$, and a square sequence $(T, a_1, \dots, a_n) \in \{\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*, \zeta, \frac{\rho_k}{\rho_l} \zeta^*\} \times (\{x, x^*, y, y^*\})^n$. Let us prove that

$$(6.4) \quad c_{n+1}(T \otimes a_1 \otimes \dots \otimes a_n) = 0.$$

For example, we can suppose that $T = \tilde{\zeta}$. If one of the a_i 's is y or y^* , then (6.4) is due to the freeness with amalgamation over \mathcal{D} . If none of the a_i 's is y or y^* , then (6.4) is due to the fact that $(\tilde{\zeta}, \frac{\rho_k}{\rho_l} \tilde{\zeta}^*)$ fulfills conjugate relations for (x, x^*) .

- In order to finish the proof, let us prove that if $(\zeta, \frac{\rho_k}{\rho_l} \zeta^*)$ is a conjugate system for (x, x^*) , then we have $x\zeta = \zeta^*x^*$. Both belong to

$$\overline{p_k \text{Alg}(\{x, x^*\} \cup \mathcal{D})}^{\|\cdot\|_2} p_k,$$

which is equal to

$$\overline{\{p_k\} \cup \{(xx^*)^n; n \geq 1\}}^{\|\cdot\|_2}.$$

Thus, since φ_k is a faithful trace state on \mathcal{A}_{kk} , it suffices that for all $n \geq 0$,

$$\varphi_k((xx^*)^n x\zeta) = \varphi_k((xx^*)^n \zeta^* x^*),$$

where $(xx^*)^0$ stands for p_k . We have

$$\begin{aligned} \varphi_k((xx^*)^n x\zeta) &= \frac{\rho_l}{\rho_k} \varphi_l(\tilde{\zeta}(xx^*)^n x) \\ &= \frac{\rho_l}{\rho_k} \sum_{i=0}^n \varphi_k((xx^*)^i) \varphi_l((xx^*)^{n-i}), \end{aligned}$$

which is a real number. Thus

$$\begin{aligned} \varphi_k((xx^*)^n \zeta^* x^*) &= \varphi_k(\zeta^* x^* (xx^*)^n) = \varphi_k(((xx^*)^n x\zeta)^*) = \\ &= \overline{\varphi_k((xx^*)^n x\zeta)} = \varphi_k((xx^*)^n x\zeta). \end{aligned}$$

So we have proved that $x\zeta = \zeta^*x^*$.

- Now, suppose that

$$\Phi_r(x, y, x^*, y^*) = \Phi_r(x, x^*) + \Phi_r(y, y^*) < \infty.$$

By theorem 2.1, in order to prove the freeness with amalgamation over \mathcal{D} of x and y , it suffices to prove that for all $n \geq 2$, for all z_1, \dots, z_n taken in the algebras $\text{Alg}(\{x, x^*\} \cup \mathcal{D})$ and $\text{Alg}(\{y, y^*\} \cup \mathcal{D})$, but not all in the same one, we have

$$(6.5) \quad c_n(z_1 \otimes \dots \otimes z_n) = 0.$$

By the formula of cumulants with products as entries (Theorem 2 of [26]), we can suppose that

$$z_1, \dots, z_n \in \{x, x^*, y, y^*\}.$$

If the sequence (z_1, \dots, z_n) is not square, then (6.5) holds by paragraph 2.2 (a). So we suppose the sequence to be square, and by equation (2.7), we can suppose that

$$(z_1, z_2) \in \{(x, y^*), (y, x^*), (x^*, y), (y^*, x)\}.$$

For example, we will treat the case where $(z_1, z_2) = (x, y^*)$.

Consider $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*, \zeta, \frac{\rho_k}{\rho_l}\zeta^*)$ a conjugate system for (x, y, x^*, y^*) . Then using the hypothesis and 1. of 6.1, we know that $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*)$, (resp. $(\zeta, \frac{\rho_k}{\rho_l}\zeta^*)$) is the conjugate system for (x, x^*) (resp. for (y, y^*)). Since $x\zeta = \zeta^*x^*$, we have

$$(6.6) \quad c_{n+1}^{(k)}(x\zeta \otimes z_2 \otimes \dots \otimes z_n \otimes z_1^*) = c_{n+1}^{(k)}(\zeta^*x^* \otimes z_2 \otimes \dots \otimes z_n \otimes z_1^*).$$

Now, we apply the formula of cumulants with products as entries (Theorem 2 of [26]) to left hand side and right hand side of (6.6).

$$\begin{aligned} LHS &= c_{n+2}^{(k)}(x \otimes \zeta \otimes y \otimes z_3 \otimes \dots \otimes z_n \otimes z_1^*) + \sum_{\substack{1 \leq i \leq n \\ i \text{ even}}} \\ &\quad \underbrace{c_i^{(l)}(\zeta \otimes z_2 \otimes z_3 \otimes \dots \otimes z_i)}_{=0 \text{ because } z_2=y \text{ (and } c_2(\zeta) = 0)} c_{n-i+2}^{(k)}(x \otimes z_{i+1} \otimes \dots \otimes z_n \otimes z_1^*) \\ &= \frac{\rho_l}{\rho_k} c_{n+2}^{(l)}(\zeta \otimes y \otimes z_3 \otimes \dots \otimes z_n \otimes z_1^* \otimes x) \\ &= 0. \end{aligned}$$

On the other side, we have

$$\begin{aligned}
RHS &= \underbrace{c_{n+2}^{(k)}(\xi^* \otimes x^* \otimes z_2 \otimes z_3 \otimes \cdots \otimes z_n \otimes z_1^*)}_{=0 \text{ because } n+2 > 2} + \sum_{\substack{1 \leq i \leq n \\ i \text{ even}}} \\
&= c_n^{(l)}(x^* \otimes z_2 \otimes z_3 \otimes \cdots \otimes z_i) \underbrace{c_{n-i+2}^{(k)}(\xi^* \otimes z_{i+1} \otimes \cdots \otimes z_n \otimes z_1^*)}_{=0 \text{ if } i < n} \\
&= c_n^{(l)}(x^* \otimes y \otimes z_3 \otimes \cdots \otimes z_n) \underbrace{c_2^{(k)}(\xi^* \otimes x^*)}_{\neq 0}.
\end{aligned}$$

Hence

$$c_n^{(l)}(x^* \otimes y \otimes z_3 \otimes \cdots \otimes z_n) = 0,$$

which is what we wanted to obtain. ■

QUESTION. It would be interesting to have a characterization of R -diagonal elements with non trivial kernel involving the Fisher's information defined in this paper. Inspired by the paper [15], we ask the following question: given a compactly supported probability measure ν on \mathbb{R}^+ , what are the elements $a \in \mathcal{A}p_1$ such that aa^* has distribution ν in (\mathcal{A}_{22}, p_2) , and such that

$$\Phi_r(p_1 a p_2, p_2 a p_2, p_2 a^* p_1, p_2 a^* p_2)$$

is minimal ?

APPENDIX A: PROOF OF PROPOSITION 5.1

Let us suppose, for example, that $\rho_k \leq \rho_l$. Fix $R > 0$ such that R^2 is more than the supremum of the support of μ . Define, for all $q \geq 1$, the map

$$\kappa_q : x \in [0, R]^q \rightarrow \frac{1}{q} \sum_{i=1}^q \delta_{x_i}.$$

The index q in κ_q will always be omitted, because no confusion will ever be possible. For P probability measure on $[0, R]^q$, we denote by $\kappa(P)$ the push-forward of P by κ . First of all, let us recall a large deviation principle. For basic definitions on large deviations, see P. 177 of [10] or many other books (e.g. [6]).

THEOREM 6.5. *Let, for $n \geq 1$, Z_n be the total mass of*

$$P_n := \Delta(x)^2 \prod_{i=1}^{q_k(n)} x_j^{q_l(n) - q_k(n)} \mathbf{1}_{[0, R]^{q_k(n)}}(x) dx.$$

Then the finite limit $B := \lim_{n \rightarrow \infty} n^{-2} \log Z_n$ exists, and the sequence $\left(\kappa \left(\frac{1}{Z_n} P_n \right) \right)_n$ satisfies a large deviation principle in the set of probability measures on $[0, R]$ endowed with

topology of weak convergence in the scale n^{-2} with the good rate function

$$I : \nu \rightarrow -\rho_k^2 \Sigma(\nu) - (\rho_k \rho_l - \rho_k^2) \int \log(x) d\nu(x) + B.$$

This theorem was proved under a slightly different hypothesis in [10] (theorem 5.5.1 p. 227, with $Q = 0$). The difference between the hypothesis above and the hypothesis of theorem 5.5.1 of [10] is that in the latter, the bound R does not appear, the measures are considered on \mathbb{R}^+ . But it is not a problem: the proof of theorem 5.5.1 can easily be adapted to this context (in fact it is more easy to work with the compact set $[0, R]$). Note that an analogous modification of a result proved for the interval \mathbb{R}^+ to the interval $[0, R]$ is done p. 240 of [10].

Note that removing the renormalization constant Z_n and the limit B , one gets the following result. Its formulation implies to have extended the notion of large deviation principle to sequences of finite measures (not only of probability measures), but it can be done without any ambiguity.

COROLLARY 6.6. *The sequence of finite measures $(\kappa(P_n))_n$ satisfies a large deviation principle in the set of probability measures on $[0, R]$ endowed with topology of weak convergence in the scale n^{-2} with the good rate function*

$$J : \nu \rightarrow -\rho_k^2 \Sigma(\nu) - (\rho_k \rho_l - \rho_k^2) \int \log(x) d\nu(x).$$

Now, we give the proof of proposition 5.1.

Step I. For all $r \geq 1, \varepsilon > 0$,

$$\chi_R(a; 2r, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n^2} \log \Lambda(\Gamma_R(a; n, 2r, \rho_k \varepsilon)) + \rho_k \rho_l \log n.$$

$\Gamma_R(a; n, r, \rho_k \varepsilon)$ is the set of matrices of $\mathfrak{M}_n(k, l)$ such that $\|M\| \leq R$ and each moment of order $\leq r$ of the spectral law of the k -th diagonal block of MM^* is ε -closed to the moment of same order of μ . Thus by the remark following lemma 5.2, $\Lambda(\Gamma_R(a; n, 2r, \rho_k \varepsilon))$ is

$$\frac{\pi^{q_k(n)q_l(n)} P_n(\{x \in [0, R]^{q_k(n)}; \forall i = 0, \dots, r, |m_i(\kappa(x)) - m_i(\mu)| \leq \varepsilon\})}{\prod_{j=1}^{q_k(n)} j! \prod_{j=q_l(n)-q_k(n)}^{q_l(n)-1} j!},$$

where P_n is the measure introduced in the previous theorem.

Step II. Let us compute the limit C , as $n \rightarrow \infty$, of

$$u_n := \frac{1}{n^2} \log \frac{\pi^{q_k(n)q_l(n)}}{\prod_{j=1}^{q_k(n)} j! \prod_{j=q_l(n)-q_k(n)}^{q_l(n)-1} j!} + \rho_k \rho_l \log n.$$

We have, by Stirling formula, $\log j! = \frac{1}{2} \log j + j(\log j - 1) + O(1)$. So

$$u_n = \frac{q_k(n)q_l(n)}{n^2} \log \pi - \frac{1}{n^2} \sum_{j=1}^{q_k(n)} \left(\frac{1}{2} \log j + j(\log j - 1) \right)$$

$$\begin{aligned}
& -\frac{1}{n^2} \sum_{j=q_l(n)-q_k(n)}^{q_l(n)-1} \left(\frac{1}{2} \log j + j(\log j - 1) \right) + \rho_k \rho_l \log n + O\left(\frac{1}{n}\right) \\
& \quad = \rho_k \rho_l \log \pi + \\
& \frac{q_k(n)(q_k(n) + 1) + (q_l(n) - 1)q_l(n) - (q_l(n) - q_k(n) - 1)(q_l(n) - q_k(n))}{2n^2} \\
& \quad - \frac{q_k(n)^2}{n^2} \frac{1}{q_k(n)} \sum_{j=1}^{q_k(n)} \frac{j}{q_k(n)} \log \frac{j}{q_k(n)} - \frac{\log q_k(n)}{n^2} \sum_{j=1}^{q_k(n)} j \\
& \quad - \frac{q_k(n)^2}{n^2} \frac{1}{q_k(n)} \sum_{j=q_l(n)-q_k(n)}^{q_l(n)-1} \frac{j}{q_k(n)} \log \frac{j}{q_k(n)} - \frac{\log q_k(n)}{n^2} \sum_{j=q_l(n)-q_k(n)}^{q_l(n)-1} j \\
& \quad \quad + \rho_k \rho_l \log n + o(1) \\
& \quad = \rho_k \rho_l (\log \pi + 1) - \rho_k^2 \int_0^1 t \log t dt - \rho_k^2 \int_{\rho_l/\rho_k-1}^{\rho_l/\rho_k} t \log t dt \\
& \quad \quad - \frac{\log q_k(n)}{2n^2} [q_k(n)(q_k(n) + 1) + (q_l(n) - 1)q_l(n) \\
& \quad \quad - (q_l(n) - q_k(n) - 1)(q_l(n) - q_k(n))] + \rho_k \rho_l \log(q_k(n) \frac{n}{q_k(n)}) + o(1) \\
& \quad = \rho_k \rho_l (\log \pi + 1 - \log \rho_k) - \rho_k^2 \int_0^1 t \log t dt - \rho_k^2 \int_{\rho_l/\rho_k-1}^{\rho_l/\rho_k} t \log t dt + \frac{\log q_k(n)}{n^2} \times \\
& \quad [q_k(n)(q_k(n) + 1) + (q_l(n) - 1)q_l(n) - (q_l(n) - q_k(n) - 1)(q_l(n) - q_k(n)) \\
& \quad \quad - 2q_k(n)q_l(n) + O(n)] + o(1).
\end{aligned}$$

Thus

$$u_n \xrightarrow{n \rightarrow \infty} C := \rho_k \rho_l (\log \pi + 1 - \log \rho_k) - \underbrace{\rho_k^2 \int_0^1 t \log t dt}_{=-1/4} - \rho_k^2 \int_{\rho_l/\rho_k-1}^{\rho_l/\rho_k} t \log t dt.$$

Step III. Now, let us denote by $F(r, \varepsilon)$ (resp. $G(r, \varepsilon)$) the set of probability measures on $[0, R]$ for which each moment of order $\leq r$ is ε -closed (resp. strictly ε -closed) to the moment of same order of μ . $F(r, \varepsilon)$ (resp. $G(r, \varepsilon)$) is closed (resp. open). Thus by the previous corollary, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(\kappa^{-1}(F(r, \varepsilon))) \leq - \inf_{F(r, \varepsilon)} J,$$

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \log P_n(\kappa^{-1}(G(r, \varepsilon))) \geq - \inf_{G(r, \varepsilon)} J.$$

But $\inf_{F(r, \varepsilon)} J = \inf_{G(r, \varepsilon)} J$, so

$$\frac{1}{n^2} \log P_n(\kappa^{-1}(F(r, \varepsilon))) \xrightarrow{n \rightarrow \infty} - \inf_{F(r, \varepsilon)} J,$$

and it follows, by Steps I and II, that

$$\chi_R(a; 2r, \varepsilon) = - \inf_{F(r, \varepsilon)} J + C.$$

As ε goes to 0 and r goes to ∞ , $\inf_{F(r, \varepsilon)} J$ goes to $J(\mu)$, and we obtain the desired result:

$$\begin{aligned} \chi_R(a) = & \\ & \rho_k^2 \Sigma(\mu) + (\rho_l - \rho_k) \rho_k \int \log(x) d\mu(x) + \rho_k \rho_l (\log \frac{\pi}{\rho_k} + 1) + \frac{\rho_k^2}{4} - \rho_k^2 \int_{\rho_l/\rho_k-1}^{\rho_l/\rho_k} t \log t dt. \end{aligned}$$

APPENDIX B: PROOF OF LEMMA 5.2

In all this proof, we shall identify elements of \mathbb{R}^q with the associated diagonal $q \times q$ matrix.

a) First of all, the fact that Ψ is an injection onto a set with negligible complementary is well known (see [11]).

b) Let P be the $q \times q'$ matrix with entry (i, j) equal to 1 if $i = j$, and to 0 in the other case. Then the set $\mathcal{U}_{q, q'}$ is $P\mathcal{U}_{q'}$. So the set $\mathcal{U}_q/T_q \times \mathbb{R}_{+, <}^q \times \mathcal{U}_{q, q'}$ is a manifold, and for $u \in \mathcal{U}_q$, $a \in \mathbb{R}_{+, <}^q$, $v \in \mathcal{U}_{q'}$, its tangent space at (uT_q, a, Pv) is the cartesian product of tangent spaces of respectively \mathcal{U}_q/T_q , $\mathbb{R}_{+, <}^q$, $\mathcal{U}_{q, q'}$ at respectively uT_q, a, Pv . The first of them can be identified, via the map $M \rightarrow u^*M$, to the set \mathfrak{U}_q^0 of anti-hermitian matrices with zeros on the diagonal, the second one is \mathbb{R}^n , and the third one can be identified, via the map $M \rightarrow Mv^*$, to the set $\mathfrak{U}_{q, q'}$ of $q \times q'$ matrices in which the submatrix of the first q columns is anti-hermitian. The differential $d\Psi_{uT_q, a, Pv}$ of Ψ at (uT_q, a, Pv) is given by the following formulae:

$$\begin{aligned} \forall X \in \mathfrak{U}_q^0, \quad d\Psi_{uT_q, a, Pv}(u^*X, 0, 0) &= u(\underbrace{Xa^{1/2} - a^{1/2}X}_{:=M_a(X)})u^*Pv, \\ \forall A \in \mathbb{R}^q, \quad d\Psi_{uT_q, a, Pv}(0, A, 0) &= \frac{1}{2}u \frac{A}{a^{1/2}} u^*Pv, \\ \forall Y \in \mathfrak{U}_{q, q'}, \quad d\Psi_{uT_q, a, Pv}(0, 0, Yv) &= ua^{1/2}u^*Yv \end{aligned}$$

c) Let \det denote the determinant on the canonical basis of $\mathfrak{M}_{q, q'}$, when considered as a real space. Define $n = q^2 - q$ and $m = 2qq' - q^2$. Fix (uT_q, a, Pv) in the manifold, and $X_1, \dots, X_n \in \mathfrak{U}_q^0$, $A_1, \dots, A_q \in \mathbb{R}^q$, and $Y_1, \dots, Y_m \in \mathfrak{U}_{q, q'}$. Now, let us compute the differential form $\Psi^*\det$ at (uT_q, a, Pv) on the family

$$((u^*X_1, 0, 0), \dots, (u^*X_n, 0, 0), (0, A_1, 0), \dots, (0, A_q, 0), (0, 0, Y_1v), \dots, (0, 0, Y_mv)).$$

It is equal to

$$\begin{aligned} \det [uM_a(X_1)u^*Pv, \dots, uM_a(X_n)u^*Pv, \frac{1}{2}u \frac{A_1}{a^{1/2}} u^*Pv, \dots, \frac{1}{2}u \frac{A_q}{a^{1/2}} u^*Pv, \\ ua^{1/2}u^*Y_1v, \dots, ua^{1/2}u^*Y_mv]. \end{aligned}$$

Define $\tilde{u} = \begin{bmatrix} u & 0 \\ 0 & I_{q'-q} \end{bmatrix} \in \mathcal{U}_{q'}$. We have $P\tilde{u} = uP$. Note that the base we chose is orthonormal for the euclidian structure we chose on $\mathfrak{M}_{q,q'}$, and the left or right multiplications by unitary elements are orthogonal, and have determinant 1 by connexity of the unitary group. So what we want to compute is equal to

$$\det(M_a(X_1)P, \dots, M_a(X_n)P, \frac{1}{2} \frac{A_1}{a^{1/2}} P, \dots, \frac{1}{2} \frac{A_q}{a^{1/2}} P, a^{1/2} u^* Y_1 \tilde{u}, \dots, a^{1/2} u^* Y_m \tilde{u}).$$

In order to compute this, let us introduce another basis of $\mathfrak{M}_{q,q'}$. Let us denote the elementary matrices of $\mathfrak{M}_{q,q'}$ by

$$E_{k,l} \quad (1 \leq k \leq q, 1 \leq l \leq q').$$

Define, for $1 \leq k < l \leq q$,

$$e_{k,l} = E_{k,l} + E_{l,k}, \quad e'_{k,l} = i(E_{k,l} - E_{l,k}),$$

let \mathcal{B}_1 be the family $(e_{k,l}, e'_{k,l})_{1 \leq k < l \leq q}$, define

$$\mathcal{B}_2 = (E_{k,k})_{1 \leq k \leq q},$$

and let \mathcal{B}_3 be any basis of $\mathfrak{U}_{q,q'}$. Note that $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ is a basis of $\mathfrak{M}_{q,q'}$. Let λ be a non-null real number such that \det is λ times the determinant on \mathcal{B} . Note that the matrix of the family

$$\mathcal{F} := (M_a(X_1)P, \dots, M_a(X_n)P, \frac{1}{2} \frac{A_1}{a^{1/2}} P, \dots, \frac{1}{2} \frac{A_q}{a^{1/2}} P, a^{1/2} u^* Y_1 \tilde{u}, \dots, a^{1/2} u^* Y_m \tilde{u})$$

on \mathcal{B} is block upper-triangular (with respect to the decomposition $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$). So $\det(\mathcal{F})$ is λ times the product of the determinants of the matrices of the families

$$(M_a(X_1)P, \dots, M_a(X_n)P), \quad \left(\frac{1}{2} \frac{A_1}{a^{1/2}} P, \dots, \frac{1}{2} \frac{A_q}{a^{1/2}} P \right), \\ (\text{Pr}(a^{1/2} u^* Y_1 \tilde{u}), \dots, \text{Pr}(a^{1/2} u^* Y_m \tilde{u}))$$

on the respective bases $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ (where Pr denotes the projection on $\text{Span}(\mathcal{B}_3) = \mathfrak{U}_{q,q'}$ in the direction of $\text{Span}(\mathcal{B}_1 \cup \mathcal{B}_2)$, i.e. the orthogonal projection on $\mathfrak{U}_{q,q'}$).

Let us compute the first determinant. M_a maps linearly \mathfrak{U}_q^0 into the set of hermitian matrices with null diagonal, so $X \rightarrow M_a(X)P$ maps linearly \mathfrak{U}_q^0 into $\text{Span}(\mathcal{B}_1)$. Let $(F_{k,l})_{1 \leq k, l \leq q}$ be the elementary matrices of $\mathfrak{M}_{q,q}$. Define, for $1 \leq k < l \leq q$,

$$f_{k,l} = F_{k,l} - F_{l,k}, \quad f'_{k,l} = i(F_{k,l} + F_{l,k}),$$

let β_1 be the family $(f_{k,l}, f'_{k,l})_{1 \leq k < l \leq q}$. The matrix of the map $X \rightarrow M_a(X)P$ between the bases β_1 and \mathcal{B}_1 is block-diagonal, with blocks

$$\begin{bmatrix} 0 & a_k^{1/2} - a_l^{1/2} \\ a_l^{1/2} - a_k^{1/2} & 0 \end{bmatrix}, \quad (1 \leq k < l \leq q).$$

So its determinant is $\Delta(a^{1/2})^2$, and the determinant of the matrix of the family $(M_a(X_1)P, \dots, M_a(X_n)P)$ in \mathcal{B}_1 is $\Delta(a^{1/2})^2$ times the determinant of the matrix of the family (X_1, \dots, X_n) in β_1 .

The second determinant is $\frac{1}{2^n} \frac{1}{(a_1 \dots a_n)^{1/2}}$ times the determinant of the matrix of the family (A_1, \dots, A_n) in \mathcal{B}_2 .

Let us compute the third determinant. In this paragraph, we shall use a two blocks-decomposition of matrices of $\mathfrak{M}_{q,q'}$. Any $q \times q'$ matrix Y will be denoted by $Y = (Y_s, Y_r)$, where Y_s is a $q \times q$ matrix, and Y_r is a $q \times (q' - q)$ matrix (s is for *square*, and r for *rectangular*). With this decomposition, Pr has a simple expression:

$\text{Pr}(Y) = \left(\frac{Y_s - Y_s^*}{2}, Y_r \right)$. Note that if $Y \in \mathfrak{U}_{q,q'}$, then Y_s is anti-hermitian, so

$$\text{Pr}(a^{1/2} u^* Y \tilde{u}) = N_a((u^* Y_s u, u^* Y_r)),$$

where

$$N_a : Y \in \mathfrak{U}_{q,q'} \rightarrow \left(\frac{a^{1/2} Y_s + Y_s a^{1/2}}{2}, a^{1/2} Y_r \right) \in \mathfrak{U}_{q,q'}.$$

Note that $Y \in \mathfrak{U}_{q,q'} \rightarrow (u^* Y_s u, u^* Y_r)$ is orthogonal, and has determinant one by connexity of \mathcal{U}_q . Let us compute the determinant of N_a . All vectors of the basis

$$(E_{k,l} - E_{l,k})_{1 \leq k < l \leq q} \cup (i(E_{k,l} + E_{l,k}))_{1 \leq k < l \leq q} \cup (E_{k,l})_{\substack{1 \leq k \leq q \\ q' - q \leq l \leq q'}} \cup (iE_{k,l})_{\substack{1 \leq k \leq q \\ q' - q \leq l \leq q'}}$$

are eigenvectors of N_a , with respective eigenvalues

$$\left(\frac{a_k^{1/2} + a_l^{1/2}}{2} \right)_{1 \leq k < l \leq q} \cup \left(\frac{a_k^{1/2} - a_l^{1/2}}{2} \right)_{1 \leq k < l \leq q} \cup (a_k^{1/2})_{\substack{1 \leq k \leq q \\ q' - q \leq l \leq q'}} \cup (a_k^{1/2})_{\substack{1 \leq k \leq q \\ q' - q \leq l \leq q'}}.$$

Thus the determinant of $N_a : \mathfrak{U}_{q,q'} \rightarrow \mathfrak{U}_{q,q'}$ is

$$(a_1 \dots a_q)^{q' - q + \frac{1}{2}} \prod_{1 \leq k < l \leq q} \left(\frac{a_k^{1/2} + a_l^{1/2}}{2} \right)^2.$$

The third determinant is this quantity times the determinant, in \mathcal{B}_3 , of the family (Y_1, \dots, Y_m) .

d) So the value of the differential form $\Psi^* \det$ at (uT_q, a, Pv) on the family

$$((u^* X_1, 0, 0), \dots, (u^* X_n, 0, 0), (0, A_1, 0), \dots, (0, A_q, 0), (0, 0, Y_1 v), \dots, (0, 0, Y_m v))$$

is

$$\lambda \Delta(a^{1/2})^2 \det_{\beta_1}(X_1, \dots, X_n) \frac{1}{2^n} \frac{1}{(a_1 \dots a_n)^{1/2}} \times \\ \det_{\text{can}}(A_1, \dots, A_n) (a_1 \dots a_q)^{q' - q + \frac{1}{2}} \prod_{1 \leq k < l \leq q} \left(\frac{a_k^{1/2} + a_l^{1/2}}{2} \right)^2 \det_{\mathcal{B}_3}(Y_1, \dots, Y_m).$$

It is well known (see, e.g., section I.5 of [5]), that it is equal, up to a multiplicative constant, to

$$\Delta(a)^2 \prod_{k=1}^q a_k^{q'-q} \omega_{uT}^{\mathcal{U}_q/T_q}(u^* X_1, \dots, u^* X_n) \det_{\text{can}}(A_1, \dots, A_n) \omega_{vP}^{\mathcal{U}_{q,q'}}(Y_1 v, \dots, Y_m v),$$

where $\omega^{\mathcal{U}_q/T_q}$ is a non-null differential n -form on \mathcal{U}_q/T_q which is invariant under the left action of the unitary group, and $\omega^{\mathcal{U}_{q,q'}}$ is a non-null differential m -form on $\mathcal{U}_{q,q'}$ which is invariant under the left and right actions of the unitary groups. So $\Psi^* \det$ is equal, up to a multiplicative constant, to

$$f \cdot \omega^{\mathcal{U}_q/T_q} \wedge \det_{\text{can}} \wedge \omega^{\mathcal{U}_{q,q'}},$$

where f is the smooth function defined on $\mathcal{U}_q/T_q \times \mathbb{R}_{+,<}^q \times \mathcal{U}_{q,q'}$ by

$$f(uT, a, Pv) = \Delta(a)^2 \prod_{k=1}^q a_k^{q'-q}.$$

Hence the push-forward, by Ψ^{-1} , of the Lebesgue measure on $\mathfrak{M}_{q,q'}$ is the tensor product $\gamma_q \otimes C\sigma_{q,q'} \otimes \gamma_{q,q'}$, where C is a positive constant.

e) Let us compute C . As noticed in the remark following the lemma, by definition of the measures γ_q and $\gamma_{q,q'}$, we can now claim that the map

$$\begin{aligned} \tilde{\Psi} : \mathcal{U}_q \times (\mathbb{R}^+)^q \times \mathcal{U}_{q'} &\rightarrow \mathfrak{M}_{q,q'} \\ (u, x, v) &\mapsto u \operatorname{diag}(x_1, \dots, x_q)^{1/2} u^* P v \end{aligned}$$

is surjective and preserves the measure $\operatorname{Haar}(\mathcal{U}_q) \otimes \frac{C}{q!} \tilde{\sigma}_{q,q'} \otimes \operatorname{Haar}(\mathcal{U}_{q'})$ (i.e. the push-forward of this measure by $\tilde{\Psi}$ is the Lebesgue measure), where $\tilde{\sigma}_{q,q'}$ is the measure on $(\mathbb{R}^+)^n$ with density given by formula (5.2).

The function $x \in \mathfrak{M}_{q,q'} \rightarrow e^{-\operatorname{Tr} x x^*}$ has integral with respect to the Lebesgue measure equal to $\pi^{qq'}$. Thus

$$\begin{aligned} \pi^{qq'} &= \\ \frac{C \pi^{qq'}}{\prod_{j=1}^q j! \prod_{j=q'-q}^{q'-1} j!} \int_{a \in (\mathbb{R}^+)^q} \int_{u \in \mathcal{U}_q} \int_{v \in \mathcal{U}_{q'}} \Delta(a)^2 \prod_{j=1}^q a_j^{q'-q} e^{-\operatorname{Tr} u a^{1/2} P v v^* P^* a^{1/2} u^*} \operatorname{d}a \operatorname{d}u \operatorname{d}v. \end{aligned}$$

Thus

$$\frac{1}{C} = \frac{1}{\prod_{j=1}^q j! \prod_{j=q'-q}^{q'-1} j!} \int_{a \in (\mathbb{R}^+)^q} \Delta(a)^2 \prod_{j=1}^q a_j^{q'-q} e^{-\sum_{i=1}^q a_i} \operatorname{d}a$$

We can now apply formula (4.1.8) p. 119 of [10], with $n = q$, $\beta = 1$, $a = q' - q + 1$, and it appears that $C = 1$.

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