Matrix Models for Random Partitions

Alexander Alexandrov

CEA, IPhT and Ecole Normale Superieure, LPT, France
ITEP, Moscow, Russia

Conference on Random Matrices
June 4, 2010

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Random Partitions vs Matrix Models

Matrix Models

$$\int [d\Phi] \ldots$$

$\updownarrow$

Random Partitions

$$\sum_\lambda \ldots$$

Motivation

Different sums over partitions play an important role in the modern mathematical physics:

- 2d Yang-Mills
- Instantonic calculus of supersymmetric gauge theories
- Hurwitz numbers
- Gromow-Witten invariants
- Chern-Simons
- 2d conformal field theories (Alday-Gaiotto-Tachikawa conjecture)

Characters

We consider $GL(\infty)$ characters, which are parameterized by an infinite set of independent times t and on the partition λ: $\chi_\lambda(t)$.
Miwa variables $t_k = \frac{1}{k}\mathrm{Tr}\,\mathbf{X}^k$, where $\mathbf{X}$ – $N \times N$ matrix.
Representations of $GL(N)$ are parameterized by partitions (Young diagrams) λ with the size $|\lambda| = \sum \lambda_i$ and the length $l(\lambda) \leq N$:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{l(\lambda)} > 0 = \lambda_{l(\lambda)+1} = \ldots$$

Weyl's formulas (Jacobi-Trudi identity)

$$\chi_\lambda(\mathbf{X}) = \frac{\det_{i,j} x_i^{\lambda_j+N-j}}{\Delta(x)}$$

$$\chi_\lambda(t) = \det_{i,j} p_{\lambda_i-i+j}(t)$$

Cauchy-Littlewood identity

$$e^{\sum_{k=1}^{\infty} k t_k \bar{t}_k} = \sum_\lambda \chi_\lambda(t)\chi_\lambda(\bar{t})$$

Dimensions

A very important role in the story is played by dimensions of the representations of the general linear

$$\dim_\lambda = \chi_\lambda(\mathbf{1}) = \dim_\lambda(GL(N)) = \prod_{0<i<j\le N} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

and symmetric groups

$$d_\lambda = \chi_\lambda(t_k = \delta_{k,1}) = \frac{\dim_\lambda(S_N)}{N!} = \prod_{0<i<j\le\infty} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

The difference between $\dim_\lambda$ and d_λ is that the first one explicitly depends on N, while the second does not. The ratio of two functions is

$$\frac{\dim_\lambda}{d_\lambda} = \prod_{i=1}^{\infty} \frac{(\lambda_i + N - i)!}{(N - i)!}$$

Character expansion: known examples without Casimirs

Character decomposition for usual unitary, complex and Hermitian matrix integral is well known.
Harish-Chandra-Itzykson-Zuber matrix integral

$$\int_{N\times N} [d\mathbf{U}]\, e^{\mathrm{Tr}\,(\mathbf{U}\mathbf{A}\mathbf{U}^\dagger \mathbf{B})} = \sum_{\lambda; l(\lambda)\leq N} \frac{d_\lambda \chi_\lambda(\mathbf{A})\chi_\lambda(\mathbf{B})}{\dim \lambda}$$

Unitary matrix model

$$\int_{N\times N} [d\mathbf{U}] \exp\left(\sum_{k=0}^{\infty} t_k \mathrm{Tr}\, \mathbf{U}^k + \bar{t}_k \mathrm{Tr}\, {\mathbf{U}^\dagger}^k\right) = \sum_{\lambda; l(\lambda)\leq N} \chi_\lambda(t)\chi_\lambda(\bar{t})$$

It is trivial to insert $e^{q|\lambda|}$ into the sums.

Character expansion: less known examples with Casimirs

What about higher Casimirs?

$$P_N(t,\bar{t};s) = \sum_{l(\lambda)\leq N} \chi_\lambda(t)\chi_\lambda(\bar{t}) \exp \sum_{i=1}^{\infty} s_i C_i$$

Not actual group Casimirs, but analogs of single trace operators.

$$C_k = \sum_{i=1}^{\infty} \left(\lambda_i - i + \frac{1}{2}\right)^k - \left(-i + \frac{1}{2}\right)^k$$

$$C_1 = |\lambda| \quad C_2 = \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i + 1)$$

Eguchi - Yang	(94-95)
Eynard et al.	(08-09)
Morozov - Shakirov	(09)

Operators

Characters – eigenfunctions, Casimirs – eigenvalues

$$\hat{C}_k \chi_\lambda = C_k \chi_\lambda$$

There are at least three different representations for operators $\hat{C}_k$

$$\hat{C}_1 = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} = \mathrm{Tr}\, \mathbf{X} \frac{\partial}{\partial \mathbf{X}^T} = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k}$$

$$\hat{C}_2 = \sum_{i=1}^{N} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{x_i x_j}{x_i - x_j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) =$$

$$= \mathrm{Tr} \left(\mathbf{X} \frac{\partial}{\partial \mathbf{X}^T} \right)^2 - N \mathrm{Tr}\, \mathbf{X} \frac{\partial}{\partial \mathbf{X}^T} =$$

$$= \sum_{k,m=1}^{\infty} kmt_k t_m \frac{\partial}{\partial t_{k+m}} + (k+m) t_{k+m} \frac{\partial^2}{\partial t_k \partial t_m}$$

Operators, acting on eigenvalues

General expression for the Casimir operators in terms of eigenvalue derivatives:

$$\hat{C}_k = \frac{1}{\widetilde{\Delta}(x)} \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^k \widetilde{\Delta}(x) - C_k^0$$

$$\widetilde{\Delta}(x) = \frac{\Delta(x)}{\det X^{N-\frac{1}{2}}}$$

$$C_k^0 = \sum_{i=1}^{N} \left(-i + \frac{1}{2} \right)^k = (-1)^k \frac{N^{k+1}}{k+1} + \ldots$$

Proof

Let us explicitly check that characters are eigenfunctions of operators with eigenvalues C_k. One has

$$\hat{C}_k \chi_\lambda(\mathbf{X}) = \widetilde{\Delta}^{-1}(x) \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^k \frac{\det_{i,j} x_i^{\lambda_j+N-j}}{\det \mathbf{X}^{N-\frac{1}{2}}} - C_k^0 \chi_\lambda(\mathbf{X}) =$$

$$= \widetilde{\Delta}^{-1}(x) \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^k \sum_{\sigma} (-1)^{|\sigma|} \prod_i x_{\sigma(i)}^{\lambda_i - i + \frac{1}{2}} - C_k^0 \chi_\lambda(\mathbf{X}) = C_k \chi_\lambda(\mathbf{X})$$

which proves the statement.
The crucial property of the expression for Casimir operators is that it can be easily exponentiated. Let us denote $x_i = e^{\varphi_i}$, then

$$\boxed{\exp \sum_{k=1}^{\infty} s_k \hat{C}_k = e^{-\sum_{k=1}^{\infty} s_k C_k^0} \frac{1}{\widetilde{\Delta}(e^{\varphi})} \exp \left(\sum_{k=1}^{\infty} s_k \sum_{i=1}^{N} \frac{\partial^k}{\partial \varphi_i^k} \right) \widetilde{\Delta}(e^{\varphi})}$$

Second Casimir

Matrix integral representation for the propagator

$$P_N(t,\bar{t}) = \sum_{l(\lambda)\le N} \chi_\lambda(t)\chi_\lambda(\bar{t}) e^{qC_1 + \frac{gC_2}{2}}$$

Operator is really simple

$$\exp\left(q\hat{C}_1 + \frac{g}{2}\hat{C}_2\right) = D_0 \exp\left(\sum_{i=1}^{N} q\frac{\partial}{\partial\varphi_i} + \frac{g}{2}\frac{\partial^2}{\partial\varphi_i^2}\right)\widetilde{\Delta}(e^{\varphi})$$

$$D_0 = \frac{\exp\left(\frac{qN^2}{2} - \frac{gN^3}{6} + \frac{gN}{24}\right)}{\widetilde{\Delta}(e^{\varphi})}$$

As usual, let us convert an operator in to the shift operator

$$\exp\left(\frac{g}{2}\frac{\partial^2}{\partial\varphi_i^2}\right) = \frac{1}{\sqrt{2\pi g}}\int_{-\infty}^{\infty} dy_i \exp\left(-\frac{1}{2g}y_i^2 + y_i\frac{\partial}{\partial\varphi_i}\right)$$

Matrix integral valued operator

Matrix integral valued operator

$$e^{q\hat{C}_1+\frac{g}{2}\hat{C}_2} = \frac{D_0}{(2\pi g)^{\frac{N}{2}}} \int_{-\infty}^{\infty} d^N y \widetilde{\Delta}\left(e^{\varphi+y+q}\right) e^{\sum_{i=1}^{N}\left((y_i+q)\frac{\partial}{\partial\varphi_i}-\frac{1}{2g}y_i^2\right)}$$

It is easy to act by this operator on the “bare partition function":

$$\exp\left(\sum_{k=1}^{\infty} t_k \mathrm{Tr}\, \mathbf{X}^k\right) = \sum_{l(\lambda)\leq N} \chi_\lambda(t)\chi_\lambda(\mathbf{X})$$

An expression for the partition function as multiple integral

$$P_N(t,\mathbf{X}) = \exp\left(q\hat{C}_1 + \frac{g}{2}\hat{C}_2\right) \exp\left(\sum_{k=1}^{\infty} t_k \mathrm{Tr}\, \mathbf{X}^k\right) =$$

$$= \frac{D_0}{(2\pi g)^{\frac{N}{2}}} \int_{-\infty}^{\infty} d^N y \widetilde{\Delta}\left(e^{\varphi+y+q}\right) \exp\sum_{i=1}^{N}\left(\sum_{k=1}^{\infty} t_k e^{k(y_i+\varphi_i+q)} - \frac{1}{2g}y_i^2\right)$$

Matrix integral

Integral over Hermitian matrixes

$$P_N(t, e^{\Phi}) \sim \int_{\mathfrak{H}} [d\mu(\mathbf{Y})] \exp\left(\frac{\mathrm{Tr}\,(\Phi + a)\mathbf{Y}}{g} - \frac{\mathrm{Tr}\,\mathbf{Y}^2}{2g} + \sum_{k=1}^{\infty} t_k \mathrm{Tr}\, e^{k\mathbf{Y}} \right)$$

Φ is a diagonal matrix, $a = q - \frac{gN}{2}$. Non-flat measure (B_k-Bernoulli numbers)

$$[d\mu(\mathbf{Y})] = \Delta(y)\Delta(e^y)\,[d\mathbf{U}] \prod_{i=1}^{N} e^{-\frac{N-1}{2} y_i} dy_i =$$

$$= \exp\left(\sum_{i;j=0, i+j>0} \frac{(-1)^j}{2(i+j)} \frac{B_{i+j}}{i!j!} \mathrm{Tr}\,\mathbf{Y}^i \mathrm{Tr}\,\mathbf{Y}^j \right) [d\mathbf{Y}] =$$

$$= \sqrt{\det \frac{\sinh\left(\frac{\mathbf{Y}\otimes\mathbf{1} - \mathbf{1}\otimes\mathbf{Y}}{2}\right)}{\left(\frac{\mathbf{Y}\otimes\mathbf{1} - \mathbf{1}\otimes\mathbf{Y}}{2}\right)}}\, [d\mathbf{Y}]$$

Matrix integral for two sets of times

To get partition function $P_N(t, \bar{t})$ from $P_N(\bar{t}, \mathbf{X})$ one can glue it with "bare propagator" $\exp \sum t_k \mathrm{Tr}\, \mathbf{X}^k$ with a help of unitary matrix integral:

$$P_N(t, \bar{t}) = \int_{\mathfrak{U}} [d\mathbf{V}]\, P_N(\bar{t}, \mathbf{V}^\dagger) \exp \sum_{k=1}^{\infty} t_k \mathrm{Tr}\, \mathbf{V}^k$$

An interplay between eigenvalues of unitary and Hermitian matrix model leads to the normal ($\left[\mathbf{Z}, \mathbf{Z}^\dagger\right] = 0$) matrix integral

$$P_N(t, \bar{t}) = \mathcal{P}^{-1} \int_{\mathfrak{N}} [d\mathbf{Z}] \exp\left(-\frac{1}{2g} \mathrm{Tr}\, \log^2 \mathbf{Z}\mathbf{Z}^\dagger\right) \times$$
$$\times \exp\left(\left(\frac{q}{g} - N - \frac{1}{2}\right) \mathrm{Tr}\, \log \mathbf{Z}\mathbf{Z}^\dagger + \sum_{k=1}^{\infty} \left(t_k \mathrm{Tr}\, \mathbf{Z}^k + \bar{t}_k \mathrm{Tr}\, \mathbf{Z}^{\dagger k}\right)\right)$$

The only special term is $\mathrm{Tr}\, \log^2 \mathbf{Z}\mathbf{Z}^\dagger$.

How does it work: $N = 1$

Let us make a simplest check of our result, namely consider $N = 1$. In this case the partition function is a simple sum of Schur polynomials:

$$P_1(t, \bar{t}) = \sum_{k=0}^{\infty} p_k(t) p_k(\bar{t}) e^{qk + \frac{g}{2}k(k-1)}$$

Matrix integral is simplified to an ordinary one

$$P_1(t, \bar{t}) = \frac{e^{-\frac{q^2}{2g} + \frac{q}{2} - \frac{g}{8}}}{\sqrt{g}} \int d^2 z \exp\left(-\frac{1}{2g}\log^2|z|^2 - \left(\frac{3}{2} - \frac{q}{g}\right)\log|z|^2\right) \times$$

$$\times \exp\left(\sum_{k=1}^{\infty}\left(t_k z^k + \bar{t}_k \bar{z}^k\right)\right) =$$

$$= \frac{e^{-\frac{q^2}{2g} + \frac{q}{2} - \frac{g}{8}}}{\sqrt{2\pi g}} \sum_{k=0}^{\infty} p_k(t) p_k(\bar{t}) \int_{-\infty}^{\infty} dR e^{-\frac{1}{2g}R^2 + \left(k + \frac{q}{g} - \frac{1}{2}\right)R}$$

Integrability

Derived partition function, as usual for matrix integrals, can be represented in the determinant form

$$P_N(t,\bar{t}) = \mathcal{P}^{-1} N! \det_{i,j=1}^{N} h_{i,j}$$

$$h_{i,j} = \int_C d^2 z z^{2-i} \bar{z}^{2-j} \exp\left(-\frac{1}{2g}\log^2|z|^2 - \left(\frac{1}{2}-\frac{q}{g}\right)\log|z|^2\right) \times \\ \times \exp\left(\sum_{k=1}^{\infty}\left(t_k z^k + \bar{t}_k \bar{z}^{\dagger k}\right)\right)$$

An obvious property

$$\frac{\partial h_{i,j}}{\partial t_k} = h_{i-k,j}, \qquad \frac{\partial h_{i,j}}{\partial \bar{t}_k} = h_{i,j-k}$$

guarantees Toda lattice integrability, where N plays a role of the discrete time.

Some specifications

Generation function of single Hurwitz numbers ($N \to \infty$).

$$P_N(t, \delta_{k,1}) = \sum_{l(\lambda)\le N} d_\lambda \chi_\lambda(t) \exp\left(qC_1 + \frac{gC_2}{2}\right) =$$

$$= \mathcal{P}^{-1} \int_{\mathfrak{N}} [d\mathbf{Z}]\, e^{-\frac{1}{2g}\mathrm{Tr}\,\log^2 \mathbf{ZZ}^\dagger - (N+\frac{1}{2}-\frac{q}{g})\mathrm{Tr}\,\log \mathbf{ZZ}^\dagger + \mathrm{Tr}\,\mathbf{Z}^\dagger + \sum_{k=1}^\infty \left(t_k \mathrm{Tr}\,\mathbf{Z}^k\right)}$$

Plancherel measure. Gromov-Witten invariants for $\mathbf{CP}^1$ with only two first times switched on. Formal series!

$$P_N(\delta_{k,1}, \delta_{k,1}) = \sum_{l(\lambda)\le N} d_\lambda^2 \exp\left(qC_1 + \frac{gC_2}{2}\right) =$$

$$= \mathcal{P}^{-1} \int_{\mathfrak{N}} [d\mathbf{Z}]\, e^{-\frac{1}{2g}\mathrm{Tr}\,\log^2 \mathbf{ZZ}^\dagger - \left(N+\frac{1}{2}-\frac{q}{g}\right)\mathrm{Tr}\,\log \mathbf{ZZ}^\dagger + \mathrm{Tr}\,\mathbf{Z} + \mathrm{Tr}\,\mathbf{Z}^\dagger}$$

All Casimirs: Miwa variables

Let us switch on all Casimirs by introduction of Miwa variables for correspondent times

$$s_k = \frac{1}{k}\mathrm{Tr}\,\mathbf{Y}^{-k}$$

where the matrix $\mathbf{Y}$ is of the size $M \times M$. For Miwa variables the propagator looks as follows:

$$P_N(t,\bar{t};\mathbf{Y}) = \sum_{l(\lambda)\leq N} \chi_\lambda(t)\chi_\lambda(\bar{t}) \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}\mathrm{Tr}\,\mathbf{Y}^{-k} C_k\right) =$$

$$\sum_{l(\lambda)\leq N} \chi_\lambda(t)\chi_\lambda(\bar{t}) \prod_{i=1}^{N}\prod_{j=1}^{M} \frac{y_j + i - \frac{1}{2}}{y_j - \lambda_i + i - \frac{1}{2}}$$

Non-eigenvalue matrix integral

$$P_N(t, e^{\Phi}; \mathbf{Y}) \sim \int_{\mathfrak{C}} [d\mathbf{Z}] \exp\left(-\mathrm{Tr}\,\mathbf{Z}\mathbf{Z}^{\dagger}\mathbf{Y} + H(\mathbf{Z}^{\dagger}\mathbf{Z} + \Phi)\right)$$

with the potential

$$H(\mathbf{A}) = -\frac{N}{2}\mathrm{Tr}\,\mathbf{A} + \sum_{k=1}^{\infty} t_k \mathrm{Tr}\, e^{k\mathbf{A}} + \sum_{i;j=0, i+j>0} \frac{(-1)^j}{2(i+j)} \frac{B_{i+j}}{i!j!} \mathrm{Tr}\,\mathbf{A}^i \mathrm{Tr}\,\mathbf{A}^j$$

where B_k are Bernoulli numbers.

Two sets of times - Normal matrix model

Further, for two sets of times we get again a *normal* matrix integral, where the eigenvalues of normal matrix are constrained by $|z| < 1$:

$$P_N(t,\bar{t};\mathbf{Y}) = \mathcal{P}_{\mathbf{Y}}^{-1} \oint_{\mathcal{C}} db_j \frac{1}{\prod_k (y_k - b_j)} \int_{\mathfrak{N},|z_i|<1} [d\mathbf{Z}] \times$$
$$\times \exp\left(\sum_{k=1}^{\infty}\left(t_k \mathrm{Tr}\,\mathbf{Z}^k + \bar{t}_k \mathrm{Tr}\,\mathbf{Z}^{\dagger k}\right) - \mathrm{Tr}\left(B + N + \frac{1}{2}\right)\log \mathbf{Z}^{\dagger}\mathbf{Z}\right)$$

Integrability

Again, we present propagator as a determinant

$$P_N(t,\bar{t};\mathbf{Y}) \sim \det_{i,j=1}^{N} h_{i,j}$$

where

$$h_{i,j} = \oint_{\mathcal{C}} db \frac{1}{\prod_k (y_k - b)} \int_{|z_i|<1} d^2 z z^{2-i} \bar{z}^{2-j} \times$$
$$\times \exp\left(\sum_{k=1}^{\infty} \left(t_k z^k + \bar{t}_k {z^\dagger}^k \right) - \left(b + \frac{1}{2} \right) \log |z|^2 \right)$$

Equations

$$\frac{\partial h_{i,j}}{\partial t_k} = h_{i-k,j}, \qquad \frac{\partial h_{i,j}}{\partial \bar{t}_k} = h_{i,j-k}$$

guarantee Toda lattice integrability with respect to times t and $\bar{t}$.

Conclusion and open questions

- Conclusion
  - Matrix integrals
  - Integrability
  - Topological expansion: $\frac{1}{N}$ and $\hbar$ corrections
- Further directions
  - Virasoro constraints (powerful Eynard technique)
  - M-theory of matrix models
  - β – deformations
  - q – deformations
  - Multiple partitions