Free convolution with a semi-circular distribution and convergence of eigenvalues of spiked deformations of large Wigner matrices

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joint work with C. Donati-Martin, D. Féral and M. Février

Definition

W_N is a $N \times N$ Wigner Hermitian matrix associated with a distribution μ of variance σ^2 and mean zero :

$(W_N)_{ii}$, $\sqrt{2}\Re e((W_N)_{ij})_{i<j}$, $\sqrt{2}\Im m((W_N)_{ij})_{i<j}$ are i.i.d, with distribution μ.

If $\mu = \mathcal{N}(0, \sigma^2)$, $W_N =: W_N^G$ is a $G.U.E$-matrix.

Notation : $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_N(X)$ eigenvalues of X.

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Theorem

Convergence of the spectral measure : *Wigner (50')*

$$\mu_{\frac{W_N}{\sqrt{N}}} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(\frac{W_N}{\sqrt{N}})} \to \mu_\sigma \quad a.s \text{ when } \quad N \to +\infty$$

$$\frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma, 2\sigma]}(x)$$

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Theorem

Convergence of the extremal eigenvalues *(Bai-Yin 1988) :*
If $\int x^4 d\mu(x) < +\infty$, *then*

$$\lambda_1(\frac{W_N}{\sqrt{N}}) \to 2\sigma \text{ and } \lambda_N(\frac{W_N}{\sqrt{N}}) \to -2\sigma \text{ a.s when } N \to +\infty.$$

$$\textbf{Finite rank deformation}: \quad M_N = \frac{1}{\sqrt{N}} W_N + A_N$$

• W_N is a $N \times N$ Wigner Hermitian matrix associated with a distribution μ of variance σ^2 and mean zero.

• A_N : a deterministic Hermitian matrix of fixed finite rank r with J distinct non-null eigenvalues (spikes) $\theta_1 > \cdots > \theta_J$ independent of N, θ_j of fixed multiplicity k_j.

$$F_X(x) := \mu_X(]-\infty; x])$$

$$\sup_x |F_{\frac{1}{\sqrt{N}}W_N + A_N}(x) - F_{\frac{1}{\sqrt{N}}W_N}(x)| \leq \frac{\mathrm{rank} A_N}{N}$$

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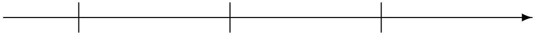
$\Longrightarrow$ Convergence of the spectral measure $\mu_{M_N} := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(M_N)}$ towards the semi-circular distribution μ_σ.

Theorem (Finite rank deformation of a G.U.E matrix : Péché 2006)

$\mu = \mathcal{N}(0, \sigma^2)$

- *If* $\theta_1 < \sigma$, $\sigma^{-1} N^{2/3}(\lambda_1(M_N^G) - 2\sigma) \xrightarrow{\mathcal{D}} F_2$ *(T.W)*
- *If* $\theta_1 = \sigma$, $\sigma^{-1} N^{2/3}(\lambda_1(M_N^G) - 2\sigma) \xrightarrow{\mathcal{D}} F_{3,k_1}$.
- *If* $\theta_1 > \sigma$, $N^{1/2}(\lambda_1(M_N^G) - \rho_{\theta_1}) \xrightarrow{\mathcal{D}}$ *the distribution of the largest eigenvalue of a* $k_1 \times k_1$ *GUE matrix, with* $\rho_{\theta_1} := \theta_1 + \frac{\sigma^2}{\theta_1}$.



Theorem

The non-Gaussian case for a PARTICULAR A_N :

Féral-Péché 2007

μ symmetric with subgaussian moments.

$$A_N := \begin{pmatrix} \frac{\theta}{N} & \cdots & \frac{\theta}{N} \\ \vdots & \cdots & \vdots \\ \frac{\theta}{N} & \cdots & \frac{\theta}{N} \end{pmatrix}.$$

The convergence and fluctuations of $\lambda_1(M_N)$ are the same as $\lambda_1(M_N^G)$ (the Gaussian case) :

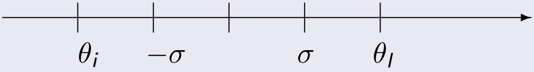
- *If $\theta < \sigma$, $\sigma^{-1}N^{2/3}(\lambda_1(M_N) - 2\sigma) \xrightarrow{\mathcal{D}} F_2$ (T.W)*
- *If $\theta = \sigma$, $\sigma^{-1}N^{2/3}(\lambda_1(M_N) - 2\sigma) \xrightarrow{\mathcal{D}} F_{3,1}$.*
- *If $\theta > \sigma$, $N^{1/2}(\lambda_1(M_N) - \rho_\theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\theta^2)$;*

$$\rho_\theta := \theta + \frac{\sigma^2}{\theta}; \quad \sigma_\theta := \sigma\sqrt{1 - (\sigma/\theta)^2}.$$

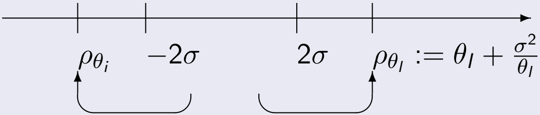
(see also Furedi-Komlós 1981)

Theorem (Capitaine, Donati-Martin, Féral (AOP 2009) Universality of the convergence)

μ symmetric, satisfying a Poincaré inequality.
A_N : any deterministic Hermitian matrix of fixed finite rank r with J distinct non-null eigenvalues $\theta_1 > \cdots > \theta_J$ independent of N, θ_j of fixed multiplicity k_j. such that



Then, almost surely,



k_i eigenvalues of M_N *k_l eigenvalues of M_N*

Theorem (Capitaine,Donati-Martin,Féral (AOP 2009) Universality of the convergence)

A_N : any deterministic Hermitian matrix of fixed finite rank r with J distinct non-null eigenvalues $\theta_1 > \cdots > \theta_J$ independent of N, θ_j of fixed multiplicity k_j.
μ symmetric, satisfying a Poincaré inequality.
Let $J_{+\sigma}$ (resp. $J_{-\sigma}$) be the number of j's such that $\theta_j > \sigma$ (resp. $\theta_j < -\sigma$). $\rho_{\theta_j} := \theta_j + \frac{\sigma^2}{\theta_j}$.

- $\forall 1 \leq j \leq J_{+\sigma}, \forall 1 \leq i \leq k_j, \lambda_{k_1+\cdots+k_{j-1}+i}(M_N) \to \rho_{\theta_j}$ a.s.,
- $\lambda_{k_1+\cdots+k_{J_{+\sigma}}+1}(M_N) \to 2\sigma$ a.s.,
- $\lambda_{k_1+\cdots+k_{J-J_{-\sigma}}}(M_N) \longrightarrow -2\sigma$ a.s.,
- $\forall j \geq J - J_{-\sigma} + 1, \forall 1 \leq i \leq k_j, \lambda_{k_1+\cdots+k_{j-1}+i}(M_N) \to \rho_{\theta_j}$ a.s.

$\Longrightarrow$ *The limiting values do not depend on μ.*

μ satisfies a **Poincaré inequality** : there exists a positive constant C such that for any $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{C}$ such that f and f' are in $L^2(\mu)$,

$$\mathbb{E}_\mu(|f - \mathbb{E}_\mu(f)|^2) \leq C\mathbb{E}_\mu(|f'|^2).$$

Poincaré inequality is just a technical condition : we conjecture that our results still hold under weaker assumptions.

I will try, dealing with a finite rank deformation, to explain how free probability may throw light on these results and thus allows to extend them to non-finite rank deformations and general Wigner matrices.

For a probability measure τ on $\mathbb{R}$, $z \in \mathbb{C}\backslash\mathbb{R}$, $g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$.
ν : a probability measure on $\mathbb{R}$. μ_σ :the centered semi-circular distribution with variance σ^2,
There exists an analytic map $F_{\sigma,\nu} : \mathbb{C}^+ \to \mathbb{C}^+$ (subordination function) such that

$$\forall z \in \mathbb{C}^+, \qquad g_{\nu \boxplus \mu_\sigma}(z) = g_\nu(F_{\sigma,\nu}(z)).$$

Theorem (P.Biane 1997)

$$F_{\sigma,\nu} : \begin{array}{l} \mathbb{C}^+ \to \{u + iv \in \mathbb{C}^+, v > v_{\sigma,\nu}(u)\} := \Omega_{\nu,\sigma} \\ z \mapsto z - \sigma^2 g_{\nu \boxplus \mu_\sigma}(z) \end{array}$$

$$v_{\sigma,\nu}(u) = \inf \left\{ v \geq 0, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} \leq \frac{1}{\sigma^2} \right\}.$$

$$H_{\sigma,\nu} : z \mapsto z + \sigma^2 g_\nu(z)$$

is a homeomorphism from $\overline{\Omega_{\nu,\sigma}}$ *to* $\mathbb{C}^+ \cup \mathbb{R}$ *with inverse* $F_{\sigma,\nu}$.

Theorem (P. Biane (1997))

The measure $\nu \boxplus \mu_\sigma$ *is absolutely continuous with respect to the Lebesgue measure with density* $p_{\sigma,\nu}$. $\Psi_{\sigma,\nu} : \mathbb{R} \to \mathbb{R}$ *being the homeomorphism defined by :*

$$\Psi_{\sigma,\nu}(u) = H_{\sigma,\nu}(u + iv_{\sigma,\nu}(u)) = u + \sigma^2 \int_{\mathbb{R}} \frac{(u-x)d\nu(x)}{(u-x)^2 + v_{\sigma,\nu}(u)^2},$$

$$p_{\sigma,\nu}(\Psi_{\sigma,\nu}(u)) = \frac{v_{\sigma,\nu}(u)}{\pi\sigma^2}.$$

$$U_{\sigma,\nu} := \{u, v_{\sigma,\nu}(u) > 0\} = \left\{u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2}\right\},$$

$$\mathrm{support}(\nu \boxplus \mu_\sigma) = \overline{\Psi_{\sigma,\nu}(U_{\sigma,\nu})}.$$

$\Psi_{\sigma,\nu}$ *is strictly increasing on* $U_{\sigma,\nu}$. *Note that* $\Psi_{\sigma,\nu} = H_{\sigma,\nu}$ *on* ${}^c U_{\sigma,\nu}$.

When $\nu = \delta_0$,

$$U_{\sigma,\nu} = \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\} = \{ u \in \mathbb{R}, |u| < \sigma \}$$

$$g_{\nu \boxplus \mu_\sigma}(z) = g_\nu(F_{\sigma,\nu}(z)) \Rightarrow F_{\sigma,\delta_0} = \frac{1}{g_{\mu_\sigma}}$$

$$H_{\sigma,\delta_0}(z) = z + \sigma^2 g_\nu(z) = z + \frac{\sigma^2}{z}$$

$$[-2\sigma; 2\sigma] = [H_{\sigma,\delta_0}(-\sigma); H_{\sigma,\delta_0}(\sigma)].$$

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Remark

- *The characterisation of the spikes of the finite rank matrix* A_N *that generate jumps of eigenvalues of* $\frac{1}{\sqrt{N}} W_N + A_N$: $\theta_i \in^c \overline{U_{\sigma,\delta_0}}$
- *The relationship between a spike* θ_i *of* A_N *such that* $|\theta_i| > \sigma$ *and the limiting point* ρ_{θ_i} *of the corresponding eigenvalues of* $\frac{1}{\sqrt{N}} W_N + A_N$:

$$\rho_{\theta_i} = \theta_i + \frac{\sigma^2}{\theta_i} = H_{\sigma,\delta_0}(\theta_i).$$

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- W_N is a $N \times N$ Wigner Hermitian matrix associated with a distribution μ of variance σ^2 and mean zero which is symmetric and satisfies a Poincaré inequality.
- A_N is a deterministic Hermitian matrix.
  $\mu_{A_N} \to \nu$ weakly , ν compactly supported.
  A_N has a number J of fixed eigenvalues (spikes) $\theta_1 > \ldots > \theta_J$ which are independent of N, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$. For any $i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$.
  A_N has $N - r$ eigenvalues $\beta_i(N)$ such that

$$\max_{i=1}^{N-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \to_{N\to\infty} 0$$

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- $\overline{U_{\sigma,\nu}} = \bigcup_{l=m}^{1} [s_l, t_l]$ with $s_m < t_m < \ldots < s_1 < t_1$.

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- Each connected component of $\overline{U_{\sigma,\nu}}$ contains a least a connected component of $\mathrm{supp}(\nu)$.

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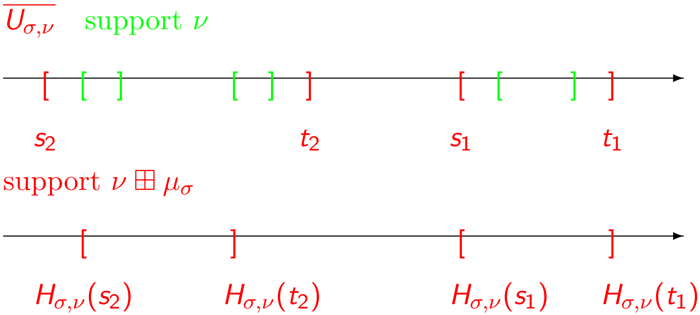
Results



$\overline{U_{\sigma,\nu}}$
support ν
s_2
t_2
s_1
t_1
support $\nu \boxplus \mu_\sigma$
$H_{\sigma,\nu}(s_2)$
$H_{\sigma,\nu}(t_2)$
$H_{\sigma,\nu}(s_1)$
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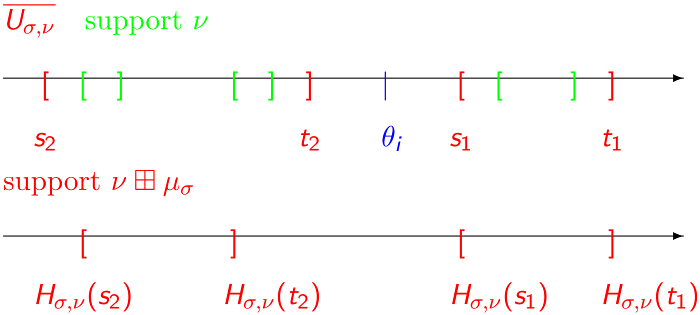
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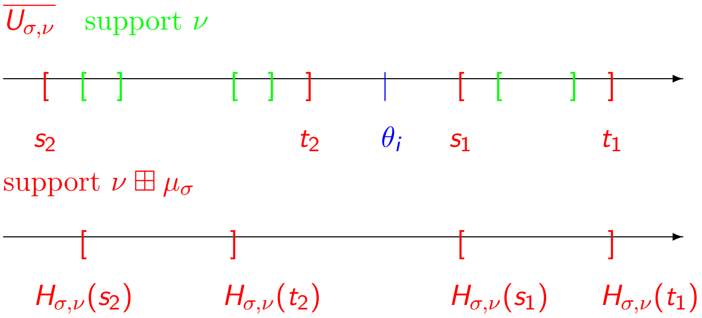
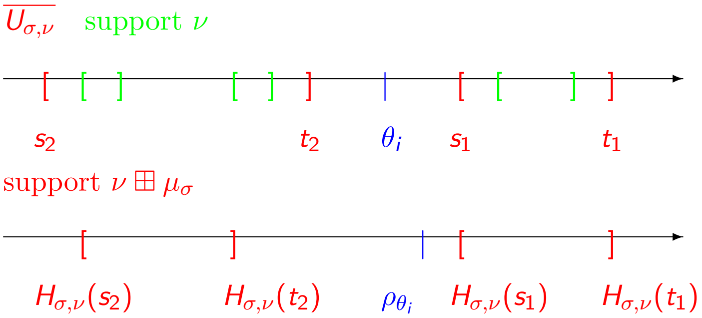


Theorem (Capitaine, Donati-Martin, Féral, Février)

$U_{\sigma,\nu} = \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$

$n_{i-1}+1, \ldots, n_{i-1}+k_i$: *the descending ranks of* θ_i *among the eigenvalues of* A_N.

(1) If $\theta_i \in {}^c\overline{U_{\sigma,\nu}}$, *the* k_i *eigenvalues* $(\lambda_{n_{i-1}+j}(M_N),\ 1 \leq j \leq k_i)$ *converge almost surely outside the support of* $\nu \boxplus \mu_\sigma$ *towards* $\rho_{\theta_i} = H_{\sigma,\nu}(\theta_i) = \theta_i + \sigma^2 g_\nu(\theta_i)$.



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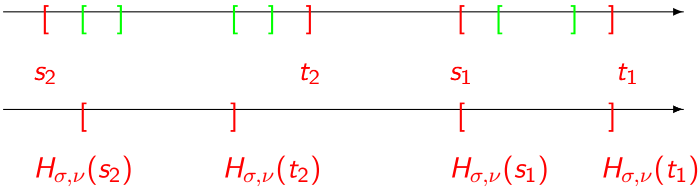
Results



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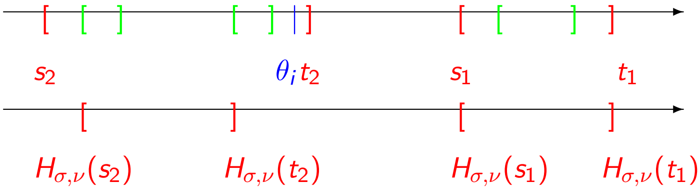

s_2
θ_i t_2
s_1
t_1
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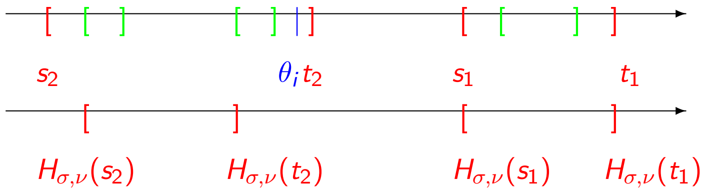

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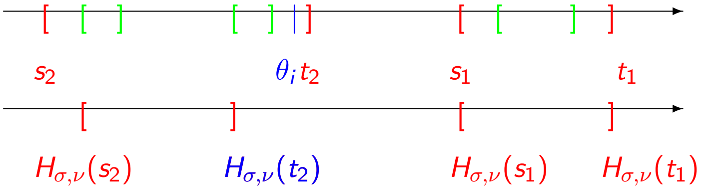
$n_{i-1}+1, \ldots, n_{i-1}+k_i$: the descending ranks of θ_i among the eigenvalues of A_N.
(2) If $\theta_i \in \overline{U_{\sigma,\nu}}$ then we let l_i be the integer number in $\{1, \ldots, m\}$ such that $[s_{l_i}, t_{l_i}]$ contains θ_i. a) If θ_i is on the right (resp. the left) of any connected component of $\mathrm{supp}(\nu)$ which is included in $[s_{l_i}, t_{l_i}]$ then the k_i eigenvalues $(\lambda_{n_{i-1}+j}(M_N), 1 \leq j \leq k_i)$ converge almost surely to $H_{\sigma,\nu}(t_{l_i})$ (resp. $H_{\sigma,\nu}(s_{l_i})$) which is a boundary point of the support of $\nu \boxplus \mu_\sigma$.



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The asymptotic behaviour of the eigenvalues of a deformed Wigner matrix comes from two phenomena :

- Inclusion of the spectrum of M_N in a ϵ-neighborhood of the support of $\mu_{A_N} \boxplus \mu_\sigma$ for all large N almost surely
- Exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, involving the subordination function $F_{\sigma,\nu}$.

For any $\epsilon > 0$,

Theorem

Almost surely, for all large N

$$Spect(\frac{1}{\sqrt{N}} W_N + A_N) \subset \epsilon\text{-neighborhood of support}(\mu_{A_N} \boxplus \mu_\sigma)$$

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Almost surely, for all large N

$$Spect(\frac{1}{\sqrt{N}} W_N + A_N) \subset \epsilon\text{-neighborhood of support}(\mu_{A_N} \boxplus \mu_\sigma)$$

$$+$$

for all large N,

$$support\,(\mu_{A_N} \boxplus \mu_\sigma) \subset \epsilon\text{-neighborhood of support } (\nu \boxplus \mu_\sigma) \bigcup_{i,|\theta_i \in^c \overline{U_{\sigma,\nu}}} \{\rho_{\theta_i}\}.$$

$$\Downarrow$$

a.s, for all large N,

$$Spect(\frac{1}{\sqrt{N}} W_N + A_N) \subset \epsilon\text{-neighborhood of support } (\nu \boxplus \mu_\sigma) \bigcup_{i,|\theta_i \in^c \overline{U_{\sigma,\nu}}} \{\rho_{\theta_i}\}.$$

Key ideas of the proof of the inclusion of the spectrum of M_N in an ϵ-neighborhood of support$(\mu_{A_N} \boxplus \mu_\sigma)$: for any z in $\mathbb{C}^+$,

$$\tilde{g_N}(z) = \int \frac{1}{z-x} d\mu_{A_N} \boxplus \mu_\sigma(x); \quad g_N(z) = \mathbb{E}\left[\int \frac{1}{z-x} d\mu_{M_N}(x)\right]$$

- g_N satisfies an approximative subordination equation :

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{1}{N} L_N(z) + O(\frac{1}{N^2}).$$

- $\Longrightarrow |g_N(z) - \tilde{g_N}(z) + \frac{E_N(z)}{N}| \leq \frac{P(|\Im z|^{-1})}{N^2}$

where E_N is the Stieltjes transform of a distribution Λ_N whose support is included in the support of $\mu_{A_N} \boxplus \mu_\sigma$.

- $\Longrightarrow$ inclusion of the spectrum by inverse Cauchy transform.

Theorem

$[a, b] \subset {}^c \left\{ \text{support } (\nu \boxplus \mu_\sigma) \bigcup_{i, |\theta_i \in {}^c \overline{U_{\sigma,\nu}}} \{\rho_{\theta_i}\} \right\}.$

Then for large N, $[F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)] \subset^c \mathrm{Spect A_N}.$

$i_N \in \{0, \ldots, N\}$ *s.t*

$$\lambda_{i_N+1}(A_N) < F_{\sigma,\nu}(a) \quad and \quad \lambda_{i_N}(A_N) > F_{\sigma,\nu}(b)$$

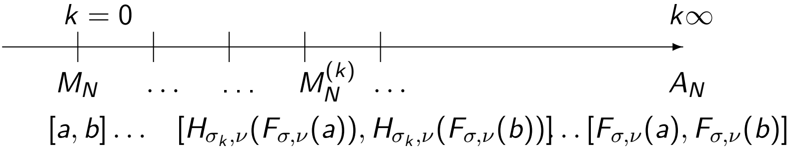
($\lambda_0 := +\infty$ *and* $\lambda_{N+1} := -\infty$*).*
Then

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \quad and \quad \lambda_{i_N}(M_N) > b, \quad for\ large \quad N] = 1.$$

Key idea of the proof of the exact separation : introduce a continuum of matrices from M_N to A_N :

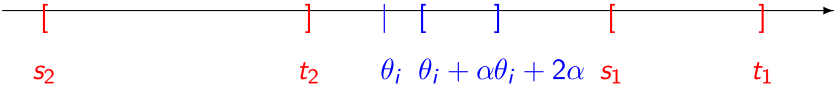
$$M_N^{(k)} := \frac{\sigma_k}{\sigma}\frac{W_N}{\sqrt{N}} + A_N,\ \sigma_0 = \sigma,\ \sigma_k \to 0 \text{ when } k \text{ goes to infinity.}$$

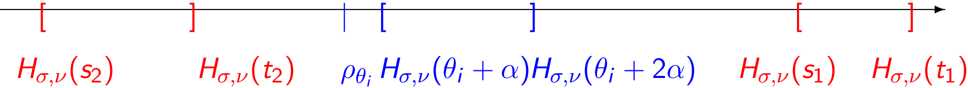
$$H_{\sigma_k,\nu}(z) = z + \sigma_k^2 g_\nu(z),\ F_{\sigma_k,\nu} = H^{-1}_{\sigma_k,\nu}.$$



• For any k, the interval $[H_{\sigma_k,\nu}(F_\sigma(a)), H_{\sigma_k,\nu}(F_{\sigma,\nu}(b))]$ splits the spectrum of $M_N^{(k)}$ in exactly the same way.

• For k large enough the interval $[H_{\sigma_k,\nu}(F_{\sigma,\nu}(a)), H_{\sigma_k,\nu}(F_{\sigma,\nu}(b))]$ splits the spectrum of $M_N^{(k)}$ as $[F_{\sigma,\nu}(a)), F_{\sigma,\nu}(b))]$ splits the spectrum of A_N,

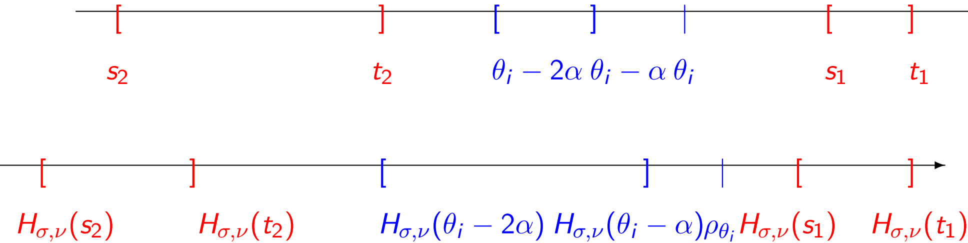




$n_{i-1} + 1, \ldots, n_{i-1} + k_i$: the descending ranks of θ_i

$\Downarrow$ Exact separation phenomenon

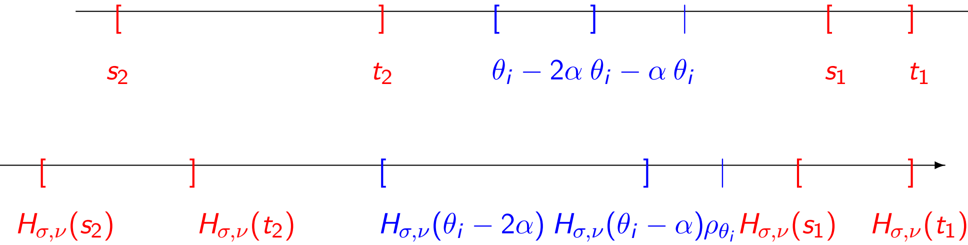
a.s. for all large N, $\lambda_{n_{i-1}+1}(M_N) \leq H_{\sigma,\nu}(\theta_i + \alpha) \leq \rho_{\theta_i} + \epsilon$



$n_{i-1}+1, \ldots, n_{i-1}+k_i$: the descending ranks of θ_i

$\Downarrow$ Exact separation phenomenon

a.s. for all large N, $\lambda_{n_{i-1}+k_i}(M_N) \geq H_{\sigma,\nu}(\theta_i - \alpha) \geq \rho_{\theta_i} - \epsilon$

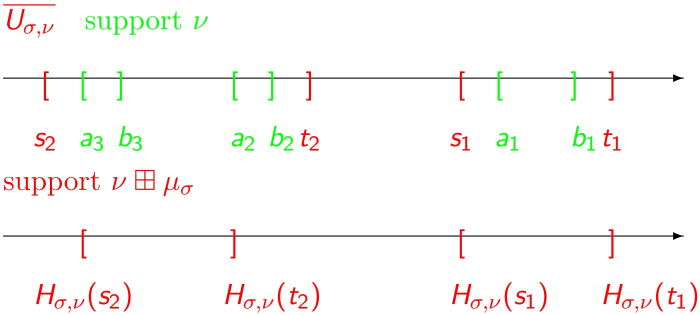


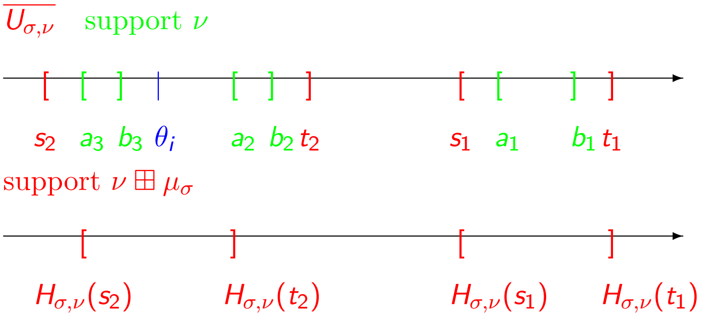
$n_{i-1}+1, \ldots, n_{i-1}+k_i$: the descending ranks of θ_i

$\Downarrow$ Exact separation phenomenon

a.s. for all large N, $\lambda_{n_{i-1}+k_i}(M_N) \geq H_{\sigma,\nu}(\theta_i - \alpha) \geq \rho_{\theta_i} - \epsilon$

$$\Rightarrow \rho_{\theta_i} - \epsilon \leq \lambda_{n_{i-1}+k_i}(M_N) \leq \ldots \leq \lambda_{n_{i-1}+1}(M_N) \leq \rho_{\theta_i} + \epsilon$$

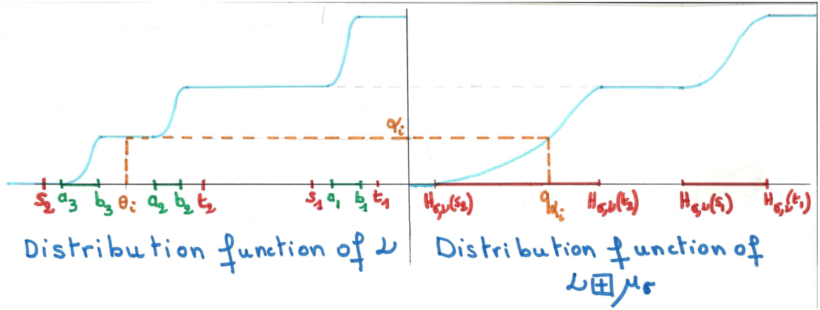

$\overline{U_{\sigma,\nu}}$
support ν
s_2 a_3 b_3 a_2 b_2 t_2 s_1 a_1 b_1 t_1
support $\nu \boxplus \mu_\sigma$
$H_{\sigma,\nu}(s_2)$ $H_{\sigma,\nu}(t_2)$ $H_{\sigma,\nu}(s_1)$ $H_{\sigma,\nu}(t_1)$


$\overline{U_{\sigma,\nu}}$ support ν
s_2 a_3 b_3 θ_i a_2 b_2 t_2 s_1 a_1 b_1 t_1
support $\nu \boxplus \mu_\sigma$
$H_{\sigma,\nu}(s_2)$ $H_{\sigma,\nu}(t_2)$ $H_{\sigma,\nu}(s_1)$ $H_{\sigma,\nu}(t_1)$

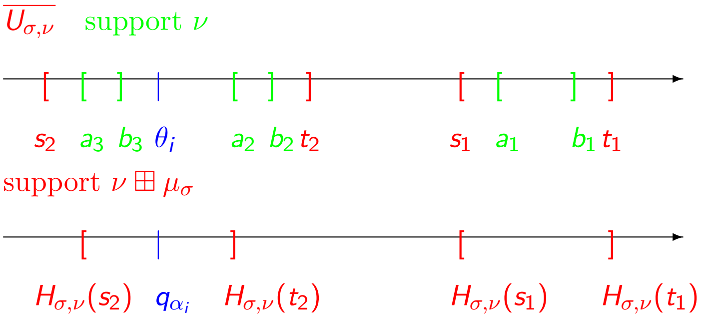
Proposition

$\forall l \in \{1, \ldots, m\}$,

$$\nu \boxplus \mu_\sigma([H_{\sigma,\nu}(s_l); H_{\sigma,\nu}(t_l)]) = \nu([s_l; t_l])$$



always continuous and strictly increasing on each $]H_{\sigma,\nu}(s_l); H_{\sigma,\nu}(t_l)[$



Theorem

If $\theta_i \in \overline{U_{\sigma,\nu}}$ then we let l_i be the integer number in $\{1, \ldots, m\}$ such that $[s_{l_i}, t_{l_i}]$ contains θ_i. If θ_i is between two connected components of $\mathrm{supp}(\nu)$ which are included in $[s_{l_i}, t_{l_i}]$ then the k_i eigenvalues $(\lambda_{n_{i-1}+j}(M_N), 1 \leq j \leq k_i)$ converge almost surely to the α_i-th quantile of $\nu \boxplus \mu_\sigma$ (that is to q_{α_i} where $\alpha_i = \nu \boxplus \mu_\sigma(]-\infty, q_{\alpha_i}])$) where α_i is such that $\alpha_i = 1 - \lim_N(n_{i-1}/N) = \nu(]-\infty, \theta_i])$.

Actually one can check that

-the results of Benaych-Georges-Rao about the convergence of the extremal eigenvalues of a matrix $X_N + A_N$, A_N being a finite rank perturbation whereas X_N is a unitarily invariant matrix with some limiting spectral compactly supported distribution μ, could be rewritten in terms of the subordination function related to the additive free convolution of δ_0 by μ.

- the results on spiked population models (Baik-Ben Arous-Péché, Baik-Silverstein, Bai-Yao) could be also fully described in terms of free probability involving the subordination function related to the multiplicative free convolution by a Marchenko-Pastur distribution.

Conclusion

Subordination property in free probability definitely sheds light on spiked deformed models.