Random embedding of ℓ_p^n into ℓ_r^N

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Local theory of Banach spaces.

Banach Mazur distance :

$$d(X,Y) = \inf\left\{ \|T\|\|T^{-1}\|, \quad T : X \to Y \text{ isomorphism} \right\}$$

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Let E be a normed space of dimension N, $(\mathbb{R}^N, \|\cdot\|)$ such that $\|x\| \le |x|_2$, and

$$M = \int_{S^{N-1}} \|\theta\| d\sigma(\theta).$$

$\forall \varepsilon \in (0,1)$, *if*

$$n \le c\, N\, M^2 \varepsilon^2 / \log(3/\varepsilon) \quad \textit{then} \quad \ell_2^n \overset{1+\varepsilon}{\hookrightarrow} E.$$

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Proofs : random methods that can be described through the use of Gaussian operators,

$$G = (g_{ij}) : \ell_2^n \to \ell_1^N \text{ where } g_{ij} \sim \mathcal{N}(0,1).$$

Other embeddings.

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Main properties :

1) if $\theta, \theta_1, \ldots, \theta_n$ are i.i.d. standard p-stable then for every $\alpha_1, \ldots, \alpha_n$,

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Consequence : $\ell_p^n \overset{1}{\hookrightarrow} L_1$

Other embeddings.

X is of stable type p iff for some (every) $r < p$, there exists $C > 0$ such that for every finite collection of vectors $x_1, \ldots, x_n$

$$\left(\mathbb{E}\left\|\sum \theta_i x_i\right\|^r\right)^{1/r} \leq C \left(\sum \|x_i\|^p\right)^{1/p}.$$

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• Milman ['71] $\forall \varepsilon \in (0,1),\ \ell_2^n \overset{1+\varepsilon}{\hookrightarrow} E_N$ where $n = n(\varepsilon, N, M)$	• Pisier ['83] $\forall \varepsilon \in (0,1),\ \ell_p^n \overset{1+\varepsilon}{\hookrightarrow} E_N$ where $n = n(\varepsilon, N, ST_p(E_n))$

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Let $0 < r < p < 2$ *and* $\frac{2p}{p+2} < r \leq 1$*,*
$\forall \eta > 0$ *and any integers* $N = (1+\eta)n$*,*
there exist explicit random operators $T : \ell_p^n \to \ell_r^N$ *such that*

$$\mathbb{P}\left\{\forall \alpha,\ c(p,r)^{1/\eta}|\alpha|_p \leq |T\alpha|_r \leq c(p,r)|\alpha|_p\right\} \geq 1 - \exp(-c(p)n)$$

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Proofs.

Definition of the random operator

First idea : X random p-stable vector in $\mathbb{R}^N$, $X = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}$

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$Y = \pm e_i$ with probability $1/2N$, Y_j independent copies of Y, λ_j independent exponential r.v. and $\Gamma_j = \sum_{i=1}^{j} \lambda_i$, let

$$\tilde{\Theta} = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j$$

Theorem [LePage, Woodroofe, Zinn '81] $\tilde{\Theta}$ *is a* p*-stable random vector and has the same distribution than*

$$\frac{s_p}{N^{1/p}} \sum_{\ell=1}^{N} \theta_\ell e_\ell$$

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$$\begin{aligned} T : \ell_p^n &\longrightarrow \ell_1^N \\ \alpha &\longmapsto \frac{\sigma_p}{N^{1/q}} \sum_{i=1}^{n} \sum_{j \geq 1} \alpha_i \, j^{-1/p} \, Y_{ij} \end{aligned}$$

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Key properties :

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1) $|\mathbb{E}|T\alpha|_1 - |\alpha|_p| \leq D_p \left(\frac{n}{N}\right)^{1/q} |\alpha|_p$ $\rightarrow$ P['83]

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Independence of the columns of the matrix T, but not of all entries.

Theorem *Let* $1 < p < 2$,

$$\mathbb{P}\left\{\forall \alpha \in S_p^{n-1}, c(p)^{1/\eta} \leq |T\alpha|_1 \leq C(p)\right\} \geq 1 - c \, \exp(-c(p)n)$$

Net argument

- Fix $\alpha \in S_p^{n-1}$,

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- It means

$$1 - t \leq |T\alpha|_1 \leq 1 + t$$

$$t = \varepsilon \in (0, 1), \quad N = C(\varepsilon)\, n$$

$$N(S_p^{n-1}, \varepsilon |\cdot|_p) \leq \exp\left(n \log\left(\frac{3}{\varepsilon}\right)\right)$$

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- But $\forall t > 0$

$$|T\alpha|_1 \leq 1 + t$$

$\rightarrow$ Concentration around the mean is useful for almost isometric result and for upper bound of $\|T\|$.

Small ball

Theorem $\forall \eta > 0$ *and integers* $N = (1+\eta)n$,

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Remarks :

• In the case $p = 2$ and $|\cdot|_2$, we would be studying the smallest singular value of $T : \ell_2^n \to \ell_2^N$

• Recent progress for sub-gaussian operators : Schechtman ['04], Litvak, Pajor, Rudelson, Tomczak-Jaegermann ['05]

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Small ball

Theorem $\forall \eta > 0$ *and integers* $N = (1+\eta)n$*,*

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• Decomposition of the sphere into two subsets on which you prove differently the small ball estimate

Almost sparse vectors

$$\mathbb{P}\left\{\left||T\alpha|_1 - |\alpha|_p\right| \geq t\right\} \leq 2\exp(-b_p N t^q)$$

- Assume $\alpha \in S_p^{n-1}$ is sparse i.e. $|\mathrm{supp}(\alpha)| \leq \delta n$.

ALMOST ISOMETRIC	$C\,n$	$\times$	n
	$(1+\eta)\,n$	$\times$	$\delta\,n$

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It means that for such vectors, if $\delta \simeq \frac{1}{C}$, then we may use the concentration around the mean to deduce the small ball estimate (with high probability).

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$\longrightarrow$ the case of $AS(\delta, \rho)$

Non-Almost sparse vectors

Basic property :
$\exists I \subset \{1, \ldots, n\}$ such that $|I| \geq \frac{1}{2}\, \delta\, n\, \rho^p$ and

$$\forall k \in I,\ \frac{\rho}{(2n)^{1/p}} \leq |\alpha_k| \leq \frac{1}{(\delta n)^{1/p}}$$

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Example : take $\alpha = (\frac{1}{n^{1/p}}, \ldots, \frac{1}{n^{1/p}})$ then

$$\begin{aligned} T\alpha &= \frac{\sigma_p}{N^{1/q}} \sum_{i=1}^{n} \sum_{j \geq 1} \alpha_i\, j^{-1/p}\ Y_{ij} \\ &\simeq \frac{1}{N} \sum_{i=1}^{n} \sum_{j \geq 1} j^{-1/p}\ Y_{ij} \end{aligned}$$

We would like to understand $\mathbb{P}\{|T\alpha|_1 \leq t\}$?

Multi-dimensional Esseen type inequality

Theorem *Let X be a random vector in $\mathbb{R}^N$, such that the function*

$$\xi \mapsto \mathbb{E}\exp(i\langle \xi, X\rangle) \in L_1(\mathbb{R}^N).$$

Let K be a compact, star-shape subset of $\mathbb{R}^N$.
Then for any $t > 0$

$$\mathbb{P}\left\{\|X\|_K \le t\right\} \le |K| \left(\frac{t}{2\pi}\right)^N \int_{\mathbb{R}^N} \left|\mathbb{E}\exp\left(i\langle \xi, X\rangle\right)\right| d\xi.$$

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- We generalize the classical Esseen inequality to the multi-dimensional case and to any norms.
- The proof is an application of Fourier analysis.

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Apply it to

$$X = N\,T\alpha = \sum_{i=1}^{n}\sum_{j\ge 1} j^{-1/p}\;Y_{ij}$$

and $K = N\,B_1^N$: $\to$ $\|X\|_K = |T\alpha|_1$ and $|K| \simeq 1$

Multi-dimensional Esseen type inequality

Apply it to

$$X = N\,T\alpha = \sum_{i=1}^{n} \sum_{j \geq 1} j^{-1/p}\; Y_{ij}$$

and $K = N\,B_1^N$: $\longrightarrow$ $\|X\|_K = |T\alpha|_1$ and $|K| \simeq C^N$

$$\mathbb{P}\left\{|T\alpha|_1 \leq t\right\} \leq C^N \left(\frac{t}{2\pi}\right)^N \int_{\mathbb{R}^N} |\mathbb{E}\exp\left(i\langle \xi, X\rangle\right)|\, d\xi.$$

Study

$$\int_{\mathbb{R}^N} |\mathbb{E}\exp\left(i\langle \xi, X\rangle\right)|\, d\xi$$

$\longrightarrow$ This is delicate but doable : $\leq C^N$

Random embedding of ℓ_p^n into ℓ_r^N

Omer Friedland and Olivier Guédon

Université Pierre et Marie Curie and Université Paris-Est Marne-la-Vallée

June 1st, 2010 Random Matrices