The single ring theorem

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Feinberg-Zee's single ring theorem

Ensemble of random, non-Normal $N \times N$ matrices with law

$$dP_N(X_N) = \frac{1}{Z_N} e^{-N \mathrm{Tr} V(X_N X_N^*)} dX_N.$$

$\{\lambda_i^N\} \in \mathbb{C}^N$-eigenvalues of X_N, $L_N^V = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N}$

Theorem (Feinberg-Zee 97')

Assume V is a polynomial. Then : L_N^V converges in probability to a deterministic, rotationally invariant μ_V whose support consists of a single ring : there exists constants $0 \leq a < b < \infty$ so that

$$\mathrm{supp}(\mu_V) = \{re^{i\theta} : a \leq r \leq b, \theta \in [0, 2\pi[\}.$$

If $V(x) = gx^2 + mx$, phase transition occurs when the support changes from a disc to an annulus.

Singular values

$$dP_N(X_N) = \frac{1}{Z_N} e^{-N \mathrm{Tr} V(X_N X_N^*)} dX_N$$

Let $\sigma_1^N \geq \sigma_2^N \geq \cdots \geq \sigma_N^N \geq 0$ denote the singular values of X_N (ev $\sqrt{X_N X_N^*}$). Their joint distribution is

$$\frac{1}{Z_N} \prod_{i<j} [(\sigma_i^N - \sigma_j^N)(\sigma_i^N + \sigma_j^N)]^2 e^{-N \sum_{i=1}^N V((\sigma_i^N)^2)} \prod_{i=1}^N \sigma_i^N d\sigma_i^N$$

One easily deduces the convergence of the empirical measure of the singular values to the probability measure which minimizes

$$\int V(x^2) d\mu(x) - \int \int \log |x^2 - y^2| d\mu(x) d\mu(y)$$

If V has k sufficiently deep wells, its support consists of at list k intervals !

Generalization

Under the Feinberg-Zee model, $X_N = U_N D_N V_N$ with D_N a real diagonal matrix with converging spectral distribution ($D_N = \mathrm{diag}(\sigma^N)$) and independent U_N, V_N, unitary matrices following Haar distribution.

What can be said about the convergence of the empirical measure $L_{X_N} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N}$ of the eigenvalues of $X_N = U_N D_N V_N$ (or $X_N = U_N D_N$) ?

Asymptotics of $X_N = U_N D_N V_N$

Let U_N, V_N follow Haar measure, be independent from D_N with spectral measure $L_{D_N} = \frac{1}{N}\sum \delta_{D_N(ii)}$ which converges to μ_D.

Theorem (G. , Krishnapur, Zeitouni '09)

a)Take $X_N = U_N D_N V_N$. Assume some technical conditions on D_N and that for some $\kappa, \kappa' > 0$,

$$\max_{\Im z \geq N^{-\kappa'}} \left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - D_N(ii)} - \int \frac{1}{z-x} d\mu_D(x) \right| \leq \frac{1}{N^{\kappa} |\Im z|}$$

with probability going to one. Then $L_{X_N} = \frac{1}{N}\sum \delta_{\lambda_i^N}$ converges to a measure μ_{X_D} which is the Brown measure of μ_D.

b)Conditions of part a) hold in Feinberg-Zee model with V of polynomial growth at infinity, smooth.
c)If D_N and D_N^{-1} are bounded uniformly and μ_D has bounded density, $L_{X_N + N^{-\kappa} G_N}$ converges to μ_{X_D} if G_N is Gaussian and $\kappa > 1/2$.

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$$\max_{\Im z \geq N^{-\kappa'}} \left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - D_N(ii)} - \int \frac{1}{z-x} d\mu_D(x) \right| \leq \frac{1}{N^{\kappa} |\Im z|}$$

with probability going to one.
Then

$$\text{supp }(X_N) \to \text{ supp }(\mu_{X_D}) \quad \text{in probability.}$$

If the above conditions hold almost surely wrt D_N, the support of the eigenvalues of X_N converges almost surely.

Generalization : Asymptotics of $X_N = U_N + D_N$

Let U_N follow Haar measure, be independent from D_N which converges in $*$-moments to D.

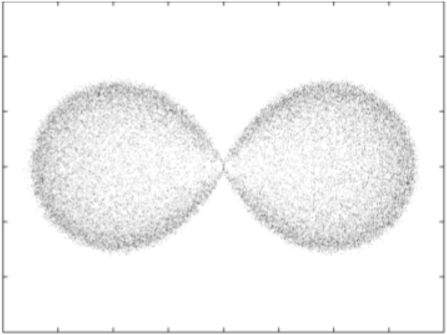
Theorem (G. , Krishnapur, Zeitouni '10)

Let D_N satisfy some technical assumptions, then the spectral measure of $U_N + D_N + N^{-\kappa} G_N$, with G_N Gaussian, $\kappa > 1/2$, converges weakly in probability towards the Brown measure of $U + D$.
These assumptions are verified if $D_N(i,j) = 1_{i=j} f(\frac{i}{N})$ with f smooth, $f' \neq 0$.

Rmk : Convergence of the spectral measure of $U_N + D_N + tN^{-1/2} G_N$ for all $t > 0$ was obtained by Sniady (01).

Generalization : Asymptotics of $X_N = U_N + D_N$

Our assumptions require that the spectral measure of $|D_N|$ converges to a measure with bounded density, which excludes the cases studied by Biane and Lehner.



Spectrum of $U_N + U\Lambda U^*$, Λ symmetry with vanishing trace

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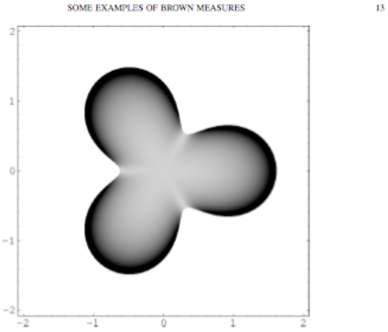

Spectrum of $U_N + U_3$

Idea of the proof

Based on the Green formula : if (λ_i^N) ev of $X_N = U_N D_N V_N$

$$\begin{aligned}\sum_{i=1}^N \psi(\lambda_i^N) &= \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\psi(z) \log|\prod_{i=1}^N (z-\lambda_i^N)| dz \\ &= \frac{N}{4\pi}\int_{\mathbb{C}} \Delta\psi(z) \int \log|x| dL_{(z-X_N)(z-X_N)^*}(x) dz\end{aligned}$$

depends on the empirical measures of the Hermitian matrices $(z-X_N)(z-X_N)^*$.

• Show the weak convergence of the spectral measure of $(z-X_N)(z-X_N)^*$ (free probability).

• Problem : Because of the singularity of log at the origin, need not only the weak convergence of $L_{(z-X_N)(z-X_N)^*}$! Example : X $N \times N$ with i.i.d entries, convergence to circular law : Girko 84', Bai 97', Gotze-Tikhomirov 07',Tao-Vu 08'

A free probability statement

Let U_N be a $N \times N$ unitary matrix distributed according to the Haar measure, and independent from D_N, bounded, diagonal matrix with real entries and spectral distribution converging to some measure μ_D.

Theorem (Voiculescu 91')

For any polynomial P,

$$\lim_{N\to\infty} \frac{1}{N} tr(P(U_N, U_N^*, D_N)) = \tau(P(U, U^*, D)) \text{ in probability.}$$

with (U, D) free under τ. Moreover $\tau(D^k) = \int x^k d\mu_D(x)$ and

$$\tau(U^{n_1}(U^*)^{m_1} U^{n_2} \cdots (U^*)^{m_k}) = 1_{\sum n_i = \sum m_i}.$$

This implies in particular the convergence of $N^{-1} tr[((z - X_N)(z - X_N)^*)^k]$ for all $k \geq 0$ if $X_N = U_N D_N$.

What is freeness ?

$\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n) \in (B(H), \tau)$ are free iff for all polynomials $P_1, \ldots, P_\ell$ and $Q_1, \cdots, Q_\ell$ so that $\tau(P_i(\mathbf{X})) = 0$ and $\tau(Q_i(\mathbf{Y})) = 0$ for all i

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- τ is uniquely determined by $\tau(P(\mathbf{X}))$ and $\tau(Q(\mathbf{Y}))$, Q, P polynomials.
- If $A \geq 0, B$ are free, the moments of $A^{\frac{1}{2}} B A^{\frac{1}{2}}$ are described by the S-transform

$$S_{A^{\frac{1}{2}} B A^{\frac{1}{2}}}(z) = S_A(z) S_B(z)$$

with $S_C(z) := \frac{1+z}{z} m_C^{-1}(z)$ if $m_C(z) = \sum_{n \geq 1} \mu_C(x^n) z^n$.

R-diagonal operators

The operators of the form UD, U, D free, are called "R-diagonal operators".

The Brown measure μ_X of an operator $X \in (B(H), \tau)$ is given by

$$\int \psi(x) d\mu_X(x) = \frac{1}{4\pi} \int \Delta\psi(z) \tau[\log(z - X)(z - X)^*] dz.$$

Theorem (Haagerup-Larsen 00')

Take $X_D := UDV$ or UD, U, V, D free, D self-adjoint with law μ_D, U, V unitaries. The Brown measure of X_D is rotation invariant and radially supported on an annulus

$$\mu_{X_D}(B(0, f(t))) = t$$

where $f(t) = 1/\sqrt{S_{D^2}(t-1)}$.

$$\mathrm{supp}(\mu_{X_D}) = \{re^{i\theta}, r \in [(\mu_D(x^{-2}))^{-\frac{1}{2}}, (\mu_D(x^2))^{\frac{1}{2}}], \theta \in [0, 2\pi[\}$$

Idea of the proof

Therefore the results amount to say that

$$\frac{1}{N}\sum_{i=1}^{N}\psi(\lambda_i^N) = \frac{1}{4\pi}\int_{\mathbb{C}}\Delta\psi(z)\frac{1}{N}\mathrm{tr}[\log((z-X_N)(z-X_N)^*)]dz$$
$$\sim \frac{1}{4\pi}\int\Delta\psi(z)\tau[\log(z-X)(z-X)^*]dz$$

and we need to show that for almost all z

$$\lim_{\epsilon\downarrow 0}\limsup_{N\uparrow\infty} E\left[\frac{1}{N}\mathrm{tr}[1_{(z-X_N)(z-X_N)^*\le\epsilon}\log((z-X_N)(z-X_N)^*)]\right]=0.$$

Strategy : • Assume this is true for $\epsilon < N^{-\gamma}$ for some $\gamma > 0$ (true if replace X_N by $X_N + N^{-\gamma'}G_N$ by Sankar and Spielman)
• Need to show the above for $n^{-\gamma} < (z-X_N)(z-X_N)^* \le \epsilon$

Center of the proof.

We shall prove that for almost all z, $\nu_N^z := E[L_{(z-X_N)(z-X_N)^*}]$ satisfies for any $x > n^{-\kappa}$ for some $\kappa > 0$, and $C_z < \infty$

$$\nu_N^z([-x,x]) \le C_z|x| \quad (*).$$

- This implies for any $\gamma > 0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{N\to\infty}\int_{n^{-\gamma}<x\le\epsilon}\log(x)d\nu_N^z(x) = 0.$$

- (*) is a consequence of

$$|\Im\left(\int\frac{1}{w-x}d\nu_N^z(x)\right)| \le C_z/2 \quad \Im w \ge n^{-\kappa}. \quad (\dagger)$$

Technical estimate : symmetrization

Want to show for $z \neq 0$,

$$|\Im\left(\int \frac{1}{w-x} d\nu_N^z(x)\right)| \leq C/2 \quad \Im w \geq n^{-\kappa}. \quad (\dagger)$$

At the limit

$$|\Im\left(\int \frac{1}{w-x} d\nu^z(x)\right)| \leq C/2 \quad \Im w \geq 0$$

with $\nu^z = L_{(z-UD)(z-UD)^*}$. Let

$$H^z := \begin{pmatrix} 0 & U + |z|^{-1} D \\ (U + |z|^{-1} D)^* & 0 \end{pmatrix}$$

Then

$$\tau((H^z - w)^{-1}) = \frac{1}{|z|^2} \int \frac{1}{|z|^{-2} w - x} d\nu^z(x)$$

so that it is enough to study the spectral measure of H^z.

Technical estimates : Master loop equations

We obtain close equations for the Stieltjes transform G of the spectral measure of $H = \begin{pmatrix} 0 & U+D \\ (U+D)^* & 0 \end{pmatrix}$ by using Master loop equations based on the invariance of the Haar measure. If P is a monomial in U, U^*, D, they read

$$\sum_{P=P_1UP_2} \tau(P_1U)\tau(P_2) = \sum_{P=Q_1U^*Q_2} \tau(Q_1)\tau(U^*Q_2).$$

They give, with $R(z) = (\sqrt{1+4z^2}-1)/2z$

$$G(z_1) = \tau\left(\left(z_2 - \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}\right)^{-1}\right) \text{ when } z_2 = z_1 - R(G(z_1))$$

G is bounded when the Stieltjes transform of D is.

Technical estimates : Master loop equations

Similarly, the Stieltjes transform G^N of the spectral measure of

$$H^N = \begin{pmatrix} 0 & U^N + D^N \\ (U^N + D^N)^* & 0 \end{pmatrix}$$

can be studied by the Master loop equations

$$E[\sum_{P=P_1UP_2} \frac{1}{N}\mathrm{tr}(P_1U)\frac{1}{N}\mathrm{tr}(P_2)] = \sum_{P=Q_1U^*Q_2} E[\frac{1}{N}\mathrm{tr}(Q_1)\frac{1}{N}\mathrm{tr}(U^*Q_2)].$$

They yield for $z_1 \gg$ with $\tilde{G}^N(z_1) = \sqrt{1+4G^N(z_1)^2+\epsilon_N(z_1)}+1$

$$G^N(z_1) = \frac{1}{2N}\mathrm{tr}\left(\left(z_2 - \begin{pmatrix} 0 & D^N \\ (D^N)^* & 0 \end{pmatrix}\right)^{-1}\right) + \epsilon_N(z_1, z_2)$$

when $z_2 = z_1 - 2G^N(z_1)/\tilde{G}^N(z_1)$. The difficulty is to show that this extends to $\Im z_1 \geq n^{-\kappa}$

Conclusion

- The Brown measure is not a continuous function of the moments. A typical example is given by

$$\Xi^N := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

  which converges in *-moments to the Haar distributed unitary matrix with spectral measure uniformly distributed on the circle whereas the Brown measure of Ξ^N is δ_0 for all N.
- Our proof heavily uses the Master loop equations and thus requires rather exact solutions. This approach is quite different from Tao and Vu (convergence to the circular law) who use a lot independence.
- Our proof generalizes to orthogonal matrices.