Multiple Orthogonal Polynomials in Random Matrix Theory

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Unitary ensembles

- **Probability measure on $n \times n$ Hermitian matrices**

$$\frac{1}{\tilde{Z}_n} e^{-n \operatorname{Tr} V(M)} \, dM$$

  - **This is GUE for** $V(M) = \frac{1}{2} M^2$

- **Explicit formula for joint density of eigenvalues**

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{n} e^{-n V(x_j)}$$

Global eigenvalue behavior

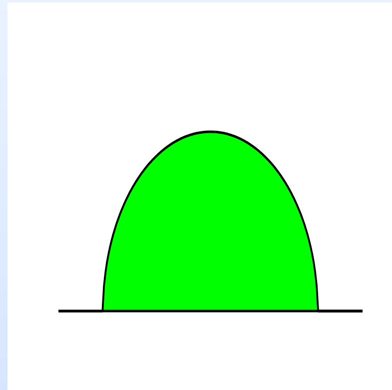
- **As $n \to \infty$, there is a limiting mean eigenvalue density $\rho_V(x)$.**
- **The probability measure $\boxed{d\mu_V(x) = \rho_V(x)dx}$ minimizes**

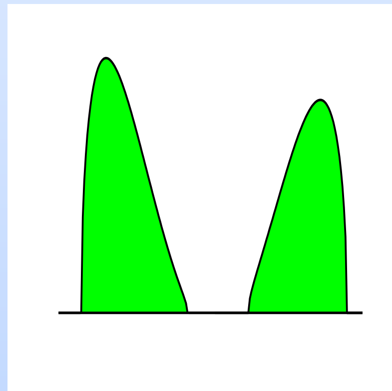
$$\iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x)$$

- **If V is real analytic, then $\mathrm{supp}(\mu_V)$ is a finite union of intervals and**

$$\rho_V(x) = \frac{1}{\pi}\sqrt{Q_-(x)}$$

$$Q(x) = \left(\frac{V'(x)}{2}\right)^2 - \int \frac{V'(x)-V'(s)}{x-s} d\mu_V(s)$$





- **Typical behavior: ρ_V is positive and real analytic on each interval and vanishes as a square root at endpoints.**

Orthogonal polynomials

- **Average characteristic polynomial**

$$P_{n,n}(x) = \mathbb{E}\left[\det(xI_n - M)\right]$$

is nth degree orthogonal polynomial with respect to $e^{-nV(x)}$ on real line

  - **Orthogonality with respect to varying weight**

- **Monic OPs $P_{k,n}(x) = x^k + \cdots$**

$$\int_{-\infty}^{\infty} P_{k,n}(x)\, x^j\, e^{-nV(x)}\, dx = h_{k,n}\delta_{j,k}, \qquad j = 0, \ldots, k.$$

Determinantal correlation functions

- Eigenvalues are determinantal point process with correlation kernel

$$K_n(x,y) = \sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}\sum_{k=0}^{n-1}\frac{P_{k,n}(x)P_{k,n}(y)}{h_{k,n}}$$

  - This means that the k point eigenvalue correlation function (which is proportional to marginal density) is given by $k \times k$ determinant

$$\det\left[K_n(x_i,x_j)\right]_{i,j=1}^{k}$$

- Global eigenvalue behavior

$$\lim_{n\to\infty}\frac{1}{n}K_n(x,x) = \rho_V(x)$$

Local eigenvalue behavior

- Local eigenvalue statistics have universal behavior as $n \to \infty$.
  - **Sine kernel** in the bulk: if $c = \rho_V(x^*) > 0$ then

$$\lim_{n\to\infty} \frac{1}{cn} K_n \left(x^* + \frac{x}{cn}, x^* + \frac{y}{cn} \right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

Pastur, Shcherbina (1997), Bleher, Its (1999)

Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

McLaughlin, Miller (2008), Lubinsky (2009)

  - **Airy kernel** at the spectral edge (if ρ_V vanishes as a square root at x^*)

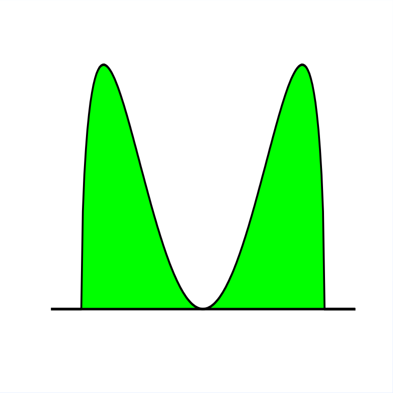
$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_n \left(x^* + \frac{x}{cn^{2/3}}, x^* + \frac{y}{cn^{2/3}} \right) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

Singular behavior

- Other limiting kernels at special points.
  - **Painlevé II kernels** at interior points where density vanishes.

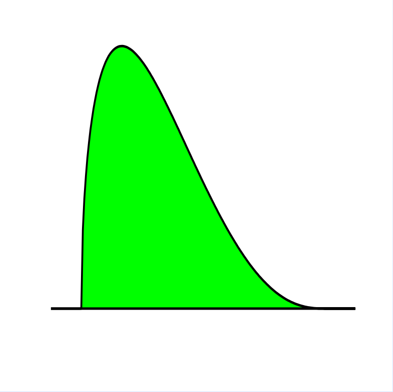
    Bleher, Its (2003), Claeys, K (2006)

    Shcherbina (2008)



  - **Painlevé I_2 kernels** at edge points where density vanishes at higher order.

    Claeys, Vanlessen (2007)



Riemann-Hilbert problem

- **Powerful tool for asymptotic analysis in case of real analytic V is the**

  Riemann-Hilbert problem for OPs **Fokas, Its, Kitaev (1992)**

  (1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic**

  (2) Y **has limiting values** $Y_\pm$ **on** $\mathbb{R}$**, satisfying**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R},$$

  (3) $Y(z) = (I + O(1/z))\,\mathrm{diag}\begin{pmatrix} z^n & z^{-n} \end{pmatrix}$ **as** $z \to \infty$**.**

- **Correlation kernel is**

$$K_n(x,y) = \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Steepest descent analysis

- Asymptotics of orthogonal polynomials can be proved by means of a steepest descent analysis of RH problem
- Essential role is played by minimizer $d\mu_V(s) = \rho_V(s)ds$ of equilibrium problem

$$\iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x)$$

- the associated g-function $\boxed{g(z) = \int \log(z-s)\rho_V(s)ds}$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ with

$$g_+(x) + g_-(x) = V(x) + \ell, \qquad x \in \mathrm{supp}(\mu_V),$$

$$g_+(x) + g_-(x) \leq V(x) + \ell, \qquad x \in \mathbb{R},$$

$$g_+(x) - g_-(x) = 2\pi i \int_x^{+\infty} \rho_V(s)ds, \qquad x \in \mathbb{R}.$$

Ultimate goal

- Extend all these results to other matrix ensembles where eigenvalues have determinantal structure
  - Random matrices with external source

$$\frac{1}{Z_n} e^{-n\,\mathrm{Tr}(V(M)-AM)} dM$$

  - Coupled random matrices (two matrix model)

$$\frac{1}{Z_n} e^{-n\,\mathrm{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} dM_1 dM_2$$

- Find extensions / analogues of
  - Orthogonal polynomials
  - Riemann-Hilbert problem
  - Equilibrium problem

External source model

$$\frac{1}{Z_n} e^{-n\,\mathrm{Tr}(V(M)-AM)} dM$$

- A **is given Hermitian matrix (the extenal source)**
- **Because of the Harish-Chandra/Itzykson-Zuber integral we can integrate out eigenvectors of** M
  - **Suppose eigenvalues** $a_1, \ldots, a_n$ **of** A **are all distinct. Then eigenvalues have joint p.d.f.**

$$\frac{1}{Z_n} \det \left[e^{n a_i x_j} \right]_{1 \le i,j \le n} \prod_{1 \le j < k \le n} (x_k - x_j) \prod_{k=1}^{n} e^{-nV(x_k)}$$

Average characteristic polynomial

- Let P_n be the average characteristic polynomial

$$P_n(x) = \mathbb{E}\left[xI_n - M\right]$$

- Suppose $a_1, \ldots, a_r$ are distinct eigenvalues of A with multiplicities $n_1, \ldots, n_r$.
  - Then P_n is the monic polynomial of degree n that satisfies

$$\int_{-\infty}^{\infty} P_n(x) x^j e^{-n(V(x)-a_k x)} dx = 0, \qquad j = 0, \ldots, n_k{-}1, \quad k = 1, \ldots, r$$

- This is an example of **multiple orthogonality**

Multiple orthogonal polynomials

- Assume we are given
  - $r \geq 2$ **weight functions** $w_1, \ldots, w_r$ **on** $\mathbb{R}$
  - **a multi-index** $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$
- **The** **multiple orthogonal polynomial** **(MOP)** $P_{\vec{n}}$ **satisfies**

$$\int_{-\infty}^{\infty} P_{\vec{n}}(x)\, x^j\, w_k(x)\, dx = 0, \qquad \text{for } j = 0, \ldots, n_k - 1, \quad k = 1, \ldots, r.$$

$$P_{\vec{n}}(x) = x^n + \cdots, \qquad \text{where } n = |\vec{n}| = n_1 + \cdots + n_r$$

  - **Existence and uniqueness is not always guaranteed.**

MOP ensemble

- **Assume p.d.f. on $\mathbb{R}^n$ of the form**

$$\frac{1}{Z_n} \det \left[f_j(x_k)\right]_{j,k=1,\dots,n} \prod_{1 \leq j < k \leq n} (x_k - x_j)$$

with $n = |\vec{n}|$ **and** $\operatorname{span}\{f_1, \dots, f_n\} =$

$$\operatorname{span}\{\, x^j w_k(x) \mid j = 0, \dots, n_k - 1,\ k = 1, \dots, r\}$$

- **Then MOP $P_{\vec{n}}$ exists, is unique, and**

$$P_{\vec{n}}(x) = \mathbb{E}\left[\prod_{j=1}^{n} (x - x_j)\right]$$

Correlation kernel

- **MOP ensemble**

$$\frac{1}{Z_n} \det \left[f_j(x_k) \right]_{j,k=1,\ldots,n} \prod_{1 \le j < k \le n} (x_k - x_j)$$

is a determinantal point process.

  - **There is a kernel K_n so that all k point correlation functions are given by determinants**

$$\det \left[K_n(x_i, x_j) \right]_{i,j=1,\ldots,k}$$

- K_n **is constructed out of MOPs and certain dual functions**

Riemann-Hilbert problem

- **Find** $(r+1)\times(r+1)$ **matrix valued function** Y **so that**
  - **(1)** $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{(r+1)\times(r+1)}$ **is analytic**
  - **(2)** Y **has limiting values** $Y_\pm$ **on** $\mathbb{R}$**, satisfying**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & \cdots & w_r(x) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \textbf{for } x \in \mathbb{R},$$

  - **(3)** $Y(z) = (I + O(1/z))\,\mathrm{diag}\begin{pmatrix} z^n & z^{-n_1} & \cdots & z^{-n_r}\end{pmatrix}$ **as** $z \to \infty$**.**

- **Then** **Van Assche, Geronimo, K (2001)**

$$P_{\vec{n}}(x) = Y_{1,1}(x)$$

Christoffel-Darboux formula

- The correlation kernel of MOP ensemble is

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & \cdots & w_r(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $x, y \in \mathbb{R}$.

  - Y contains MOPs
  - The inverse matrix Y^{-1} contains the dual functions
  - The formula is based on a Christoffel-Darboux formula for MOPs

Daems, K (2004)

- RH problem is also useful for asymptotic analysis

External source model

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M) - AM)} dM$$

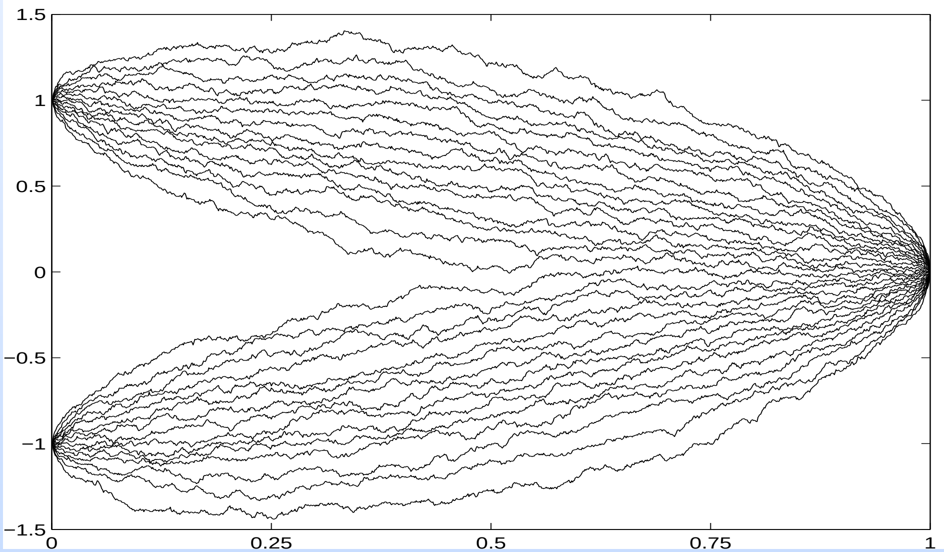
- Special case $V(M) = \frac{1}{2}M^2$
- Gaussian model with external source is equivalent with
  - $M = M_0 + A$ where M_0 is GUE matrix (deformed GUE)
  - Non-intersecting Brownian paths with several starting points and one ending point

Non-intersecting Brownian motion

- Assume r different starting points $a_1, \dots, a_r$, and one ending point at 0.
- The positions of the Brownian paths at time $t \in [0, T]$ is

$$w_j(x) = e^{-\frac{T}{2t(T-t)}x^2 + \frac{a_j}{t}x}, \qquad j = 1, \dots, r,$$

and multi-index $(n_1, \dots, n_r)$ if n_j of the paths start at a_j

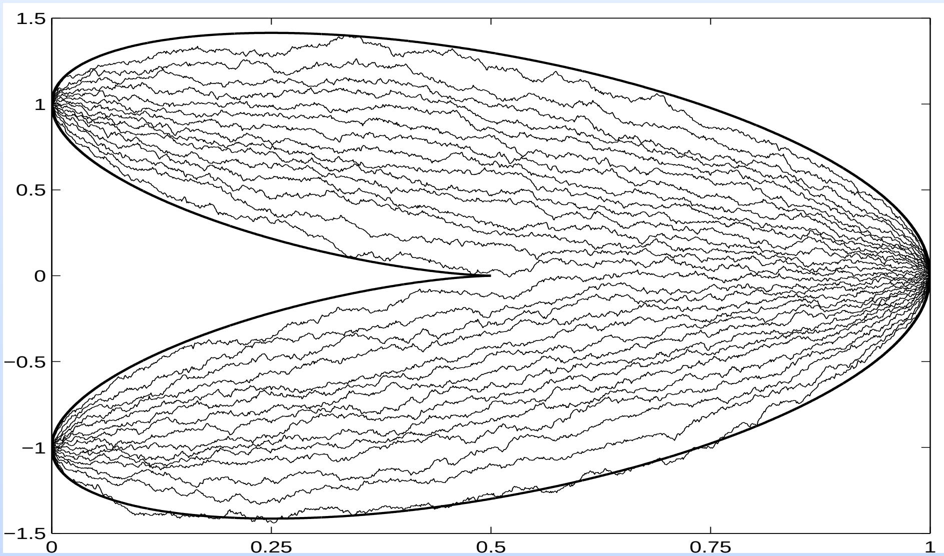


Non-intersecting Brownian motion

- **Two starting positions $\pm a$ and one endpoint at 0**
- **MOP ensemble with two weights**

$$\exp\left(-\frac{T}{2t(T-t)}x^2 \pm \frac{a}{t}x\right)$$

- **Rescale time variables $T \mapsto 1/n$, $t \mapsto t/n$, so that $0 < t < 1$.**



Large n behavior

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M) - AM)} dM$$

- **Assume n is even and**

$$A = \operatorname{diag}(\underbrace{a, \ldots, a}_{n/2 \text{ times}}, \underbrace{-a, \ldots, -a}_{n/2 \text{ times}})$$

Global eigenvalue behavior

- Limiting mean eigenvalue density is

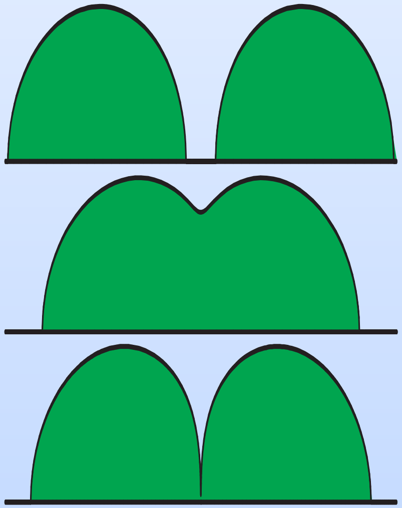
$$\rho(x;a) = \frac{1}{\pi}\,\mathrm{Im}\,\xi_1(x)$$

where $\xi_1(x)$ is a solution of the Pastur equation   Pastur (1972)

$$\xi^3 - x\xi^2 + (1-a^2)\xi + a^2 x = 0$$

Three cases

- For $a > 1$: two intervals
- For $0 < a < 1$: one interval
- For $a = 1$: transition with with density $\boxed{\rho(x) \approx c|x|^{1/3}}$ near $x = 0$



RH problem

- **Steepest descent analysis of 3×3 RH problem**
  - Y **is analytic in $\mathbb{C} \setminus \mathbb{R}$ with**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-n(\frac{1}{2}x^2 - ax)} & e^{-n(\frac{1}{2}x^2 + ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R},$$

  - $Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ **as $z \to \infty$.**

- **We can analyze the RH problem in all three cases**
  - **We do not need an equilibrium problem, since we have the Pastur equation.**

Local eigenvalue behavior

- We find the usual **sine kernel** in the bulk and the **Airy kernel** at the regular edge points
- New family of kernels for the critical case $a = 1$.
  - Pearcey ODEs

$$p'''(x) = xp(x) - sp'(x) \quad \text{and} \quad q'''(y) = yq(y) + sq'(y)$$

  - Double scaling limit at $x^* = 0$ are the **Pearcey kernels**

$$\lim_{n\to\infty} \frac{1}{n^{3/4}} K_n\left(\frac{x}{n^{3/4}}, \frac{y}{n^{3/4}}; a = 1 + \frac{s}{2\sqrt{n}}\right)$$
$$= \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - sp(x)q(y)}{x - y}$$

Brézin, Hikami (1998), Tracy, Widom (2006)

Adler, Van Moerbeke (2007), Okounkov, Reshetikhin (2007)

Bleher, K (2007)

External source model

$$\frac{1}{Z_n} e^{-n\,\mathrm{Tr}(V(M)-AM)}\,dM, \qquad A = \mathrm{diag}(\underbrace{a,\ldots,a}_{n/2 \textbf{ times}}, \underbrace{-a,\ldots,-a}_{n/2 \textbf{ times}})$$

- **How to analyze the RH problem for more general V**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-n(V(x)-ax)} & e^{-n(V(x)+ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \textbf{ for } x \in \mathbb{R},$$

$$Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix} \quad \textbf{as } z \to \infty.$$

- **We do not have an algebraic equation, but there is an equilibrium problem if V is an even polynomial ...**

Vector equilibrium problem

- **Minimize the energy functional**

$$\iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y)$$
$$- \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y)$$
$$+ \int (V(x) - a|x|)\, d\mu_1(x)$$

over pairs (μ_1, μ_2) of measures, where

- μ_1 **is on** $\mathbb{R}$ **with** $\int d\mu_1 = 1$**,**
- μ_2 **is on** $i\mathbb{R}$ **with** $\int d\mu_2 = 1/2$**,**
- $\mu_2 \leq \sigma$**, where** σ **has constant density** $\dfrac{d\sigma}{|dz|} = \dfrac{a}{\pi}$

Structure of minimizer

- There is a unique minimizer (μ_1, μ_2)
  - The support of μ_1 is a **finite union of intervals**
  - The support of μ_2 is **full imaginary axis**
  - The constraint σ for μ_2 is active along symmetric interval around 0, which can be empty

$$\text{supp}(\sigma - \mu_2) = (-i\infty, -ic] \cup [ic, i\infty), \qquad c \geq 0$$

Theorem

- The density of the measure μ_1 is the limiting mean eigenvalue density

$$\frac{d\mu_1(x)}{dx} = \lim_{n\to\infty} \frac{1}{n} K_n(x,x)$$

- We find the usual **sine kernel** in the bulk and the **Airy kernel** at the regular edge points

  Bleher, Delvaux, K (2010)

  - From equilibrium problem we obtain two g-functions

$$g_j(z) = \int \log(z-s)\,d\mu_j(s), \qquad j = 1,2$$

that satisfy a number of (in)equalities

$$g_{1,+}(x) + g_{1,-}(x) - g_2(x) = V(x) + \ell, \qquad x \in \operatorname{supp}(\mu_1),$$

$$g_{2,+}(x) + g_{2,-}(x) - g_1(x) = 0, \qquad x \in \operatorname{supp}(\sigma - \mu_2),$$

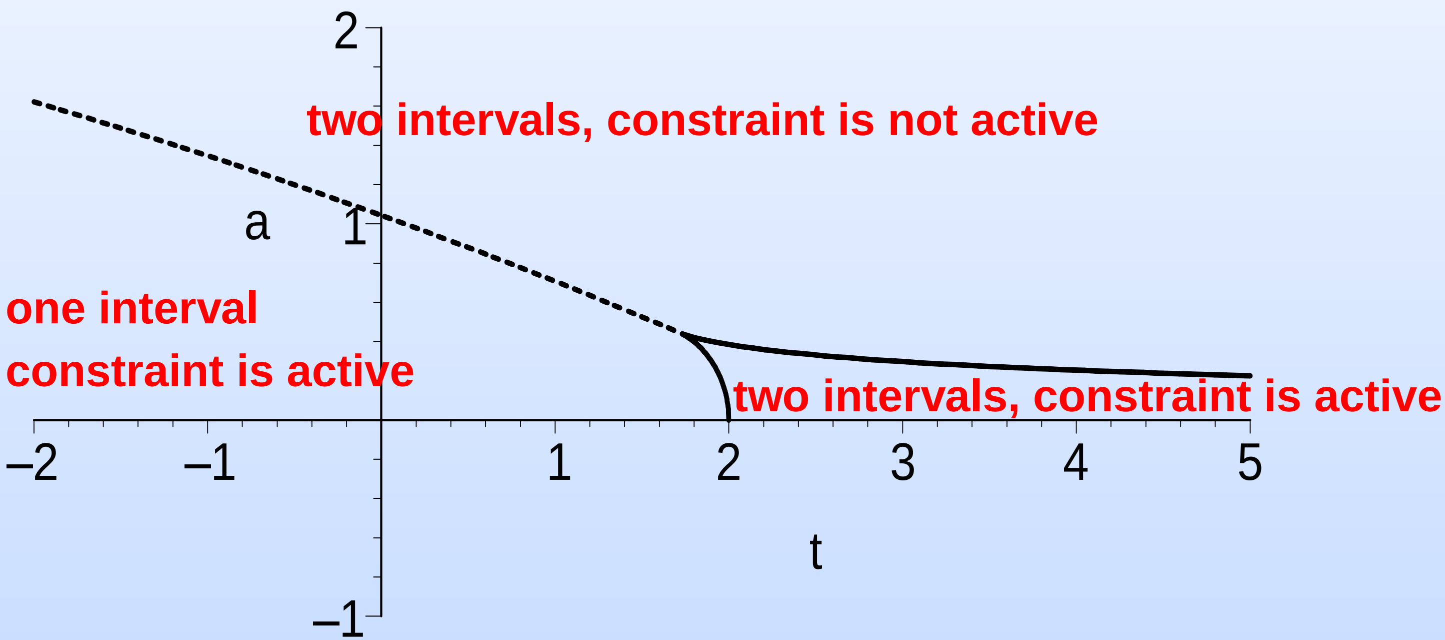
that are used in the steepest descent analysis of RH problem.

Quartic potential

- **We can completely analyze the external source model with quartic potential**

$$V(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

- **Phase diagram in ta-plane**



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