More about extreme eigenvalues of perturbed random matrices

F. Benaych-Georges – A. Guionnet – M. Maïda

LPMA, Univ Paris 6 – UMPA, ENS Lyon – LM Orsay, Univ Paris-Sud

Conference on Random Matrices – ANR GranMa
Chevaleret - June 2010

Outline of the talk

Outline of the talk

- Presentation of the models

Outline of the talk

- Presentation of the models
- Recall on almost sure convergence

Outline of the talk

- Presentation of the models
- Recall on almost sure convergence
- Fluctuations far from the bulk

Outline of the talk

- Presentation of the models
- Recall on almost sure convergence
- Fluctuations far from the bulk
- Fluctuations near the bulk

Outline of the talk

- Presentation of the models
- Recall on almost sure convergence
- Fluctuations far from the bulk
- Fluctuations near the bulk
- Large deviation principle

Presentation of the models

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

$$(H1) \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_X,$$

with μ_X compactly supported,

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

$$(H1) \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_X, \lambda_1^n \longrightarrow a, \lambda_n^n \longrightarrow b$$

with μ_X compactly supported, with edges of support a and b.

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

$$(H1) \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_X, \lambda_1^n \longrightarrow a, \lambda_n^n \longrightarrow b$$

with μ_X compactly supported, with edges of support a and b.

R_n finite rank perturbation

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

$$(H1) \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_X, \lambda_1^n \longrightarrow a, \lambda_n^n \longrightarrow b$$

with μ_X compactly supported, with edges of support a and b.

R_n finite rank perturbation

$$\widetilde{X_n} = X_n + R_n = X_n + \sum_{j=1}^{r} \theta_i G_i^n (G_i^n)^*,$$

with $\sqrt{n} G_i^n$ vectors with iid entries with law ν satisfying log-Sobolev

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

$$(H1) \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_X, \lambda_1^n \longrightarrow a, \lambda_n^n \longrightarrow b$$

with μ_X compactly supported, with edges of support a and b.

R_n finite rank perturbation

$$\widetilde{X_n} = X_n + R_n = X_n + \sum_{j=1}^{r} \theta_i U_i^n (U_i^n)^*,$$

with $\sqrt{n} G_i^n$ vectors with iid entries with law ν satisfying log-Sobolev
(or U_i^n orthonormalized version of the vectors G_i^n)

Presentation of the models

X_n deterministic self-adjoint with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$

$$(H1) \quad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_X, \lambda_1^n \longrightarrow a, \lambda_n^n \longrightarrow b$$

with μ_X compactly supported, with edges of support a and b.

R_n finite rank perturbation

$$\widetilde{X_n} = X_n + R_n = X_n + \sum_{j=1}^{r} \theta_i U_i^n (U_i^n)^*,$$

with $\sqrt{n} G_i^n$ vectors with iid entries with law ν satisfying log-Sobolev (or U_i^n orthonormalized version of the vectors G_i^n) and

$$\theta_1 \geqslant \cdots \geqslant \theta_{r_0} > 0 > \theta_{r_0+1} \geqslant \cdots \geqslant \theta_r.$$

Almost sure convergence of extreme eigenvalues

Almost sure convergence of extreme eigenvalues

Main tool :

$$f_n(z) = det\left(\left[G^n_{i,j}(z)\right]^r_{i,j=1} - diag\left(\theta_1^{-1}, \ldots, \theta_r^{-1}\right)\right),$$

with

$$G^n_{i,j}(z) = \langle U^n_i, (z - X_n)^{-1} U^n_j \rangle.$$

Almost sure convergence of extreme eigenvalues

Main tool :

$$f_n(z) = det\left(\left[G^n_{i,j}(z)\right]^r_{i,j=1} - diag\left(\theta_1^{-1}, \dots, \theta_r^{-1}\right)\right),$$

with

$$G^n_{i,j}(z) = \langle U^n_i, (z - X_n)^{-1} U^n_j \rangle.$$

Key point :

$$G^n_{i,j}(z) \longrightarrow \mathbf{1}_{i=j} G_{\mu_X}(z) := \mathbf{1}_{i=j} \int \frac{1}{z-x} d\mu_X(x)$$

Almost sure convergence of extreme eigenvalues

Main tool :

$$f_n(z) = det\left(\left[G^n_{i,j}(z)\right]^r_{i,j=1} - diag\left(\theta_1^{-1}, \ldots, \theta_r^{-1}\right)\right),$$

with

$$G^n_{i,j}(z) = \langle U^n_i, (z - X_n)^{-1} U^n_j \rangle.$$

Key point :

$$G^n_{i,j}(z) \longrightarrow \mathbf{1}_{i=j} G_{\mu_X}(z) := \mathbf{1}_{i=j} \int \frac{1}{z-x} d\mu_X(x)$$

$$f_n(z) \longrightarrow \prod_{i=1}^{r} \left(G_{\mu_X}(z) - \frac{1}{\theta_i} \right)$$

Almost sure convergence of extreme eigenvalues

Almost sure convergence of extreme eigenvalues

We define

$$\rho_\theta := \begin{cases} G_{\mu_X}^{-1}(1/\theta) & \text{if } \theta \in (-\infty, \underline{\theta}) \cup (\overline{\theta}, +\infty), \\ a & \text{if } \theta \in [\underline{\theta}, 0) \\ b & \text{if } \theta \in (0, \overline{\theta}] \end{cases}$$

Almost sure convergence of extreme eigenvalues

We define

$$\rho_\theta := \begin{cases} G_{\mu_X}^{-1}(1/\theta) & \text{if } \theta \in (-\infty, \underline{\theta}) \cup (\overline{\theta}, +\infty), \\ a & \text{if } \theta \in [\underline{\theta}, 0) \\ b & \text{if } \theta \in (0, \overline{\theta}] \end{cases}$$

and almost sure convergence of the extreme eigenvalues is governed by

Theorem

For all $i \in \{1, \ldots, r_0\}$ we have

$$\widetilde{\lambda}_i^n \xrightarrow{a.s.} \rho_{\theta_i}$$

and for all $i > r_0$,

$$\widetilde{\lambda}_i^n \xrightarrow{a.s.} b.$$

Gaussian fluctuations outside the bulk

Gaussian fluctuations outside the bulk

Let $\alpha_1 > \cdots > \alpha_q > 0$ be the different values of the θ_i's such that $\rho_{\theta_i} > b$.

Gaussian fluctuations outside the bulk

Let $\alpha_1 > \cdots > \alpha_q > 0$ be the different values of the θ_i's such that $\rho_{\theta_i} > b$.
For each j, let I_j be the set of indices i so that $\theta_i = \alpha_j$. Set $k_j = |I_j|$.

Gaussian fluctuations outside the bulk

Let $\alpha_1 > \cdots > \alpha_q > 0$ be the different values of the θ_i's such that $\rho_{\theta_i} > b$.
For each j, let I_j be the set of indices i so that $\theta_i = \alpha_j$. Set $k_j = |I_j|$.

Theorem

The random vector

$$\left(\gamma_i := \sqrt{n}(\widetilde{\lambda}_i^n - \rho_{\theta_i}), i \in I_j\right)_{1 \leqslant j \leqslant q}$$

Gaussian fluctuations outside the bulk

Let $\alpha_1 > \cdots > \alpha_q > 0$ be the different values of the θ_i's such that $\rho_{\theta_i} > b$.
For each j, let I_j be the set of indices i so that $\theta_i = \alpha_j$. Set $k_j = |I_j|$.

Theorem

The random vector

$$\left(\gamma_i := \sqrt{n}(\widetilde{\lambda}_i^n - \rho_{\theta_i}), i \in I_j\right)_{1 \leqslant j \leqslant q}$$

converges in law to the eigenvalues of $(c_j M_j)_{1 \leqslant j \leqslant q}$ *with independent matrices* $M_j \in$*GOE*(k_j)

Gaussian fluctuations outside the bulk

Let $\alpha_1 > \cdots > \alpha_q > 0$ be the different values of the θ_i's such that $\rho_{\theta_i} > b$.
For each j, let I_j be the set of indices i so that $\theta_i = \alpha_j$. Set $k_j = |I_j|$.

Theorem

The random vector

$$\left(\gamma_i := \sqrt{n}(\widetilde{\lambda}_i^n - \rho_{\theta_i}), i \in I_j\right)_{1 \leqslant j \leqslant q}$$

converges in law to the eigenvalues of $(c_j M_j)_{1 \leqslant j \leqslant q}$ with independent matrices $M_j \in$GOE(k_j) (or $M_j + D_j$ depending on $\kappa_4(\nu)$).

Fluctuations outside the bulk : sketch of proof

Fluctuations outside the bulk : sketch of proof

We have to study, for $\rho_n := \rho_\alpha + \frac{x}{\sqrt{n}}$,

$$M_{i,j}^n(\alpha, x) := \quad \left(G_{i,j}^n(\rho_n) - \frac{1}{\alpha} 1_{i=j} \right)$$

Fluctuations outside the bulk : sketch of proof

We have to study, for $\rho_n := \rho_\alpha + \frac{x}{\sqrt{n}}$,

$$M^n_{i,j}(\alpha, x) := \sqrt{n}\left(G^n_{i,j}(\rho_n) - \frac{1}{\alpha}1_{i=j}\right)$$

Fluctuations outside the bulk : sketch of proof

We have to study, for $\rho_n := \rho_\alpha + \frac{x}{\sqrt{n}}$,

$$M^n_{i,j}(\alpha, x) := \sqrt{n}\left(G^n_{i,j}(\rho_n) - \frac{1}{\alpha} 1_{i=j} \right) =: M^{n,1}_{i,j}(x)$$

where

$$M^{n,1}_{i,j}(x) \quad := \quad \sqrt{n}\left(\langle G^n_i, (\rho_n - X_n)^{-1} G^n_j \rangle - 1_{i=j} \frac{1}{n} \mathrm{tr}((\rho_n - X_n)^{-1}) \right),$$

Fluctuations outside the bulk : sketch of proof

We have to study, for $\rho_n := \rho_\alpha + \frac{x}{\sqrt{n}}$,

$$M^n_{i,j}(\alpha, x) := \sqrt{n}\left(G^n_{i,j}(\rho_n) - \frac{1}{\alpha} 1_{i=j} \right) =: M^{n,1}_{i,j}(x) + M^{n,2}_{i,j}(x)$$

where

$$\begin{aligned} M^{n,1}_{i,j}(x) &:= \sqrt{n}\left(\langle G^n_i, (\rho_n - X_n)^{-1} G^n_j \rangle - 1_{i=j} \frac{1}{n} \mathrm{tr}((\rho_n - X_n)^{-1}) \right), \\ M^{n,2}_{i,j}(x) &:= 1_{i=j} \sqrt{n} \left(\frac{1}{n}\mathrm{tr}((\rho_n - X_n)^{-1}) - \frac{1}{n}\mathrm{tr}((\rho_\alpha - X_n)^{-1}) \right), \end{aligned}$$

Fluctuations outside the bulk : sketch of proof

We have to study, for $\rho_n := \rho_\alpha + \frac{x}{\sqrt{n}}$,

$$M^n_{i,j}(\alpha, x) := \sqrt{n}\left(G^n_{i,j}(\rho_n) - \frac{1}{\alpha}1_{i=j}\right) =: M^{n,1}_{i,j}(x) + M^{n,2}_{i,j}(x) + M^{n,3}_{i,j}(x)$$

where

$$\begin{aligned}
M^{n,1}_{i,j}(x) &:= \sqrt{n}\left(\langle G^n_i, (\rho_n - X_n)^{-1} G^n_j\rangle - 1_{i=j}\frac{1}{n}\mathrm{tr}((\rho_n - X_n)^{-1})\right), \\
M^{n,2}_{i,j}(x) &:= 1_{i=j}\sqrt{n}\left(\frac{1}{n}\mathrm{tr}((\rho_n - X_n)^{-1}) - \frac{1}{n}\mathrm{tr}((\rho_\alpha - X_n)^{-1})\right), \\
M^{n,3}_{i,j}(x) &:= 1_{i=j}\sqrt{n}\left(\frac{1}{n}\mathrm{tr}((\rho_\alpha - X_n)^{-1})) - G_{\mu_X}(\rho_\alpha)\right).
\end{aligned}$$

Non universality of the fluctuations near the bulk

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i*'s is critical,*

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i's is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b are at distance at most $n^{-1+\varepsilon}$ of the extreme eigenvalues of X_n, for any $\varepsilon > 0$.

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i's is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b are at distance at most $n^{-1+\varepsilon}$ of the extreme eigenvalues of X_n, for any $\varepsilon > 0$.

Rough explanation : for fixed values of the θ_i's,

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i's is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b are at distance at most $n^{-1+\varepsilon}$ of the extreme eigenvalues of X_n, for any $\varepsilon > 0$.

Rough explanation : for fixed values of the θ_i's, we have a repulsion phenomenon from the eigenvalues of X_n at the edge.

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i's is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b are at distance at most $n^{-1+\varepsilon}$ of the extreme eigenvalues of X_n, for any $\varepsilon > 0$.

Rough explanation : for fixed values of the θ_i's, we have a repulsion phenomenon from the eigenvalues of X_n at the edge.

- If the repulsion is very strong, the extreme ev of $\widetilde{X}_n$ converge away from the bulk.

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i's is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b are at distance at most $n^{-1+\varepsilon}$ of the extreme eigenvalues of X_n, for any $\varepsilon > 0$.

Rough explanation : for fixed values of the θ_i's, we have a repulsion phenomenon from the eigenvalues of X_n at the edge.

- If the repulsion is very strong, the extreme ev of $\widetilde{X}_n$ converge away from the bulk.
- If the repulsion is milder, the extreme ev of $\widetilde{X}_n$ stick to the edge of the bulk.

Non universality of the fluctuations near the bulk

Theorem

Under additional hypotheses, if none of the θ_i's is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b are at distance at most $n^{-1+\varepsilon}$ of the extreme eigenvalues of X_n, for any $\varepsilon > 0$.

Rough explanation : for fixed values of the θ_i's, we have a repulsion phenomenon from the eigenvalues of X_n at the edge.

- If the repulsion is very strong, the extreme ev of $\widetilde{X}_n$ converge away from the bulk.
- If the repulsion is milder, the extreme ev of $\widetilde{X}_n$ stick to the edge of the bulk.
- If the repulsion is even milder, the extreme ev of $\widetilde{X}_n$ stick to the extreme ev of X_n even at the level of fluctuations.

Fluctuations near the bulk : precise statement

Fluctuations near the bulk : precise statement

If none of the θ_i's is critical,

Fluctuations near the bulk : precise statement

If none of the θ_i's is critical, if there exists $m_n = O(n^\alpha)$ with $\alpha \in (0,1)$, $\eta, \eta' > 0$ such that for any $\delta > 0$,

$$\sum_{i=m_n+1}^{n} \frac{1}{(\lambda_r - \lambda_i)^2} \leqslant n^{2-\eta}, \quad \sum_{i=m_n+1}^{n} \frac{1}{(\lambda_r - \lambda_i)^4} \leqslant n^{4-\eta'}$$

and

$$\sum_{i=m_n+1}^{n} \frac{1}{\lambda_r - \lambda_i} \leqslant \frac{1}{\overline{\overline{\theta}}} + \delta$$

Fluctuations near the bulk : precise statement

If none of the θ_i's is critical, if there exists $m_n = O(n^\alpha)$ with $\alpha \in (0,1)$, $\eta, \eta' > 0$ such that for any $\delta > 0$,

$$\sum_{i=m_n+1}^{n} \frac{1}{(\lambda_r - \lambda_i)^2} \leqslant n^{2-\eta}, \quad \sum_{i=m_n+1}^{n} \frac{1}{(\lambda_r - \lambda_i)^4} \leqslant n^{4-\eta'}$$

and

$$\sum_{i=m_n+1}^{n} \frac{1}{\lambda_r - \lambda_i} \leqslant \frac{1}{\overline{\overline{\theta}}} + \delta$$

then if I_b is the set of indices for which $\rho_{\theta_i} = b$, for any $\alpha' > \alpha$, with overwhelming probability,

$$\max_{i \in I_b} \min_k |\widetilde{\lambda}_i - \lambda_k| \leqslant n^{-1+\alpha'}.$$

Fluctuations near the bulk : sketch of proof

Fluctuations near the bulk : sketch of proof

On the set $\Omega_n := \{z / \min_k |z - \lambda_k| > n^{-1+\alpha'}\}$,

Fluctuations near the bulk : sketch of proof

On the set $\Omega_n := \{z / \min_k |z - \lambda_k| > n^{-1+\alpha'}\}$, for $i \neq j$, there is $k > 0$, so that

$$\sup_{z \in \Omega_n} |G^n_{i,j}| \leqslant n^{-\kappa}$$

with overwhelming probability.

Fluctuations near the bulk : sketch of proof

On the set $\Omega_n := \{z / \min_k |z - \lambda_k| > n^{-1+\alpha'}\}$, for $i \neq j$, there is $k > 0$, so that

$$\sup_{z \in \Omega_n} |G^n_{i,j}| \leqslant n^{-\kappa}$$

with overwhelming probability.
Therefore

$$f_n(z) = \prod_{i=1}^{r} \left(G^n_{i,i} - \frac{1}{\theta_i} \right) + O(n^{-\kappa})$$

Fluctuations near the bulk : sketch of proof

On the set $\Omega_n := \{z / \min_k |z - \lambda_k| > n^{-1+\alpha'}\}$, for $i \neq j$, there is $k > 0$, so that

$$\sup_{z \in \Omega_n} |G^n_{i,j}| \leqslant n^{-\kappa}$$

with overwhelming probability.
Therefore

$$f_n(z) = \prod_{i=1}^{r} \left(G^n_{i,i} - \frac{1}{\theta_i} \right) + O(n^{-\kappa})$$

but with overwhelming probability,

$$\sup_{z \in \Omega_n} \max_{1 \leqslant i \leqslant r} G^n_{i,i} \leqslant \frac{1}{\overline{\theta}} + \delta$$

Fluctuations near the bulk : sketch of proof

On the set $\Omega_n := \{z / \min_k |z-\lambda_k| > n^{-1+\alpha'}\}$, for $i \neq j$, there is $k > 0$, so that

$$\sup_{z\in\Omega_n} |G^n_{i,j}| \leqslant n^{-\kappa}$$

with overwhelming probability.
Therefore

$$f_n(z) = \prod_{i=1}^{r} \left(G^n_{i,i} - \frac{1}{\theta_i} \right) + O(n^{-\kappa})$$

but with overwhelming probability,

$$\sup_{z\in\Omega_n} \max_{1\leqslant i\leqslant r} G^n_{i,i} \leqslant \frac{1}{\underline{\theta}} + \delta$$

so that

$$G^n_{i,i} - \frac{1}{\theta_i} \leqslant \frac{1}{\underline{\theta}} - \frac{1}{\theta_i} + \delta < 0.$$

Possible generalisations

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Consequences :

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Consequences :

- Wigner, Wishart matrices with entries having a fourth moment (some band Hermitian matrices, non-white Wishart) : Gaussian fluctuations away from the bulk

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Consequences :

- Wigner, Wishart matrices with entries having a fourth moment (some band Hermitian matrices, non-white Wishart) : Gaussian fluctuations away from the bulk
  Cf Péché, Féral-Péché, Capitaine-Donati-Féral

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Consequences :

- Wigner, Wishart matrices with entries having a fourth moment (some band Hermitian matrices, non-white Wishart) : Gaussian fluctuations away from the bulk
  Cf Péché, Féral-Péché, Capitaine-Donati-Féral
- GUE, GOE, LUE, LOE : inheritance of Tracy-Widom laws.

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Consequences :

- Wigner, Wishart matrices with entries having a fourth moment (some band Hermitian matrices, non-white Wishart) : Gaussian fluctuations away from the bulk
  Cf Péché, Féral-Péché, Capitaine-Donati-Féral
- GUE, GOE, LUE, LOE : inheritance of Tracy-Widom laws.

Our perturbation has delocalized eigenvectors.

Possible generalisations

If the hypotheses hold in probability and X_n is independent of the perturbation, the theorems still hold.

Consequences :

- Wigner, Wishart matrices with entries having a fourth moment (some band Hermitian matrices, non-white Wishart) : Gaussian fluctuations away from the bulk
  Cf Péché, Féral-Péché, Capitaine-Donati-Féral
- GUE, GOE, LUE, LOE : inheritance of Tracy-Widom laws.

Our perturbation has delocalized eigenvectors.

Open question : fluctuations for critical θ_i's.

Large deviation principle

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i and U_i^n obtained by orthonormalization.

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i and U_i^n obtained by orthonormalization.

In the iid case, f_n depends polynomially on the entries of $K^n(z)$ with

$$(K^n(z))_{ij} := \frac{1}{n} \sum_{k=1}^{n} \frac{g_i(k) g_j(k)}{z - \lambda_k}$$

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i and U_i^n obtained by orthonormalization.

We can find H^n having the same zeroes as f_n and depending polynomially on the entries of $K^n(z)$ and C^n

$$(K^n(z))_{ij} := \frac{1}{n} \sum_{k=1}^{n} \frac{g_i(k) g_j(k)}{z - \lambda_k} \quad \text{and} \quad (C^n)_{ij} := \frac{1}{n} \sum_{k=1}^{n} g_i(k) g_j(k).$$

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i and U_i^n obtained by orthonormalization.

We can find H^n having the same zeroes as f_n and depending polynomially on the entries of $K^n(z)$ and C^n

$$(K^n(z))_{ij} := \frac{1}{n} \sum_{k=1}^{n} \frac{g_i(k) g_j(k)}{z - \lambda_k} \quad \text{and} \quad (C^n)_{ij} := \frac{1}{n} \sum_{k=1}^{n} g_i(k) g_j(k).$$

Theorem

The law of the r_0 largest eigenvalues of $\widetilde{X_n}$ satisfies a LDP in the scale n with a good rate function.

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i and U_i^n obtained by orthonormalization.

We can find H^n having the same zeroes as f_n and depending polynomially on the entries of $K^n(z)$ and C^n

$$(K^n(z))_{ij} := \frac{1}{n} \sum_{k=1}^{n} \frac{g_i(k) g_j(k)}{z - \lambda_k} \quad \text{and} \quad (C^n)_{ij} := \frac{1}{n} \sum_{k=1}^{n} g_i(k) g_j(k).$$

Theorem

The law of the r_0 largest eigenvalues of $\widetilde{X_n}$ satisfies a LDP in the scale n with a good rate function. It has a unique minimizer towards which we have almost sure convergence.

Large deviation principle

Consider the following model :
X_n diagonal, deterministic, satisfying $(H1)$.
$G = (g_1, \ldots, g_r)$ a random vector satisfying that $\mathbb{E}(e^{\alpha \sum |g_i^2|}) < \infty$ for some $\alpha > 0$ (and not charging an hyperplane)
G_i^n random vector whose entries are $1/\sqrt{n}$ times independent copies of g_i and U_i^n obtained by orthonormalization.

We can find H^n having the same zeroes as f_n and depending polynomially on the entries of $K^n(z)$ and C^n

$$(K^n(z))_{ij} := \frac{1}{n} \sum_{k=1}^{n} \frac{g_i(k) g_j(k)}{z - \lambda_k} \quad \text{and} \quad (C^n)_{ij} := \frac{1}{n} \sum_{k=1}^{n} g_i(k) g_j(k).$$

Theorem

The law of the r_0 largest eigenvalues of $\widetilde{X_n}$ satisfies a LDP in the scale n with a good rate function. It has a unique minimizer towards which we have almost sure convergence.

Remark : minimizers depend on G only through its covariance matrix.

Large deviation principle : sketch of proof I

Large deviation principle : sketch of proof I

Starting point :

$$H^n(z) = P_\Theta(K^n(z), C^n)$$

Large deviation principle : sketch of proof I

Starting point :

$$H^n(z) = P_\Theta(K^n(z), C^n)$$

First step : fix $\mathcal{K}$ a compact interval contained in (b, ∞), the law of $(K^n(z), C^n)$ on $\mathcal{C}(\mathcal{K}, H_r) \times H_r$ equipped with the uniform topology satisfies an LDP

Large deviation principle : sketch of proof I

Starting point :

$$H^n(z) = P_\Theta(K^n(z), C^n)$$

First step : fix $\mathcal{K}$ a compact interval contained in (b, ∞), the law of $(K^n(z), C^n)$ on $\mathcal{C}(\mathcal{K}, H_r) \times H_r$ equipped with the uniform topology satisfies an LDP with good rate function

$$\mathbf{I}(K(.), C) = \sup_{P,X,Y} \left\{ Tr\left(\int K'(z)P(z)dz + K(z^*)X + CY \right) - \Gamma(P, Y, X) \right\}$$

where $\Gamma(P, Y, X)$ is given by the formula

$$\Gamma(P, Y, X) = \int \Lambda \left(-\int \frac{1}{(z-x)^2} P(z)dz + \frac{1}{z^* - x} X + Y \right) d\mu_X(x)$$

and the supremum is taken over piecewise constant functions P with values in H_r and X, Y in H_r.

Large deviation principle : sketch of proof II

Large deviation principle : sketch of proof II

By contraction, the law of H^n satisfies an LDP with good rate function, for a continuous$f$,

$$J_{\mathcal{K}}(f) = \inf\{\mathbf{I}(F) : F \in \mathcal{C}(\mathcal{K}, H_r) \times H_r, P_\Theta(F(z)) = f(z), \ \forall z \in \mathcal{K}\}.$$

Large deviation principle : sketch of proof II

By contraction, the law of H^n satisfies an LDP with good rate function, for a continuous$f$,

$$J_{\mathcal{K}}(f) = \inf\{\mathbf{I}(F) : F \in \mathcal{C}(\mathcal{K}, H_r) \times H_r, P_\Theta(F(z)) = f(z),\ \forall z \in \mathcal{K}\}.$$

Theorem

The law of $\widetilde{\lambda}_1^{(n)}, \ldots, \widetilde{\lambda}_m^{(n)}$ of $\widetilde{X}_n$ satisfies a LDP with good rate function L, defined for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, by

$$L(\alpha) = \begin{cases} \lim_{\varepsilon \downarrow 0} \inf_{\cup_{\gamma>0} S^{\varepsilon}_{(\alpha_1,\ldots,\alpha_{m-k}),\gamma}} J_{K_\varepsilon} & \textit{if } \alpha \in \mathbb{R}^m_\downarrow(b), \alpha_{m-k+1} = b \textit{ and} \\ & \alpha_{m-k} > b, \\ +\infty & \textit{otherwise.} \end{cases}$$

Large deviation principle : sketch of proof II

By contraction, the law of H^n satisfies an LDP with good rate function, for a continuous$f$,

$$J_{\mathcal{K}}(f) = \inf\{\mathbf{I}(F) : F \in \mathcal{C}(\mathcal{K}, H_r) \times H_r, P_\Theta(F(z)) = f(z),\ \forall z \in \mathcal{K}\}.$$

Theorem

The law of $\widetilde{\lambda}_1^{(n)}, \ldots, \widetilde{\lambda}_m^{(n)}$ of $\widetilde{X}_n$ satisfies a LDP with good rate function L, defined for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, by

$$L(\alpha) = \begin{cases} \lim_{\varepsilon \downarrow 0} \inf_{\cup_{\gamma>0} S^{\varepsilon}_{(\alpha_1,\ldots,\alpha_{m-k}),\gamma}} J_{K_\varepsilon} & \textit{if } \alpha \in \mathbb{R}^m_{\downarrow}(b), \alpha_{m-k+1} = b \textit{ and} \\ & \alpha_{m-k} > b, \\ +\infty & \textit{otherwise.} \end{cases}$$

with

$$S^{\varepsilon}_{\alpha,\gamma} \quad := \quad \left\{ f \in \mathcal{C}(K_\varepsilon) : f(z) = s.g(z) \prod_{i=1}^{m-k} (z - \alpha_i) \textit{ with } g \geqslant \gamma \right\},$$

Study of the minimizers

Study of the minimizers

L good rate function, vanishes at minimizers $(\lambda_1^*, \ldots, \lambda_m^*)$. Let k be such that $\lambda_{m-k}^* > b$ and $\lambda_{m-k+1}^* = b$. By compacity, one can find f vanishing at $(\lambda_1^*, \ldots, \lambda_{m-k}^*)$ such that $J_{K_\varepsilon}(f) = 0$ for any $\varepsilon > 0$. It also means that $f(z) = P_\Theta(K, C)$, with (K, C) minimizing $\mathbf{I}$.

Study of the minimizers

L good rate function, vanishes at minimizers $(\lambda_1^*, \dots, \lambda_m^*)$. Let k be such that $\lambda_{m-k}^* > b$ and $\lambda_{m-k+1}^* = b$. By compacity, one can find f vanishing at $(\lambda_1^*, \dots, \lambda_{m-k}^*)$ such that $J_{K_\varepsilon}(f) = 0$ for any $\varepsilon > 0$. It also means that $f(z) = P_\Theta(K, C)$, with (K, C) minimizing $\mathbf{I}$.

$$\Bigg| \mathbb{E}\left(e^{\varepsilon Tr\left(-\int \frac{1}{(z-x)^2} P(z) z + \frac{1}{z^* - x} X + Y\right) Z} \right)$$
$$- \mathbb{E}\left(1 + \varepsilon Tr\left(\left(- \int \frac{1}{(z-x)^2} P(z) dz + \frac{1}{z^* - x} X + Y \right) Z \right) \right) \Bigg| \leqslant \varepsilon^2 L,$$

Study of the minimizers

L good rate function, vanishes at minimizers $(\lambda_1^*, \ldots, \lambda_m^*)$. Let k be such that $\lambda_{m-k}^* > b$ and $\lambda_{m-k+1}^* = b$. By compacity, one can find f vanishing at $(\lambda_1^*, \ldots, \lambda_{m-k}^*)$ such that $J_{K_\varepsilon}(f) = 0$ for any $\varepsilon > 0$. It also means that $f(z) = P_\Theta(K, C)$, with (K, C) minimizing $\mathbf{I}$.

$$\left| \mathbb{E}\left(e^{\varepsilon Tr\left(-\int \frac{1}{(z-x)^2} P(z)z + \frac{1}{z^*-x} X + Y\right) Z} \right) \right.$$
$$\left. - \mathbb{E}\left(1 + \varepsilon Tr\left(\left(-\int \frac{1}{(z-x)^2} P(z)dz + \frac{1}{z^*-x} X + Y\right) Z\right)\right) \right| \leqslant \varepsilon^2 L,$$

so that

$$\Gamma(\varepsilon P, \varepsilon X, \varepsilon Y) = \varepsilon Tr\left(\int (K^*)'(z)P(z)dz + K^*(z^*)X + C^* Y\right) + O(\varepsilon^2)$$

with

$$(K^*(z))_{ij} = \int \frac{(C^*)_{ij}}{z-\lambda} d\mu_X(\lambda) \quad \text{and} \quad (C^*)_{ij} = \mathbb{E}[g_i g_j].$$

Study of the minimizers : last remark

Study of the minimizers : last remark

In the case when $(g_1, \ldots, g_r)$ are independent centered variables with variance one, one can check that $C^* = I_r$, $K^*(z) = \int \frac{1}{z-x} \mu_X(x).I_r$ and

$$H(z) = \prod_{i=1}^{r} \left(\frac{1}{\theta_i} - \int \frac{1}{z-x} \mu_X(x) \right)$$