On the Shuffling Algorithm for the Aztec Diamond

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Background

Shuffling algorithm

Warren's Process

Aztec diamond point process

Asymptotics

Borodin & Ferrari

Dyson's Brownian Minor process

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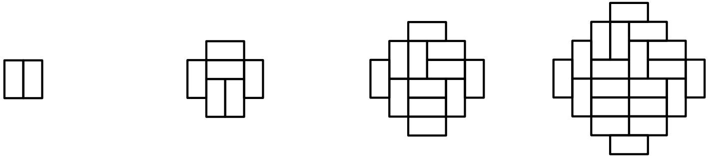
Université Catholique de Louvain, Belgium

Université Pierre et Marie Curie, 3 June 2010

The Aztec Diamond

Aztec diamonds of orders 1, 2, 3 and 4.



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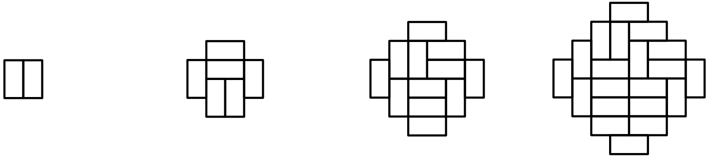
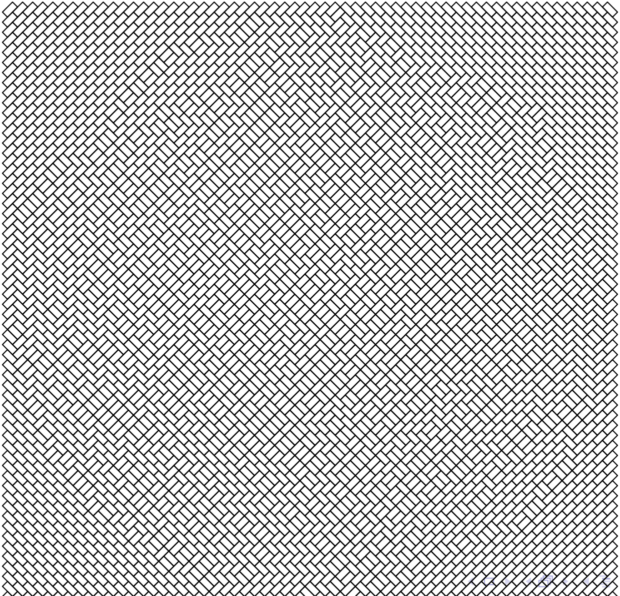


The diamond of order n can be tiled in $2^{n(n+1)/2}$ ways.
Elkies et al, '92

The Aztec Diamond



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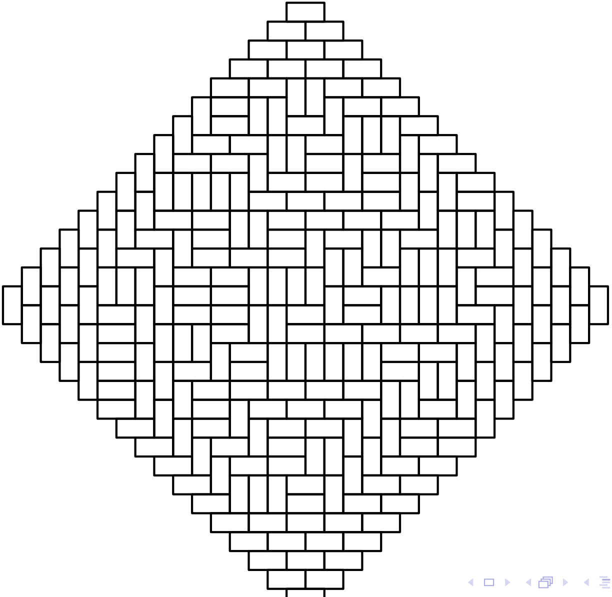

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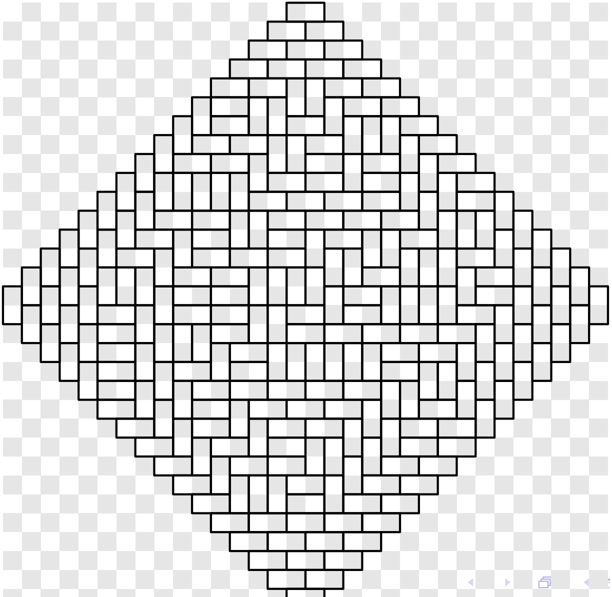

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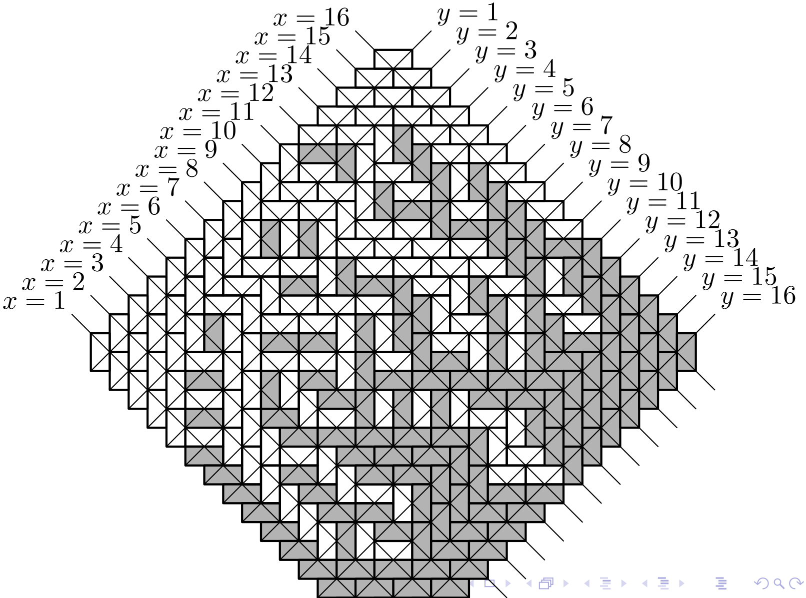

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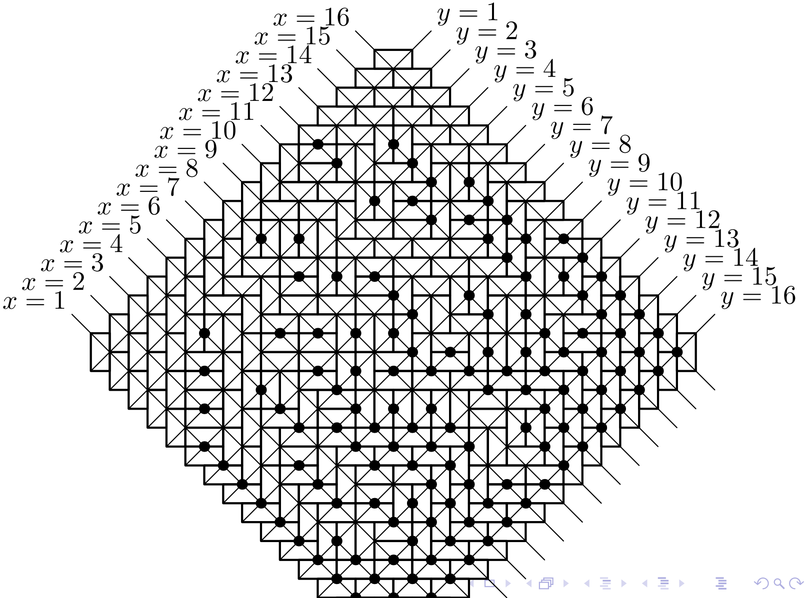
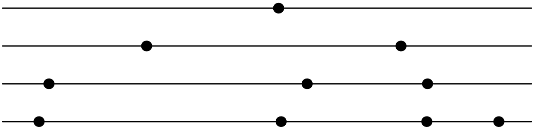


GUE Minor Process

Let H be a large GUE matrix, i.e. a random matrix with probability density $Z^{-1}e^{-\operatorname{Tr} H^2}$.
Let $H_n = [H_{i,j}]_{1\le i,j\le n}$ be the n:th minor of H.
Let H_n have eigenvalues $\lambda_1^n, \dots, \lambda_n^n$



Theorem (Johansson&N '06)

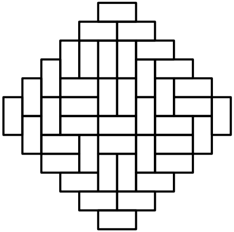
The Aztec diamond point process in a suitable rescaling converges to the GUE minor process.

1. The shuffling algorithm
2. Warren's Interlaced Brownian motions
3. Asymptotics
4. Borodin & Ferrari
5. Dyson Brownian minor process
6. Extension to Boutillier's bead kernel

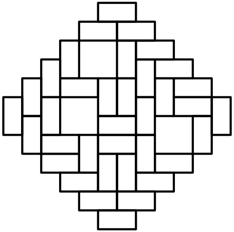
Three phases of the algorithm.

1. Delete
2. Shuffle
3. Create

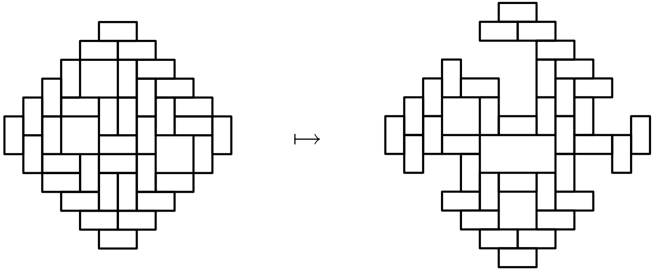
Delete



$\mapsto$



Shuffle



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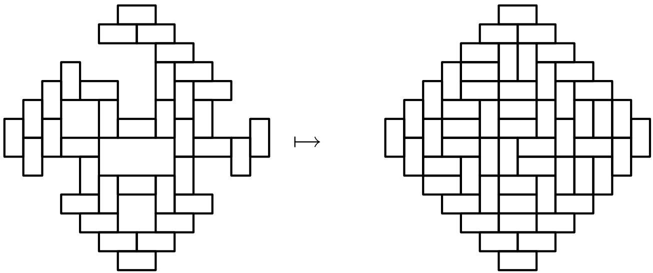
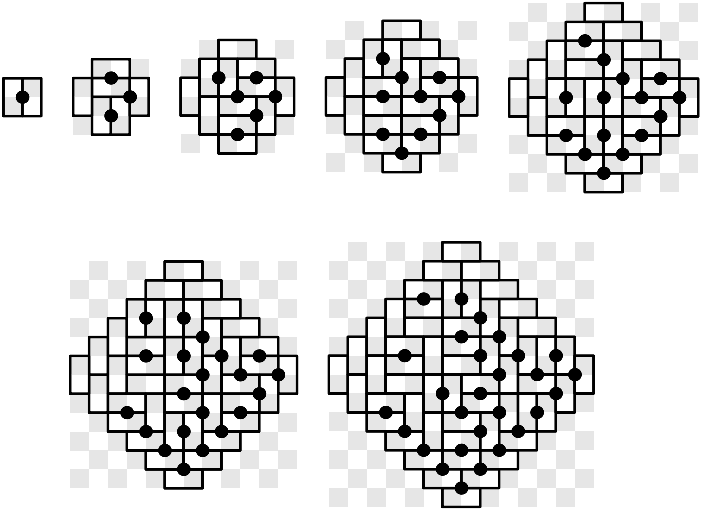
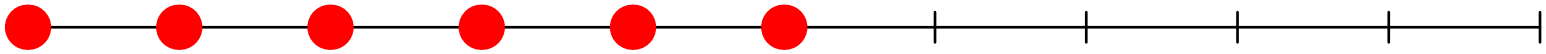
Create



Particle dynamics



TASEP with step initial condition



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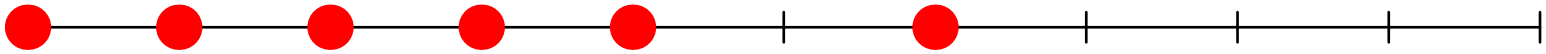

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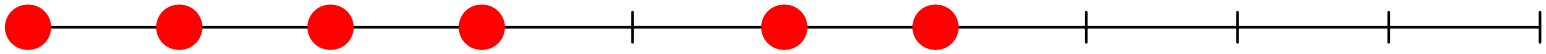
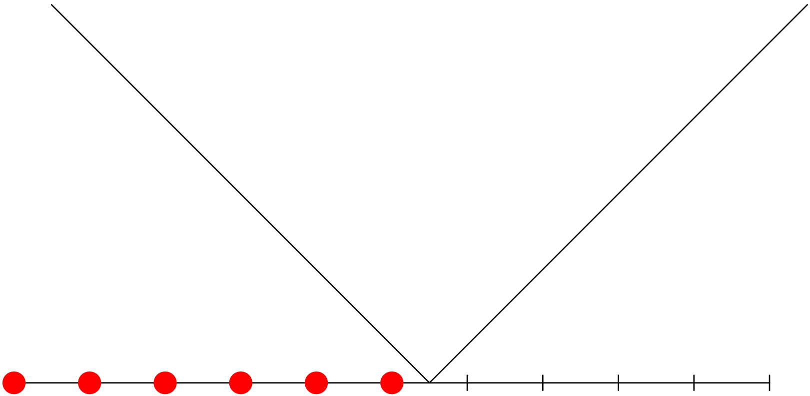


TASEP with step initial condition



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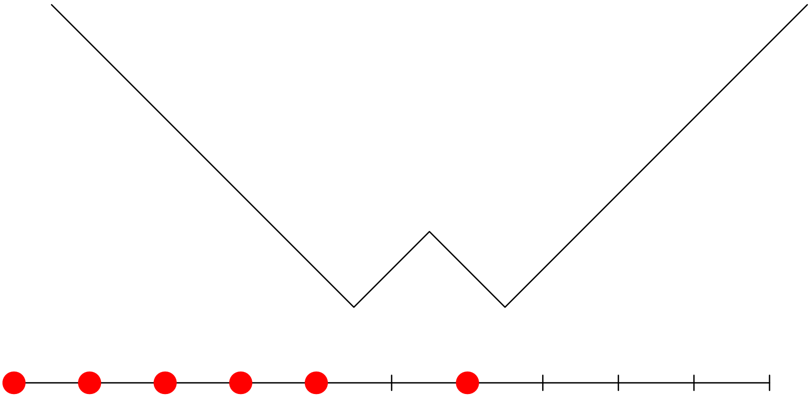
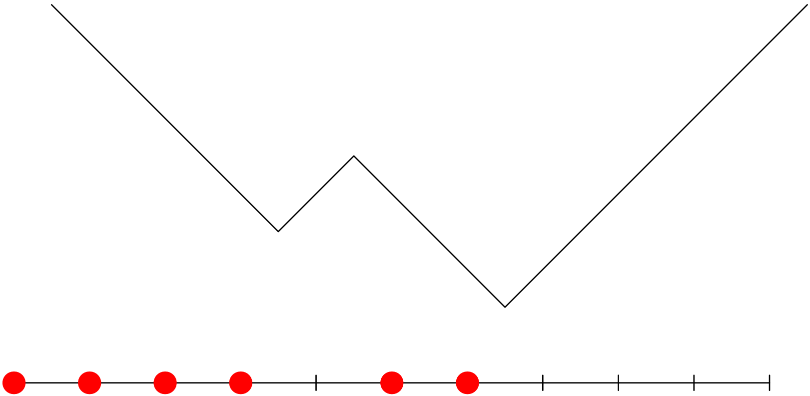
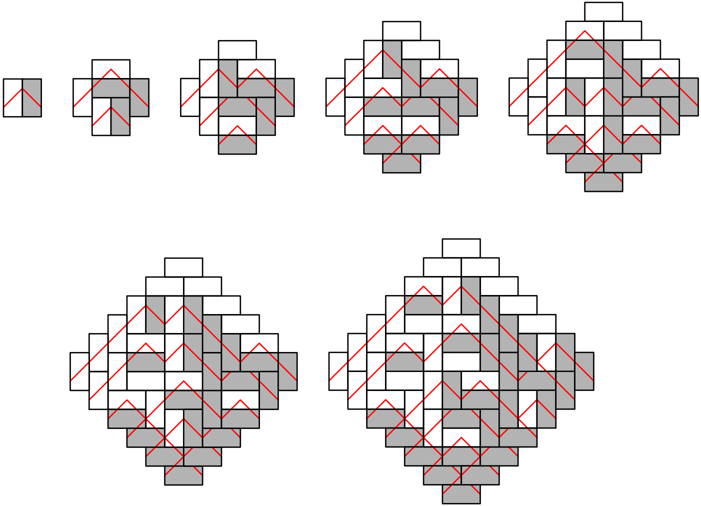


TASEP with step initial condition



Particle dynamics



Particle dynamics

$$
\begin{aligned}
x_1^1(t) &= x_1^1(t-1) + \gamma_1^1(t) \\
x_1^j(t) &= x_1^j(t-1) + \gamma_1^j(t) - \mathbf{1}\{x_1^j(t-1) + \gamma_1^j(t) = x_1^{j-1}(t-1) + 1\} \\
x_j^j(t) &= x_j^j(t-1) + \gamma_j^j(t) + \mathbf{1}\{x_j^j(t-1) + \gamma_j^j(t) = x_{j-1}^{j-1}(t-1)\} \\
x_i^j(t) &= x_i^j(t-1) + \gamma_i^j(t) - \mathbf{1}\{x_i^j(t-1) + \gamma_i^j(t) = x_j^{j-1}(t-1) + 1\} \\
&\qquad + \mathbf{1}\{x_i^j(t-1) + \gamma_i^j(t) = x_{j-1}^{j-1}(t-1)\}.
\end{aligned}
$$

Here, $\gamma_i^j(t)$ are i.i.d. fair coin tosses and initial conditions $x_i^j(j) = i$ for $j = 1, 2, \ldots$ and $1 \leq i \leq j$. At each time t,

$$
x_i^j(t) \leq x_i^{j-1}(t) \leq x_{i+1}^j(t).
$$

Aztec diamond particle dynamics

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Perform substitution

$$X_i^j(t) = x_i^j(t-j) \tag{1}$$

For all t,

$$X_i^j(t) \leq X_i^{j-1}(t) < X_{i+1}^j(t) \tag{2}$$

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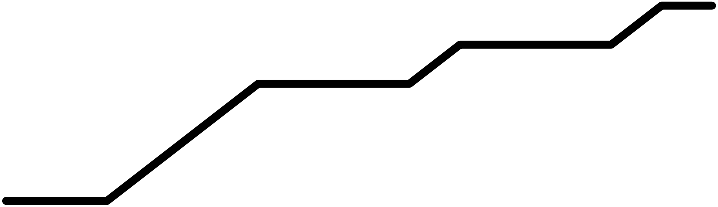

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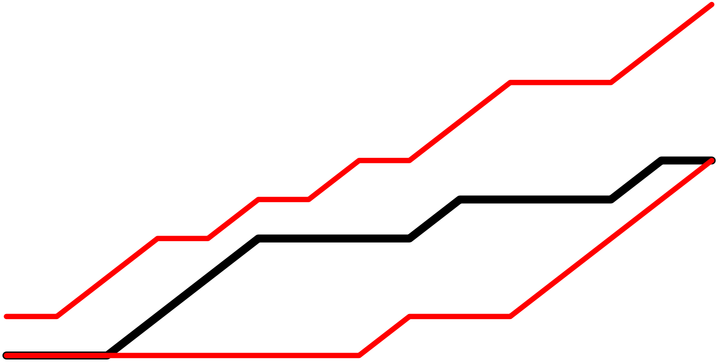

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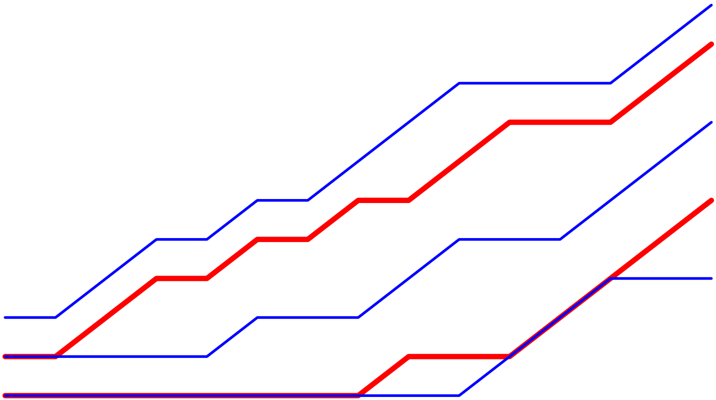
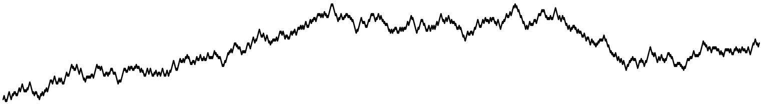


Warren's Process



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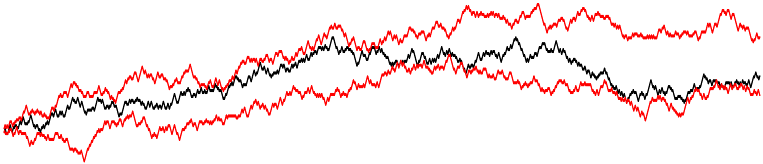
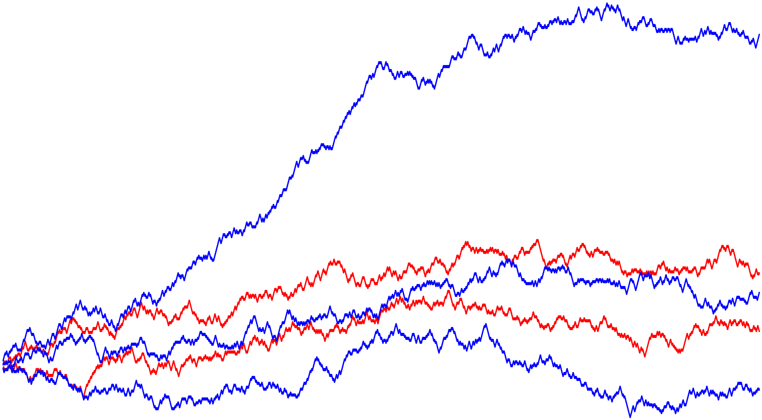


Warren's Process



Transition Density for Dyson's BM

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Let $W_n = \{x \in \mathbb{R}^n : x_1 \le x_2 \le \cdots \le x_n\}$. For $x,\ x' \in W_n$

$$p_t^{n,+}(x, x') = \frac{h_n(x')}{h_n(x)} \det\left[\varphi_t(x_i' - x_j)\right] \tag{3}$$

where

$$h_n(x) = \prod_{i<j}(x_j - x_i) \tag{4}$$

and

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \tag{5}$$

Transition Density for Warren's Process

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Let $W_{n,n+1} = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n : x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_n \leq x_{n+1}\}$.
For (x,y) and $(x',y') \in W_{n,n+1}$,

$$q_t^{n,+}((x,y),(x',y')) = \frac{h_n(y')}{h_n(y)} \det \begin{bmatrix} A_t(x,x') & B_t(x,y') \\ C_t(y,x') & D_t(y,y') \end{bmatrix} \quad (6)$$

where
$[A_t(x,x')]_{ij} = \varphi_t(x'_i - x_j)$,
$[B_t(x,y')]_{ij} = \Phi_t(y'_i - x_j) - \mathbf{1}(j \geq i)$,
$[C_t(y,x')]_{ij} = \varphi'_t(y'_i - x_j)$ and
$[D_t(y,y')]_{ij} = \varphi_t(y'_i - y_j)$.
where $\Phi_t(x) = \int_{-\infty}^{x} \phi_t(y)\, dy$.

Characterisation of Warren's process

At level n there are n components of the process: $X_1^n(t), \dots, X_n^n(t)$.
Interlacing: $X_i^n(t) \leq X_i^{n-1}(t) \leq X_{i+1}^n(t)$
Can be constructed as follows. Construct $X_1^1(t)$, it is an ordinary Brownian motion.

1. Start with X^n, a Dyson Brownian Motion of n particles.
2. Construct $X^{k+1} = (X_1^{n+1}, \dots, X_{n+1}^{n+1})$ so that (X^n, X^{n+1}) has transition densities $q_t^{n,+}$.
3. Then X^{n+1} is a DBM of $n+1$ particles.

Discrete Dyson BM

Let $\mathcal{W}_n = \{x \in \mathbb{N}^n : x_1 \leq x_2 \leq \cdots \leq x_n\}$. For x, $x' \in \mathcal{W}_n$

$$p_t^{n,+}(x, x') = \frac{h_n(x')}{h_n(x)} \det\left[\phi^t(x'_i - x_j)\right] \tag{7}$$

where

$$\phi^1(x) = \phi(x) = \begin{cases} 1/2 & \text{if } x = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and

$$\phi^t(z) = (\phi * \phi^{t-1})(z) = \sum_{x+y=z} \phi(x)\phi^{t-1}(y) \tag{9}$$

Transition Probabilities for Aztec Diamond Process

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Let
$\mathcal{W}_{n,n+1} = \{(x,y) \in \mathbb{N}^{n+1} \times \mathbb{N}^n : x_1 \leq y_1 < x_2 \leq \cdots \leq y_n < x_{n+1}\}$.
For (x,y) and $(x',y') \in W_{n,n+1}$,

$$q_t^{n,+}((x,y),(x',y')) = \frac{h_n(y')}{h_n(y)} \det \begin{bmatrix} A_t(x,x') & B_t(x,y') \\ C_t(y,x') & D_t(y,y') \end{bmatrix} \qquad (10)$$

where $[A_t(x,x')]_{ij} = \phi^t(x'_i - x_j)$,
$[B_t(x,y')]_{ij} = \Delta^{-1}\phi^t(y'_i - x_j) - \mathbf{1}(j \geq i)$,
$[C_t(y,x')]_{ij} = \Delta\phi^t(y'_i - x_j)$ and
$[D_t(y,y')]_{ij} = \phi^t(y'_i - y_j)$,
$\phi = \phi^1 = \frac{1}{2}(\delta_0 + \delta_1)$, $\Delta\phi(x) = \phi(x) - \phi(x-1)$,
$\Delta^{-1} = \sum_{y=-\infty}^{x} \phi(y)$ and $\phi^{t+1} = \phi * \phi^t$.

Asymptotics

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Let $\mathbf{X}(t)$ be Warren's process.
Let $\mathcal{X}(t)$ be the process from the shuffling algorithm.

Theorem (N '08)

The process $(X^n(t), X^{n+1}(t))$ from $\mathcal{X}$, extended by interpolation to non-integer times t, rescaled according to

$$\tilde{X}_i^n(t) = \frac{X_i^n(Nt) - \frac{1}{2}Nt}{\frac{1}{2}\sqrt{N}} \tag{11}$$

converges weakly to the process $(X^n(t), X^{n+1}(t))$ from $\mathbf{X}$ as $N \to \infty$.

Theorem (N '08)

The process $\mathcal{X}(t) = (X^1(t), \dots, X^n(t))$, *rescaled according to*

$$\tilde{X}_i^n(t) = \frac{X_i^n(Nt) - \frac{1}{2}Nt}{\frac{1}{2}\sqrt{N}} \tag{12}$$

converges weakly to $\mathbf{X}(t)$ *as* $N \to \infty$.

Remark: $\mathbf{X}(1)$ is the GUE minor process.

Conjecture

Consider the process $(\mathcal{X}(t))_{t=0,1,\dots}$ rescaled according to

$$\tilde{X}_i^n(t) = \frac{X_i^n(Nt) - \frac{1}{2}Nt}{\frac{1}{2}\sqrt{N}} \tag{13}$$

and defined by linear interpolation for non-integer values of Nt. The process $\tilde{\mathcal{X}}(t)$ converges weakly to Warren's process $\mathbf{X}(t)$ as $N \to \infty$.

Λ-chain (Sequential update)

Consider Markov operators satisfying $\Delta = \Lambda P = P^* \Lambda$.

$$\begin{array}{ccc} S^* & \xrightarrow[\Lambda]{} & S \\ \downarrow{\scriptstyle P^*} & & \downarrow{\scriptstyle P} \\ S^* & \xrightarrow[\Lambda]{} & S \end{array}$$

$$S_\Lambda = \{(x^*, x) \in S^* \times S : \Lambda(x^*, x) > 0\}$$

$$P_\Lambda((x^*, x), (y^*, y)) = \begin{cases} \frac{P(x,y)P^*(x^*,y^*)\Lambda(y^*,y)}{\Delta(x^*,y)} & \Delta(x^*, y) > 0 \\ 0 & \text{otherwize} \end{cases}$$

Δ-chain (Parallel update)

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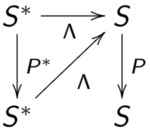
Consider Markov operators satisfying $\Delta = \Lambda P = P^*\Lambda$.



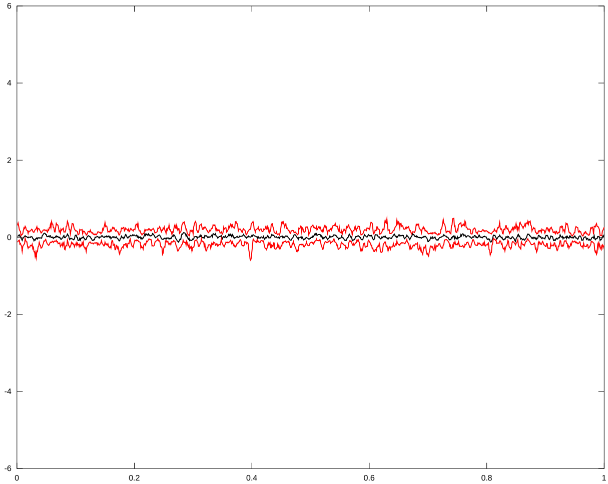
$$S_\Delta = \{(x^*, x) \in S^* \times S : \Delta(x^*, x) > 0\}$$

$$P_\Delta((x^*, x), (y^*, y)) = \frac{P(x, y)P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}$$

Dyson's Brownian Minor process

Take an $N \times N$ GUE matrix and let the elements evolve according to independent Ornstein-Uhlenbeck processes. Consider all the $N(N+1)/2$ eigenvalues of this matrix and it's minors.

- The eigenvalues on one level are Markovian and evolve as O-U processes conditioned never to intersect. [Dyson '62]
- The eigenvalues on two consecutive levels are Markovian. [Adler, vMoerbeke, N. 2010]
- All the $\binom{N}{2}$ eigenvalues together are not Markovian for $N > 2$.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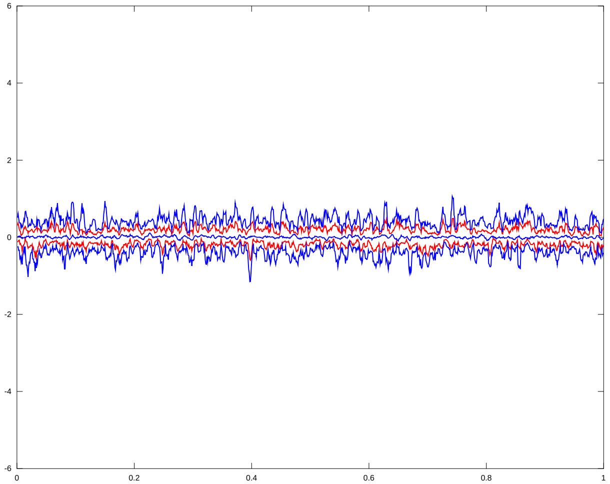
-4
-6
0
0.2
0.4
0.6
0.8
1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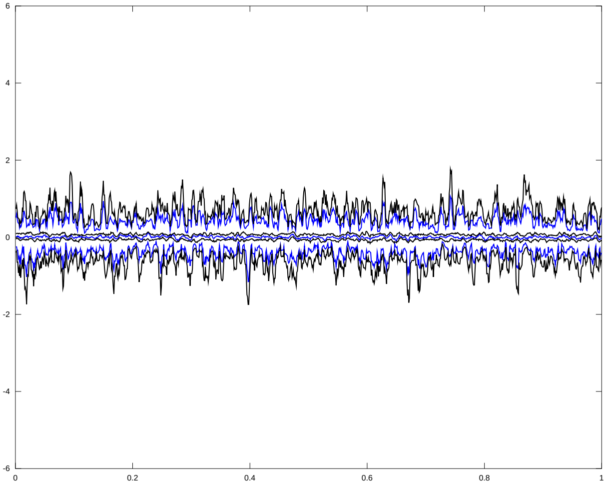
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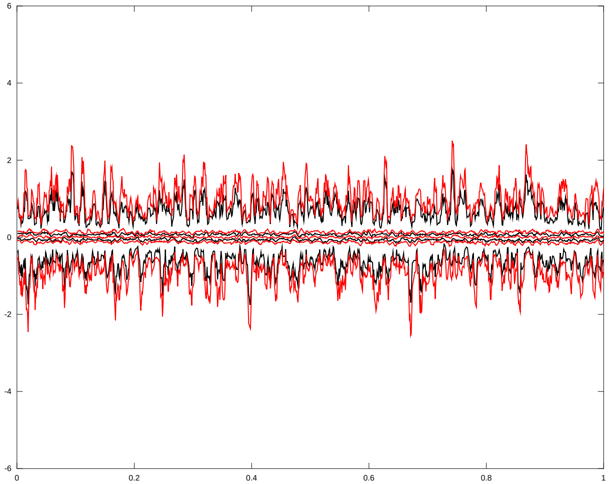
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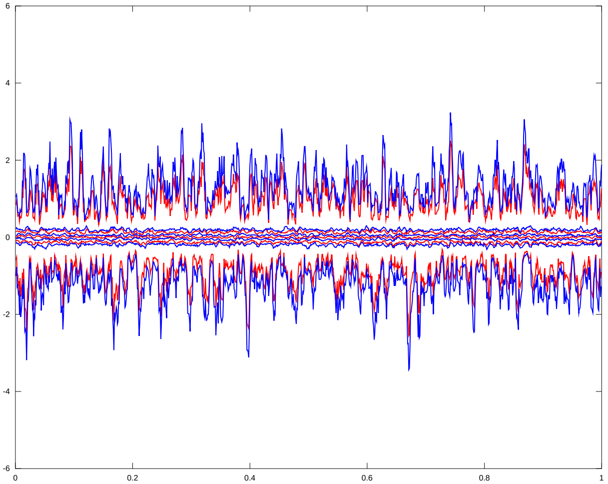
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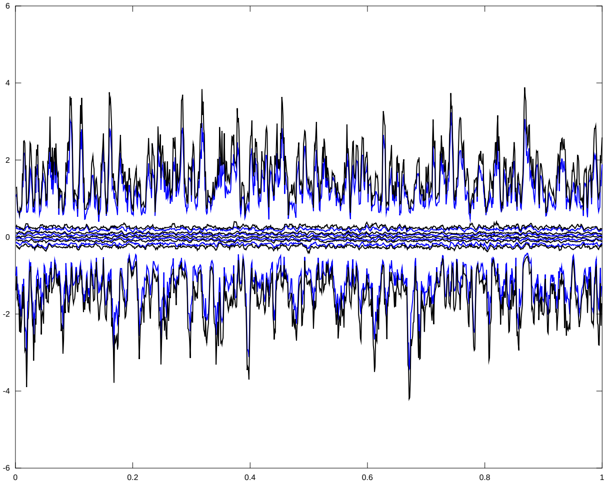
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-6
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0.2
0.4
0.6
0.8
1

Theorem

Take a sequence $\{(n_i, t_i, x_i)\}_{i=1}^{k}$ of times, levels and positions. Let them follow a space like path, i.e.

$$t_1 \leq t_2 \leq \cdots \leq t_k, \tag{14}$$

$$n_1 \geq n_2 \geq \cdots \geq n_k. \tag{15}$$

Then the density of the event that there is a particle at time t_i on level n_i at position x_i in the Dyson Brownian minor process is

$$\rho((n_1, t_1, x_1), \ldots, (n_k, t_k, x_k)) = \det[K^{OU}((n_i, t_i, x_i), (n_j, t_j, x_j))]_{i,j=}^{k} \tag{16}$$

Kernel for the Dyson Brownian minor process

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Background
Shuffling algorithm
Warren's Process
Aztec diamond point process
Asymptotics
Borodin & Ferrari
Dyson's Brownian Minor process

For $n \leq m$ or $s \geq t$,

$$K^{OU}((n,x,s),(m,y,t)) = \sum_{k=-\infty}^{-1} \sqrt{\frac{(n+k)!}{(m+k)!}} e^{-k(t-s)} h_{n+k}(x) h_{m+k}(y) e^{-y^2},$$

and otherwise,

$$K^{OU}((n,x,s),(m,y,t)) = -\sum_{k=0}^{\infty} \sqrt{\frac{(n+k)!}{(m+k)!}} e^{-k(t-s)} h_{n+k}(x) h_{m+k}(y) e^{-y^2}.$$

Kernel for the Warren process

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Background
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Asymptotics
Borodin & Ferrari
Dyson's Brownian Minor process

For $n \leq m$ or $s \geq t$,

$$K^W((n,x,s),(m,y,t)) =$$
$$\frac{1}{\sqrt{s}} \sum_{k=-\infty}^{-1} \sqrt{\frac{(n+k)!}{(m+k)!}} \left(\frac{s}{t}\right)^{k/2} h_{n+k}(x/\sqrt{s}) h_{m+k}(y/\sqrt{t}) e^{-y^2/t},$$

and otherwise,

$$K^W((n,x,s),(m,y,t)) =$$
$$-\frac{1}{\sqrt{s}} \sum_{k=0}^{\infty} \sqrt{\frac{(n+k)!}{(m+k)!}} \left(\frac{s}{t}\right)^{k/2} h_{n+k}(x/\sqrt{s}) h_{m+k}(y/\sqrt{t}) e^{-y^2/t},$$

Theorem

Let a be a real number on the interval $(-1,1)$. In the bulk scaling limit around $a\sqrt{2N}$ the Dyson Brownian minor process converges, along spacelike paths, to a time dependent Bead kernel with parameter a. More precisely,

$$K_a^{\mathrm{Bead}}((n,x,s),(m,y,t)) = \lim_{N\to\infty} e^{-N(t-s)}(4N)^{\frac{1}{2}(n-m)}(2N)^{-\frac{1}{2}}\times \times K^{\mathcal{DBM}}((N+n,\sqrt{2N}a+\frac{x}{\sqrt{2N}},\frac{s}{2N}),(N+m,\sqrt{2N}a+\frac{y}{\sqrt{2N}},\frac{t}{2N})) \tag{17}$$

The limit holds uniformly on compact sets.

$$K_a^{\mathrm{Bead}}((n,x,s),(m,y,t)) := -\phi_a^{\mathrm{Bead}}((n,x,s),(m,y,t)) + \frac{1}{2\pi i}\int_{u_-}^{u_+} u^{m-n} e^{\frac{1}{2}(t-s)(u^2-2au)+u(x-y)}\,du \quad (18)$$

where, for $n \le m$ or $s \ge t$,

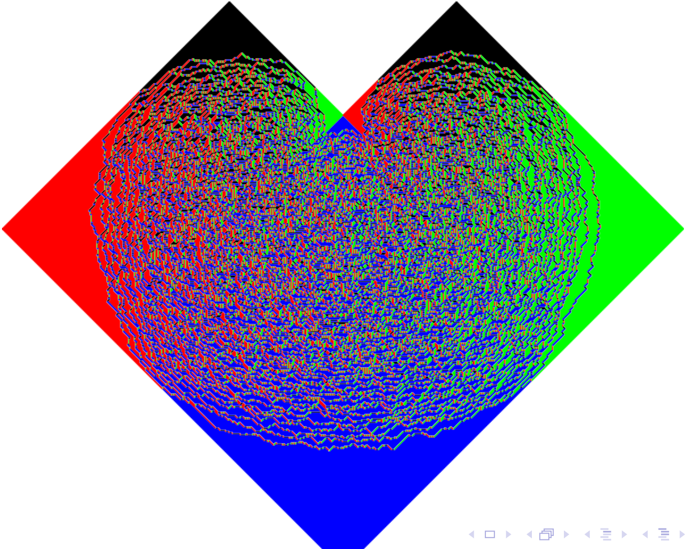
$$\phi_a^{\mathrm{Bead}}((n,x,s),(m,y,t)) = 0 \quad (19)$$

otherwise

$$\phi_a^{\mathrm{Bead}}((n,x,s),(m,y,t)) = 2^{\frac{1}{2}(n-m)} \int_{\mathbb{R}} H^{n-m}(x-z) p_{\frac{1}{2}(t'-t)}(z, y-a(t-s))\,dz \quad (20)$$

and

$$u_\pm = a \pm i\sqrt{1-a^2}. \quad (21)$$



Related work

- Dieker & Warren, *Determinantal Transition Kernels for some Interacting Particles on a Line*, arXiv:0707.1843v2.
- Johansson, *A Multi-Dimenstional Markov Chain and the Meixner Ensemble*, arXiv:0707.0098v1.
- Borodin & Ferrari, *Anisotropic growth of random surfaces in 2+1 dimensions*, arXiv:0804.3035.
- Borodin & Gorin, *Shuffling algorithm for boxed plane partitions*, arXiv:0804.3071.
- Metcalfe, O'Connell & Warren, *Interlaced processes on the circle*, arXiv:0804.3142.
- Warren & Windridge, *Some Examples of Dynamics for Gelfand Tsetlin Patterns*, arXiv:0812.0022.

Thank you for your attention.

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- N, *On the Shuffling Algorithm for Domino Tilings*, arXiv:0802.2592, Electron. J. of Probab. 15 (2010), no. 3, pp. 75-95
- Mark Adler, N, Piere van Moerbeke, *The Dyson Brownian Minor process*, work in progress.
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