# A probabilistic view of the quantum Toda lattice

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# Prelude

Weyl chamber:

$$C_k = \{x \in \mathbb{R}^k : \; x_1 \geq x_2 \geq \cdots \geq x_k\}.$$

Interlacing: for $x \in C_k$, $y \in C_{k-1}$ write $x \preceq y$ if

$$x_1 \geq y_1 \geq x_2 \geq \cdots y_{k-1} \geq x_k.$$

For $x \in \mathbb{R}^N$, denote

$$\Gamma_N(x) = \{(T_{k,i})_{1 \leq i \leq k \leq N} \in \mathbb{R}^{N(N-1)/2} : \; T_{N,i} = x_i, \; 1 \leq i \leq N\}$$

Gelfand-Tsetlin polytope (assuming $x \in C_N$):

$$\mathrm{GT}_N(x) = \{T \in \Gamma_N(x) : \; x \equiv T_{N,\cdot} \preceq T_{N-1,\cdot} \preceq \cdots T_{2,\cdot} \preceq T_{1,\cdot}\}$$

# Prelude

It is well-known that

$$Vol(GT_N(x)) = \int_{\Gamma_N(x)} \prod_{k=1}^{N-1} \prod_{i=1}^{k} 1_{T_{k,i} \le T_{k+1,i}} 1_{T_{k+1,i+1} \le T_{k,i}}$$

$$= \left( \prod_{k=1}^{N-1} k! \right)^{-1} h(x)$$

where $h(x) = \prod_{i<j}(x_i - x_j)$.

# Prelude

Note that

$$1_{x \leq y} = \lim_{\beta \to \infty} \exp(-e^{\beta(x-y)}) \qquad x \neq y$$

Consider the substitution:

$$1_{x \leq y} \iff \exp(-e^{x-y})$$

# The quantum Toda lattice

The quantum Toda lattice is a quantum integrable system with Hamitonian given by the Schrödinger operator

$$H = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{N-1} e^{x_{i+1}-x_i}.$$

It is closely associated with the Lie group $GL(N, \mathbb{R})$.

More generally,

$$H = \Delta_{\mathfrak{a}} - 2 \sum_{\text{simple } \alpha} e^{-\alpha(x)}.$$

# Givental's integral formula

Givental (1997): The eigenfunctions of $H$ are given by

$$\psi_\nu(x) = \int_{\Gamma_N(x)} e^{\mathcal{F}_\nu(T)} \prod_{k=1}^{N-1} \prod_{i=1}^{k} dT_{k,i},$$

where

$$\mathcal{F}_\nu(T) = \sum_{k=1}^{N} \nu_k \left( \sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) - \sum_{k=1}^{N-1} \sum_{i=1}^{k} \left( e^{T_{k,i}-T_{k+1,i}} + e^{T_{k+1,i+1}-T_{k,i}} \right).$$

# Recursive structure

Write $H = H^{(N)}$, $\psi_\nu = \psi_\nu^{(N)}$. Set $H^{(1)} = d^2/dx^2$, $\psi_\lambda^{(1)}(x) = e^{\lambda x}$.

Define a kernel on $\mathbb{R}^N \times \mathbb{R}^{(N-1)}$ by

$$Q_\theta^{(N)}(x,y) = \exp\left(\theta\left(\sum_{i=1}^N x_i - \sum_{i=1}^{N-1} y_i\right) - \sum_{i=1}^{N-1}\left(e^{y_i - x_i} + e^{x_{i+1}-y_i}\right)\right);$$

$$Q_\theta^{(N)} f(x) := \int_{\mathbb{R}^{N-1}} Q_\theta^{(N)}(x,y) f(y) dy.$$

Givental's formula is equivalent to:

$$\psi^{(N)}_{\nu_1,\ldots,\nu_N} = Q^{(N)}_{\nu_N} \psi^{(N-1)}_{\nu_1,\ldots,\nu_{N-1}}.$$

# An intertwining relation

Gerasimov et al (2006):

$$(H^{(N)} - \theta^2) \circ Q_\theta^{(N)} = Q_\theta^{(N)} \circ H^{(N-1)}.$$

Equivalently,

$$(H_x^{(N)} - \theta^2) Q_\theta^{(N)}(x,y) = H_y^{(N-1)} Q_\theta^{(N)}(x,y).$$

Combining this with

$$\psi^{(N)}_{\nu_1,\dots,\nu_N} = Q^{(N)}_{\nu_N} \psi^{(N-1)}_{\nu_1,\dots,\nu_{N-1}}$$

yields the eigenvalue equation

$$H^{(N)} \psi_\nu^{(N)} = \left( \sum_{i=1}^N \nu_i^2 \right) \psi_\nu^{(N)}.$$

# Some probability

The Schrödinger operator

$$\frac{1}{2}H = \frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{N-1} e^{x_{i+1}-x_i}$$

is the infinitesimal generator of a Brownian motion in $\mathbb{R}^N$ killed at rate

$$V(x) = \sum_{i=1}^{N-1} e^{x_{i+1}-x_i}$$

when it is at position $x$.

# Some probability

This process can be conditioned to survive forever by a Doob transform via the (positive) $H$-harmonic function $\psi_0$.

The conditioned process has infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\psi_0(x)^{-1} H \psi_0(x) = \frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla.$$

Its transition kernel is given by

$$q_t(x, y) = \frac{\psi_0(y)}{\psi_0(x)} p_t(x, y),$$

where $p_t$ is the sub-Markov transition kernel associated with $H/2$.

# Some probability

The intertwining relation becomes

$$\mathcal{L}^{(N)} \circ K^{(N)} = K^{(N)} \circ \mathcal{L}^{(N-1)},$$

where $K^{(N)}$ is the Markov kernel/operator

$$K^{(N)}(x,y) = \psi_0^{(N)}(x)^{-1} Q_0^{(N)}(x,y).$$

This suggests that we can couple the Markov processes with generators $\mathcal{L}^{(N)}$ and $\mathcal{L}^{(N-1)}$.

# The coupled processes

Define a Markov process $((X(t), Y(t)), t \geq 0)$ taking values in $\mathbb{R}^N \times \mathbb{R}^{(N-1)}$, as follows.

The process $Y$ evolves as an autonomous Markov process with generator $\mathcal{L}^{(N-1)}$. Let $W$ be standard one-dim. Brownian motion, independent of $Y$, and define the evolution of $X$ via the SDEs

$$
\begin{aligned}
dX_1 &= dY_1 + e^{X_2 - Y_1} dt \\
dX_2 &= dY_2 + \left( e^{X_3 - Y_2} - e^{X_2 - Y_1} \right) dt \\
&\vdots \\
dX_{N-1} &= dY_{N-1} + \left( e^{X_N - Y_{N-1}} - e^{X_{N-1} - Y_{N-2}} \right) dt \\
dX_N &= dW - e^{X_N - Y_{N-1}} dt.
\end{aligned}
$$

# The coupled processes

**Theorem**

*Let $(X, Y)$ be the above Markov process, started with initial law*

$$\lambda^x = \delta_x \times K^{(N)}(x, \cdot).$$

*Then $X$ is a Markov process (in its own filtration) with generator $\mathcal{L}^{(N)}$ started at $x$. Moreover, for each $t \geq 0$, the conditional law of $Y(t)$, given $\{X(s),\ s \leq t;\ X(t) = x\}$, is given by $K^{(N)}(x, \cdot)$.*

# The coupled processes

**Proof.**

The intertwining

$$\mathcal{L}^{(N)} \circ K^{(N)} = K^{(N)} \circ \mathcal{L}^{(N-1)}$$

extends to

$$\mathcal{L}^{(N)} \circ \tilde{K}^{(N)} = \tilde{K}^{(N)} \circ \mathcal{G}^{(N)},$$

where

$$\tilde{K}^{(N)}(x,(x,y)) = \delta_x \times K^{(N)}(x,\cdot)$$

and

$$\mathcal{G}^{(N)}_{x,y} = \mathcal{L}^{(N-1)}_y + \mathcal{I}^{(N)}_{x,y}$$

is the generator of the process $(X, Y)$. □

# A representation theorem

With a bit of extra work, it follows that a certain 'exponential functional' $\mathcal{T}W$ of a Brownian motion $W$ in $\mathbb{R}^N$ is Markov with generator $\mathcal{L}^{(N)}$.

The first coordinate of $\mathcal{T}W(t)$ is given by $\log Z_t^N$ where

$$Z_t^N = \int_{0=s_0<s_1<\cdots<s_{N-1}<s_N=t} \exp\left(\sum_{i=1}^N W_i(s_i) - W_i(s_{i-1})\right) ds_1 \ldots ds_{N-1}.$$

This is the partition function for (1+1)-dimensional directed polymer in a random environment which was introduced and studied in O'C-Yor (2001), Moriarty-O'C (2007).

# Definition of $\mathcal{T}$

For $i = 1, \dots, N-1$, and continuous $\eta : (0, \infty) \to \mathbb{R}^N$, define

$$(\mathcal{T}_i \eta)(t) = \eta(t) + \left( \log \int_0^t e^{\eta_{i+1}(s) - \eta_i(s)} ds \right) (e_i - e_{i+1}),$$

where $e_1, \dots, e_N$ denote the standard basis vectors in $\mathbb{R}^N$.

The operator $\mathcal{T}$ is defined by

$$\mathcal{T} = (\mathcal{T}_1 \circ \cdots \circ \mathcal{T}_{N-1}) \circ \cdots \circ (\mathcal{T}_1 \circ \mathcal{T}_2) \circ \mathcal{T}_1.$$

The operators $\mathcal{T}$ and $\mathcal{T}_i$ are studied extensively in the papers Biane-Bougerol-O'C (05,09), in a more general setting, where various Lie-theoretic interpretations are given.

# The case $N = 2$

When $N = 2$, the eigenfunctions $\psi_\nu$ are given by

$$\psi_\nu(x) = 2\exp\left(\frac{1}{2}(\nu_1+\nu_2)(x_1+x_2)\right) K_{\nu_1-\nu_2}\left(2e^{(x_2-x_1)/2}\right)$$

and we recover the following:

**Theorem (Matsumoto-Yor '99)**

*Let $(B_t, t \geq 0)$ be a one-dimensional Brownian motion and*

$$Z_t = \int_0^t e^{2B_s - B_t} ds.$$

*Then $\log Z$ is a diffusion with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{d}{dx}\log K_0(e^{-x})\right)\frac{d}{dx}.$$

# Connection to random matrices I

Let $\beta > 0$, and define

$$H_\beta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2\beta^2 \sum_{i=1}^{N-1} e^{\beta(x_{i+1}-x_i)},$$

$$\mathcal{L}_\beta = \frac{1}{2}\psi_0(\beta x)^{-1} H_\beta \psi_0(\beta x) = \frac{1}{2}\Delta + \nabla \log \psi_0(\beta \cdot) \cdot \nabla.$$

# Connection to random matrices I

As $\beta \to \infty$,

$$\beta^{-N(N-1)/2} \psi_0(\beta x) \to \left( \prod_{k=1}^{N-1} k! \right)^{-1} h(x)$$

where

$$h(x) = \prod_{1 \le i < j \le N} (x_i - x_j).$$

Thus,

$$\mathcal{L}_\beta \to \frac{1}{2} h(x)^{-1} \Delta_C h(x) = \frac{1}{2} \Delta + \nabla \log h \cdot \nabla$$

where $\Delta_C$ is the Dirichlet Laplacian in the Weyl chamber

$$C = \{x \in \mathbb{R}^N : \; x_1 > \cdots > x_N\}.$$

# Connection to random matrices II

The diffusion with generator $\mathcal{L}$ has an entrance law given by

$$\mu_t(dx) = \psi_0(x)\vartheta_t(x)dx, \qquad t > 0,$$

where $\vartheta_t$ is characterized by

$$\int_{\mathbb{R}^N} \psi_\lambda(x)\vartheta_t(x)dx = \exp\left(\frac{1}{2}\sum_{i=1}^N \lambda_i^2 t\right), \qquad \lambda \in \iota\mathbb{R}^N.$$

The law of $\mathcal{T}W(t)$ is given by $\mu_t$ and

$$P(\log Z_t^N \le u) = \mu_t(\{x \in \mathbb{R}^N : \; x_1 \le u\}).$$

# Connection to random matrices II

The fact that this characterizes $\vartheta_t$ follows from the Plancherel theorem of Semenov-Tian-Shansky (cf. Kharchev-Lebedev) which states that

$$f \mapsto \int_{\mathbb{R}^N} f(x)\psi_\lambda(x)dx$$

is an isometry from $L_2(\mathbb{R}^N, dx)$ to $L_2(\iota\mathbb{R}^N, s(\lambda)d\lambda)$, where $s(\lambda)$ is the *Sklyanin measure* defined by

$$s(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j\neq k} \Gamma(\lambda_j - \lambda_k)^{-1}.$$

In particular,

$$\mu_t(dx) = \psi_0(x) \left( \int_{\iota\mathbb{R}^N} \psi_\lambda(x) e^{\sum_i \lambda_i^2 t/2} s(\lambda) d\lambda \right) dx.$$

# Connection to random matrices II

The probability measure on $\iota\mathbb{R}^N$ with density proportional to

$$\exp(\sum_i \lambda_i^2 t/2) s(\lambda)$$

can be interpreted (up to factor of $\iota$) as the law, at time $1/t$, of the radial part of a Brownian motion in the symmetric space of positive definite $N \times N$ Hermitian matrices or, equivalently, the law of the eigenvalues, at time $1/t$, of an $N \times N$ Hermitian Brownian motion with drift

$$\text{diag}\,(N-1, N-3, \ldots, 3-N, 1-N).$$

# The law of the partition function

By Plancherel theorem, the functions $\lambda \mapsto \psi_\lambda(x)$, $x \in \mathbb{R}^N$, are an ONB for $L_2(\iota\mathbb{R}^N, s(\lambda)d\lambda)$. This fact, combined with a Mellin-Barnes type integral formula for $\psi_\lambda$ due to Kharchev and Lebedev (1999) yields:

## Corollary

*The probability density of* $\log Z_t^N$ *is given by*

$$p_t(a) = \int_{\iota\mathbb{R}^N} \int s_{N-1}(\gamma) Q(\gamma, 0) Q(\gamma, \lambda) e^{a(\sum \lambda_i - 2\sum \gamma_i)} e^{\sum \lambda_i^2 t/2} s_N(\lambda) d\gamma d\lambda,$$

*where second integral is along vertical lines with* $\Re\gamma_i < \Re\lambda_j$ *for all* $i, j$,

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}, \quad Q(\gamma, \lambda) = \prod_{i,j} \Gamma(\lambda_i - \gamma_j).$$

# Hypergroup property

## Corollary

*For each* $x, y \in \mathbb{R}^N$,

$$\frac{\psi_\lambda(x)}{\psi_0(x)} \frac{\psi_\lambda(y)}{\psi_0(y)} = \int_{\mathbb{R}^N} \frac{\psi_\lambda(z)}{\psi_0(z)} \gamma^{x,y}(dz)$$

*where* $\gamma^{x,y}$ *is a probability measure on* $\mathbb{R}^N$.

The probability measure $\gamma^{x,y}$ can be interpreted as the conditional law of $\mathcal{T}W(s+t)$ given $\mathcal{T}W(s) = x$, $(\mathcal{T}\tau_s W)(t) = y$, where $0 < s < t$ and $\tau_s W(\cdot) = W(s+\cdot) - W(s)$.

$\longrightarrow$ 'Tropical' version of Bessel-Kingman hypergroup.

cf. Biane, Bougerol, O'C (2009).

# Hypergroup property

In the case $N = 2$ this is equivalent to:

$$K_\nu(z)K_\nu(w) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}[t+(z^2+w^2)/t]} K_\nu\left(\frac{zw}{t}\right)\frac{dt}{t}.$$

(Dixon and Ferrar 1933)

# The wider context

The map $W \mapsto (\mathcal{T}W, \ldots)$ is in fact a variation of the so-called RSK correspondence, a combinatorial algorithm in the theory of Young tableaux. It can be thought of as a 'tropicalization' of a certain 'specialization' of RSK introduced and studied in Bougerol-Jeulin (02), O'C-Yor (02), Biane-Bougerol-O'C (05,09).

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Alternatively, it can be regarded as a specialization of 'tropical RSK' (Kirillov '00, Noumi-Yamada '04, . . . )

Tropical RSK is closely related to Dodgson's (who's he?) condensation method for computing determinants.

# The wider context

When $\beta \to \infty$ we recover the multidimensional version of Pitman's $2M - X$ theorem obtained in Bougerol-Jeulin (02), O'C-Yor (02).

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The quantum Toda lattice has a $q$-analogue which appears to have a similar structure, and is an interesting direction for future research. In the rank 1 case, the $q$-Hermite polynomials play a central role.

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