Limiting distributions for Tasep.

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Plan

- I. Tasep: a review of definitions and some known results.
- II. Connection with last passage percolation and queues.
- III. Limiting distribution in/out of equilibrium.
- IV. Some ideas of the proof.

Tasep

The Totally Asymmetric Simple Exclusion Process (TASEP) non-reversible interacting particle system:
Consider a configuration of particles $\eta_t \in \{0,1\}^{\mathbb{Z}}$, $t \geq 0$ with the following meaning

$\eta_t(i) = 1$: there is a particle at site i at time t;

There is at most one particle at each site (simple).

Given η_o, the dynamics is defined as follows:
Particles can jump to the neighboring right site only (Asymmetric)
provided that the site is empty (exclusion).

Jumps are independent of each other and take place after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty.

Standard initial conditions

-step initial condition : $\eta_o(i) = 0$ if $i > 0$ and $\eta_o(i) = 1$ if $i \leq 0$;
-flat initial condition : $\eta_o(i) = 0$ if i is odd and $\eta_o(i) = 1$ if i is even.
-Invariant measures:

- $\eta_o(i)$, $i \in \mathbb{Z}$ i.i.d. Bernoulli with a given density $\rho \in [0, 1]$ (translation invariant) known as equilibrium Tasep

- blocking measure (all sites occupied to the right of some site i)

-two sided initial condition: Bernoulli independent random variables with density ρ_- (resp. ρ_+) on $\mathbb{Z}_-$ (resp. $\mathbb{Z}_+$).

What is the large time behavior ?

Some quantities of interest I

The height function

$$h_t(j) = \begin{cases} 2N_t + \sum_{i=1}^{j}(1-2\eta_i(t)), & \text{for } j \geq 1, \\ 2N_t, & \text{for } j = 0, \\ 2N_t - \sum_{i=j+1}^{0}(1-2\eta_i(t)), & \text{for } j \leq -1, \end{cases}.$$

where N_t is the number of particles which jumped from site 0 to site 1 during the time-span $[0,t]$.

Assign label 0 to the particle sitting at the smallest positive integer site initially. Then use the ordering $\cdots < \mathbf{x}_2(0) < \mathbf{x}_1(0) < 0 \leq \mathbf{x}_0(0) < \mathbf{x}_{-1}(0) < \cdots$. Then $\mathbf{x}_k(t) > \mathbf{x}_{k+1}(t)$ for all $t \geq 0$.

$$\mathbb{P}(\cap_{k=1}^{m}\{h_{t_k}(x_k - y_k) \geq x_k + y_k\}) = \mathbb{P}(\cap_{k=1}^{m}\{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}).$$

Some quantities of interest II

Current of particles past an observer moving at speed x

$$\begin{aligned} J(xt,t) := \quad & \sharp \text{ particles to the left of } 0 \text{ at } t=0 \text{ and to the right of } xt \text{ at time } t \\ & -\sharp \text{ particles to the right of } 0 \text{ at } t=0 \text{ and to the left of } xt \text{ at time } t \ . \end{aligned}$$

Relationship to height function: $h_t(j) = 2J(j,t) + j$.

One fundamental result: for step initial condition.

$$J(k,t) := \sum_{j>k} \eta_j(t) > m \equiv \text{ particle started at } -m \text{ has made } \geq m+k+1 \text{ steps at } t.$$

Theorem Rost ('81) Johansson ('98) For each $u \in [0,1)$,

$$\lim_{t\to\infty} \mathbb{P}\left(J([ut],t) \leq \frac{t(1-u^2)}{4} + \frac{(1-u)^{2/3}}{(1+u)^{1/3}} x t^{1/3} \right) = 1 - F_{GUE}(-x),$$

where $F_{GUE}(x)$ is the GUE Tracy-Widom distribution.

Equilibrium Tasep

For equilibrium Tasep : product (Bernoulli (ρ)) as initial condition

Theorem LLN and CLT (Ferrari-Fontes (94))

$$\lim_{t\to\infty} \frac{J(xt,t)}{t} = \rho(1-\rho) - x\rho;$$
$$\frac{J(xt,t) - \mathbb{E}J(xt,t)}{\sqrt{t}} \xrightarrow{d} \mathcal{N}\left(0, \rho(1-\rho)|1-2\rho-x|\right).$$

A critical velocity: $x = 1 - 2\rho$.

Coming soon: Ferrari-Spohn (2005)

If $x = 1 - 2\rho$ fluctuations are in the order of Or$(t^{1/3})$.

Connections and translation to a Last Passage Percolation problem

Queues

Suppose that there are infinitely many servers with FIFO policy:

- the service time of customers at each server i.i.d. $\mathrm{Exp}(1)$.

- once a customer is served at the server i, she joins at the $(i+1)$th queue.

System in equilibrium with parameter ρ: the arrival process at each queue is independent Poisson process of rate ρ. Then (Burke) the departure process at each queue is also independent Poisson process of rate ρ.

Consider a fixed time $t = 0$ and arbitrary select one customer: assign label 0 to that customer and the queue where she is the 0th queue. We assign labels to the other customers so that the labels decreases for the customers ahead in the queues.

Equilibrium Tasep and queues

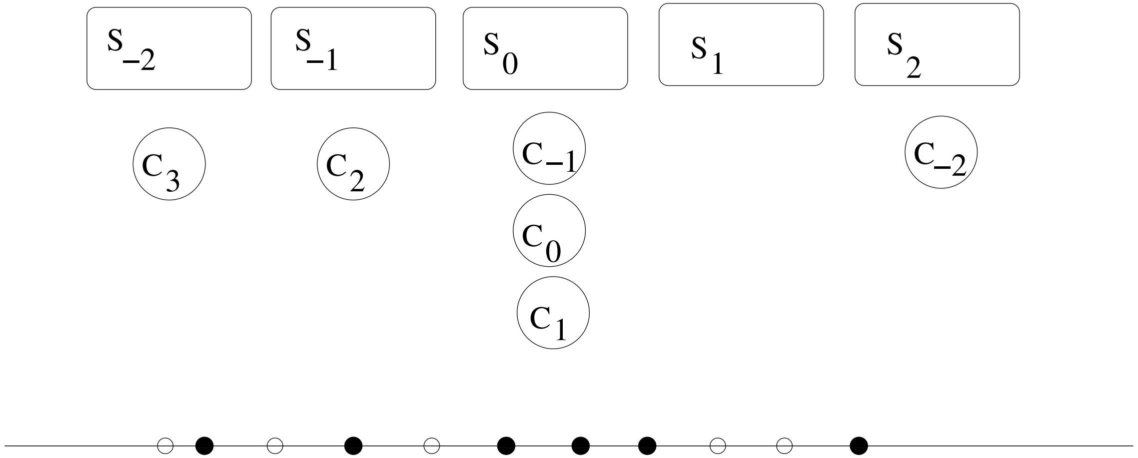


Figure 1: Queues in tandem in equilibrium. The black dots represents the customers and at every white dots one changes to the next counter.

Let $Q_j(t)$ denote the label of the queue in which the jth customer is in at time t. Then

$$\mathbf{x}_j(t) = Q_j(t) - j.$$

Call also $E_j(i)$ be the time the jth customer exits the queue i, then we find that

$$\mathbb{P}(\cap_{k=1}^m \{E_{y_k}(x_k - 1) \leq t_k\}) = \mathbb{P}(\cap_{k=1}^m \{Q_{y_k}(t_k) \geq x_k\}) = \mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}).$$

Last passage percolation

At each site $(i,j) \in \mathbb{N}^2$, a random variable w_{ij} is attached. The w_{ij}'s are independent (waiting time) not necessarily identically distributed.

An up-right path π from $(0,0)$ to $(x,y) \in \mathbb{N}^2$ is a sequence of points $(\pi_k \in \mathbb{Z}^2, k = 0, \dots, x+y)$, with $\pi_0 = (0,0)$ and $\pi_{x+y} = (x,y)$, and satisfying $\pi_{k+1} - \pi_k \in \{(1,0),(0,1)\}$.

Set $L(\pi) = \sum_{(i,j)\in\pi} w_{i,j}$. Then, the last passage time is defined by

$$G(x,y) = \max_{\pi:(0,0)\to(x,y)} L(\pi).$$

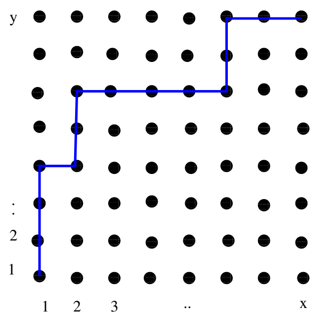
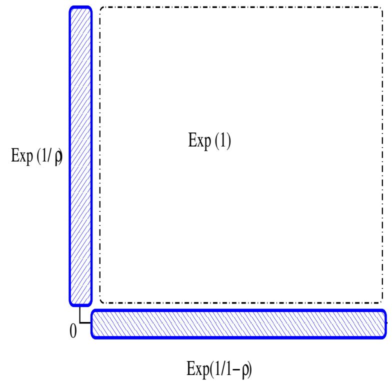


Figure 2: An upright path.

Equilibrium Tasep and LPP

Let $w_{i,j}$, $i,j \geq 0$, $i,j \in \mathbb{Z}$, be independent random variables with

$$
\begin{aligned}
&w_{0,0} = 0, \\
&w_{0,j} \sim \text{ exponential with mean } 1/\rho,\ j \geq 1, \\
&w_{i,0} \sim \text{ exponential with mean } 1/(1-\rho),\ i \geq 1, \\
&w_{i,j} \sim \text{ exponential with mean } 1, \quad i,j \geq 1.
\end{aligned}
$$



Associated last passage time : $G(x,y) = \max_{\pi:(0,0)\to(x,y)} L(\pi)$.

Equilibrium Tasep: initial configuration η_o is the Bernoulli ρ product measure.

If $\underline{x_k, y_k \to \infty}$,

$$\lim \mathbb{P}(\cap_{k=1}^{m}\{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}) = \lim \mathbb{P}(\cap_{k=1}^{m}\{G(x_k,y_k) \leq t_k\}).$$

Explanation: two sided boundary condition

Assume that η_o is the product measure of Bernoulli with parameter $\rho^{\pm}$ on $\mathbb{Z}^{\pm}$.

Theorem Praehofer-Spohn (2001)
Let ζ^+ (resp. ζ^-) be geometric random variables with parameter $1-\rho^+$ (resp. ρ_-). The $\{w(i,j),(i,j)\in\mathbb{N}^2\}$ are independent:

$$
\begin{aligned}
&w(i,j) \text{ is exponential with mean } 1, \forall i,j\geq 1, \quad w(0,0)=0,\\
&w(j,0)=0,\ \forall 0\leq j\leq \zeta^+, \quad w(0,j)=0,\ \forall 0\leq j\leq \zeta^-,\\
&w(j,0) \text{ is exponential with mean } (1-\rho^+)^{-1}, j>\zeta^+,\\
&w(0,j) \text{ is exponential with mean } (\rho^-)^{-1}, j>\zeta^-.
\end{aligned}
$$

Define $\widehat{G}(x,y)$ to be the last passage time to (x,y) in this LPP model. Then,

$$\mathbb{P}(\cap_{k=1}^m\{\mathbf{x}_{y_k}(t_k)\geq x_k-y_k\})=\mathbb{P}(\cap_{k=1}^m\{\widehat{G}(x_k,y_k)\leq t_k\}).$$

Multipoint fluctuation results for LPP and then Tasep

LPP Away from the critical direction

A crucial role is played by the critical direction (which corresponds to the characteristic line of TASEP),

$$\frac{y}{x} = \gamma_c = \frac{\rho^2}{(1-\rho)^2}.$$

Along a direction other than the critical direction, the fluctuations of $G(x, y)$ are Gaussian on the $N^{1/2}$ scale: "competition" of two one source models.

Define $Q(a, b) = G((a, b), (x, y))$ to be the passage time from (a, b) to (x, y). Then

$$G(x, y) = \max\{Q(0, 1), Q(1, 0)\}.$$

$Q(0, 1)$ cannot "see" the first line thus one source model: LPP with exponential r.v. with mean 1 except on the first column only.
$Q(0, 1)$ has the same distribution as the largest eigenvalue of a well-chosen Wishart random matrix.

Gaussian fluctuations

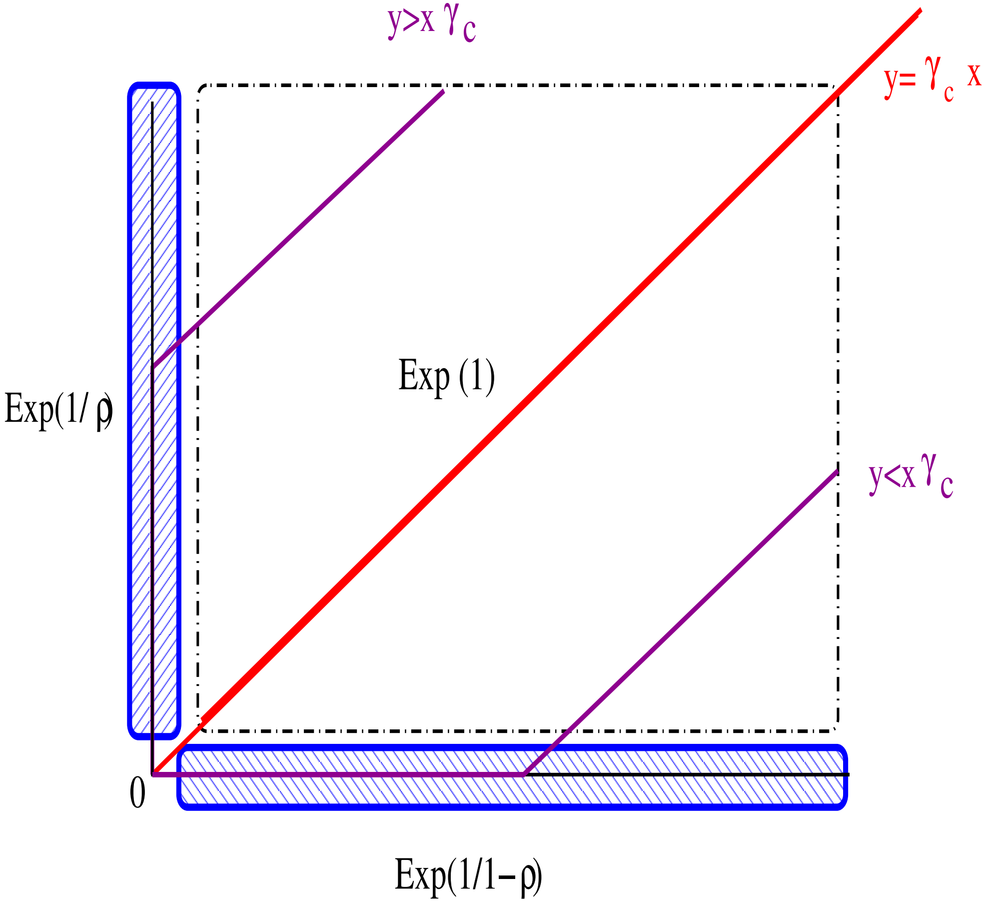
Assume that $\gamma > \gamma_c$ and set $x = \frac{\gamma}{1+\gamma}N$, $\quad y = \frac{1}{1+\gamma}N$, and
$c_1 = \frac{\gamma}{1+\gamma}(\frac{1}{\rho} + \frac{1}{\gamma(1-\rho)})$, $\quad c_1' = \frac{\gamma}{1+\gamma}(1+\frac{1}{\sqrt{\gamma}})^2$.

Theorem Baik-GBA-Peche (2005) If $\gamma > \gamma_c$, there exist constants c_2, c_2' such that

$$\lim_{N\to\infty} \mathbb{P}(Q(1,0) \le c_1 N + c_2 s N^{1/2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} dx\, e^{-x^2/2} \equiv \Phi(s),$$
$$\lim_{N\to\infty} \mathbb{P}(Q(0,1) \le c_1' N + c_2' s N^{2/3}) = F_{\mathrm{GUE}}(s), \text{ Tracy-Widom.}$$

As $c_1' < c_1$ we get that
$$\lim_{N\to\infty} \mathbb{P}(G(x,y) \le c_1 N + c_2 s N^{1/2}) = \Phi(s).$$

Away from the critical direction



Multipoint limiting distribution at equilibrium

Theorem Ferrari-Spohn (2005) Baik-Ferrari-Peche (2010). Set $\chi := \rho(1-\rho)$.

$$x(\tau) = \left\lfloor (1-\rho)^2 T + \tau \frac{2T^{\frac{2}{3}}\chi^{\frac{4}{3}}}{1-2\chi} \right\rfloor \quad y(\tau) = \left\lfloor \rho^2 T - \tau \frac{2T^{\frac{2}{3}}\chi^{\frac{4}{3}}}{1-2\chi} \right\rfloor,$$

$$\ell(\tau, s) = T - \tau \frac{2(1-2\rho)\chi^{\frac{1}{3}}}{1-2\chi} T^{\frac{2}{3}} + s \frac{T^{\frac{1}{3}}}{\chi^{\frac{1}{3}}}.$$

Then given $m \in \mathbb{N}$ and real numbers $\tau_1 < \tau_2 < \ldots < \tau_m$ and $s_1, \ldots, s_m$,

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{G(x(\tau_k), y(\tau_k)) \le \ell(\tau_k, s_k)\} \right)$$
$$= \sum_{k=1}^{m} \frac{\partial}{\partial s_k} \left(g_m(\tau, s) \det\left(\mathbb{1} - P_s \widehat{K}_{\mathrm{Ai}} P_s\right)_{L^2(\{1,\ldots,m\}\times\mathbb{R})} \right),$$

where $L^2(\{1, \ldots, m\} \times \mathbb{R})$ is equipped with the standard measure $\nu \otimes dx$ where ν is the counting measure on $\{1, \ldots .m\}$. P_s denotes the projection operator $P_s(k, x) = \mathbb{1}_{[x > s_k]}$, and $\widehat{K}_{\mathrm{Ai}}$ is the so-called extended Airy kernel with shifted entries defined by the kernel

$$\widehat{K}_{\mathrm{Ai}}((i,x),(j,y)) := [\widehat{K}_{\mathrm{Ai}}]_{i,j}(x,y) = \begin{cases} \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda+\tau_i^2)\mathrm{Ai}(y+\lambda+\tau_j^2)e^{-\lambda(\tau_j-\tau_i)}, & \text{if } \tau_i \le \tau_j, \\ -\int_{-\infty}^0 d\lambda \mathrm{Ai}(x+\lambda+\tau_i^2)\mathrm{Ai}(y+\lambda+\tau_j^2)e^{-\lambda(\tau_j-\tau_i)}, & \text{if } \tau_i > \tau_j. \end{cases}$$

The function $g_m(\tau, s)$ is defined by

$$g_m(\tau,s) = \mathcal{R} + \langle \rho P_s \Phi, P_s \Psi \rangle = \mathcal{R} + \sum_{i=1}^m \sum_{j=1}^m \int_{s_i}^\infty dx \int_{s_j}^\infty dy\, \Psi_j(y)\rho_{j,i}(y,x)\Phi_i(x), \text{ where}$$

$$\mathcal{R} = s_1 + e^{-\frac{2}{3}\tau_1^3} \int_{s_1}^\infty dx \int_0^\infty dy\, \mathrm{Ai}(x+y+\tau_1^2)e^{-\tau_1(x+y)},$$

$$\Psi_j(y) = e^{\frac{2}{3}\tau_j^3+\tau_j y} - \int_0^\infty dx\, \mathrm{Ai}(x+y+\tau_j^2)e^{-\tau_j x},$$

$$\Phi_i(x) = e^{-\frac{2}{3}\tau_1^3} \int_0^\infty d\lambda \int_{s_1}^\infty dy\, e^{-\lambda(\tau_1-\tau_i)} e^{-\tau_1 y} \mathrm{Ai}(x+\tau_i^2+\lambda)\mathrm{Ai}(y+\tau_1^2+\lambda)$$

$$+ \quad \mathbb{1}_{[i\ge 2]} \frac{e^{-\frac{2}{3}\tau_i^3-\tau_i x}}{\sqrt{4\pi(\tau_i-\tau_1)}} \int_{-\infty}^{s_1-x} dy\, e^{-\frac{y^2}{4(\tau_i-\tau_1)}} - \int_0^\infty dy\, \mathrm{Ai}(y+x+\tau_i^2)e^{\tau_i y}.$$

for $i, j = 1, 2, \ldots, m$, where Ai denotes the Airy function and $\rho := (\mathbb{1} - P_s \widehat{K}_{\mathrm{Ai}} P_s)^{-1}, \qquad \rho_{j,i}(y,x) := \rho((j,y),(i,x)),$

Translation to Tasep

Consider the scaling

$$\begin{aligned} J(\tau) &= \lfloor (1-2\rho)T + 2\tau\chi^{1/3}T^{2/3} \rfloor, \\ H(\tau) &= \lfloor (1-2\chi)T + 2\tau(1-2\rho)\chi^{1/3}T^{2/3} - 2s\chi^{2/3}T^{1/3} \rfloor. \end{aligned}$$

Fix $m \in \mathbb{N}$. For real numbers $\tau_1 < \tau_2 < \ldots < \tau_m$ and $s_1, \ldots, s_m$. Then,

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{h_T(J(\tau_k)) \geq H(\tau_k)\} \right) = \sum_{k=1}^{m} \frac{\partial}{\partial s_k} \Big(g_m(\tau, s) \det \Big(\mathbb{1} - P_s \widehat{K}_{\mathrm{Ai}} P_s \Big) \Big).$$

:: Proof

Some ideas of the proof

We first consider a slightly different directed percolation model.

$$\begin{aligned}
&\widetilde{w}_{0,0} \sim \text{Exp}(1/(a+b)), \\
&\widetilde{w}_{i,0} \sim \text{Exp}(1/(1/2+b)), \qquad i \geq 1, \\
&\widetilde{w}_{0,j} \sim \text{Exp}(1/(1/2+a)), \qquad j \geq 1, \\
&\widetilde{w}_{i,j} \sim \text{Exp}(1), \qquad i,j \geq 1,
\end{aligned} \tag{1}$$

where the parameters a and b satisfy

$$a, b \in (-1/2, 1/2), \qquad a+b > 0. \tag{2}$$

We call $G^+(a,b)$ the associated last passage time.
We would like to choose $a = \rho - 1/2$ and $b = 1/2 - \rho$. Problem at the the origin!
But the process is determinantal...

Ideas of the proof II

More general version of directed percolation model: w_{ij} independent exponential random variables with mean $\frac{1}{\pi_i + \tilde{\pi}_j}$.

There are two "cuts" along which we obtain determinantal random point processes:
-the cut $x + y = N$ (the one used in the proof, initial idea of Praehofer-Spohn),
-the cut x constant: Consider an infinite array

$$A(N) = (A_{ij}), 1 \leq i \leq N, j \geq 1$$

where the A_{ij} are independent complex Gaussian random variables which are centered and of variance :

$$1/(\pi_i + \tilde{\pi}_j).$$

Define $A(N, n)$ to be the matrix obtained by considering the first n columns from A.
Theorem Borodin-Peche (2008) DiekerWarren (2009)
The largest eigenvalue process induced by $M(n) := A(N, n)A(N, n)^*$ and the process of last passage times $G(N, n), n \geq 1$ in the LPP model have the same distribution.

Ideas of the proof III

Let $G^+_{a,b}(x,y)$ (resp. $G_{a,b}(x,y)$) be the last passage time from $(0,0)$ to (x,y) for the modified (resp. initial) model. Note that the original model corresponds to the case where $a+b=0$ and $\widetilde{w}_{0,0}=0$.

(1) Shift argument: for $a,b\in(-1/2,1/2)$ with $a+b>0$, we have that

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{G_{a,b}(x_k,y_k)\le u_k\}\right)=\left(1+\frac{1}{a+b}\sum_{k=1}^{m}\frac{\partial}{\partial u_k}\right)\mathbb{P}\left(\bigcap_{k=1}^{m}\{G^+_{a,b}(x_k,y_k)\le u_k\}\right).$$

(2) Analytic continuation: find an expression for $G_{a,b}$ which can be analytically continued in all $a,b\in(-1/2,1/2)$.

(3) Choice of parameter: finally we set $a=\rho-1/2$ and $b=1/2-\rho$.

(4) Asymptotic analysis : saddle point.

:: Proof

Extension to points not aligned

The points $(x(\tau), y(\tau))$ are on the line $x + y = (1 - 2\chi)T$. We can extend to results to points which are not necessarily on the same line. This is a difficulty in principle: we have determinantal formulae only for points "really aligned".

We use slow-decorrelation: let $\nu \in [0, 1)$. If we compare the last passage time G at two points (x, y) and (x', y') with $(x' - x, y' - y) = \theta T^\nu \cdot ((1 - \rho)^2, \rho^2)$, their fluctuation will be $\theta T^\nu + \mathrm{Or}(T^{\nu/3})$.
Then the multipoint fluctuation theorem is unchanged with

$$x(\tau, \theta) = \left\lfloor (1 - \rho)^2 (T + \theta T^\nu) + \tau \frac{2T^{\frac{2}{3}}\chi^{\frac{4}{3}}}{1 - 2\chi} \right\rfloor \quad y(\tau, \theta) = \left\lfloor \rho^2 (T + \theta T^\nu) - \tau \frac{2T^{\frac{2}{3}}\chi^{\frac{4}{3}}}{1 - 2\chi} \right\rfloor,$$

$$\ell(\tau, s, \theta) = T + \theta T^\nu - \tau \frac{2(1 - 2\rho)\chi^{\frac{1}{3}}}{1 - 2\chi} T^{\frac{2}{3}} + s \frac{T^{\frac{1}{3}}}{\chi^{\frac{1}{3}}}.$$

Slow decorrelation

Theorem Ferrari-Spohn (2005)
Let $A = (c_1 T, c_2 T)$ for some $c_1, c_2 > 0$. Let then $B = A + r((1-\rho)^2, \rho^2)$ with $r \sim T^\nu$ with $0 < \nu < 1$. Then, for any $\beta \in (\nu/3, \frac{1}{3})$, $\lim_{T\to\infty} \mathbb{P}\left(|G(B) - G(A) - r| \leq T^\beta\right) = 1$.

In other words:

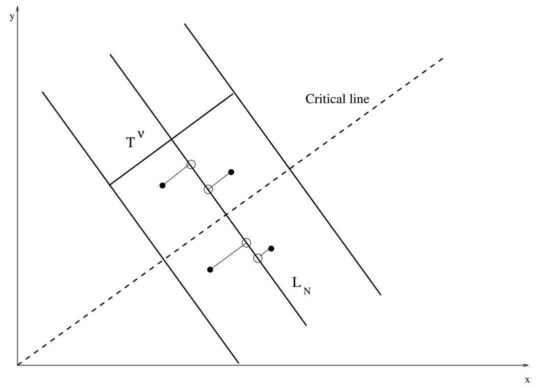


Figure 3: The black dots are Or(T^ν) for some $\nu < 1$ away from the line L_N. Then, fluctuations of the passage time at the locations of the black dots are, on the $T^{1/3}$ scale, the same as the one of their projection to L_N along the critical direction, the white dots.

Extension to two-sided boundary conditions

Tasep Initial condition: Bernoulli $\rho^{\pm}$ on $\mathbb{Z}^{\pm}$ or equivalently LPP with 2 sources.

Theorem Corwin-GBA(2009): limiting one point fluctuation for the LPP with two sources (and corresponding result for Tasep).

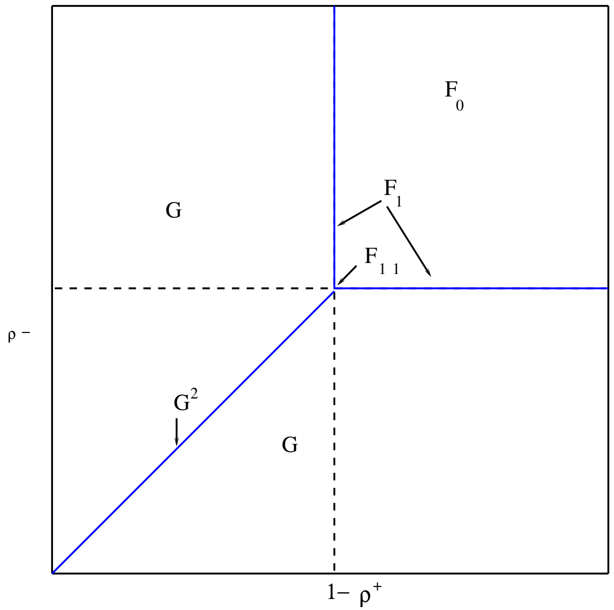
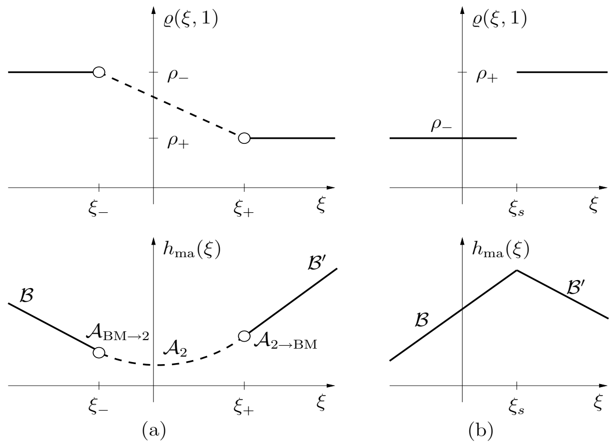


Figure 4: Limiting distribution for $G(x, y)$ when $y/x \to 1$.

Multipoint limiting fluctuation in Corwin-Ferrari-Peche (2010)

Fluctuations for Tasep



The asymptotic density ϱ and the limit shape in the cases (a) $\rho_- > \rho_+$ and (b) $\rho_- < \rho_+$. Transitions happen at $\xi_\pm = 1 - 2\rho_\pm$ and shockwave at $\xi_s = 1 - (\rho_- + \rho_+)$.

(a) $\rho_- > \rho_+$. The asymptotic density decreases linearly from $1-\rho_-$ to $1-\rho_+$ over the region from $(1-2\rho_-)t$ to $(1-2\rho_+)t$ (rarefaction fan). In this region the height fluctuations live on a $t^{1/3}$ scale and are governed by the Airy$_2$ process. Around positions $(1-2\rho_\pm)t$ there is a transition process from Airy$_2$ to Brownian Motion. To the left and right of the rarefaction fan, they are governed by Brownian Motion.
(b) $\rho_- < \rho_+$. For large t there is a macroscopic shock with density jump from ρ_- to ρ_+ around the position $(1-\rho_- -\rho_+)t$ The fluctuations on the left and on the right of the shock are independent.